BLACK HOLE PHYSICS

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6. PHYSICS IN ACCELERATED FRAMES

Curvature at the surface of a black hole of very large mass M is: $\Re \sim \nabla\nabla\varphi / c^2 \sim r_g^{-2} \sim M^{-2}$

(Show that $\Re$ is equal to a tidal force on the surface of the Earth for a black hole of mass $\sim 6\times 10^7 M_\odot$)

In the limit $M \to \infty$ an observer at rest near the horizon, at distance ℓ from it, has 4-acceleration equal to $a = c^2/\ell$. In this limit a ST in his/her vicinity is practically flat, and hence such an observer is equivalent to a uniformly accelerated observer in Minkowsky ST. This is nothing but the **equivalence principle.**

Uniformly accelerated frames

Spacetime model in special relativity is a 4D affine space M^4.

ST events are represented by points in this space $p \in M^4$.

A vector $\vec{a}$ connecting two points (events) p and q, $\vec{a} = q - p$, belongs to a linear space R^4 with a scalar product [in Cartesian coordinates (T, X, Y, Z)] $(\vec{a}, \vec{b}) = -a^T b^T + a^X b^X + a^Y b^Y + a^Z b^Z$.

The ST interval between two close points X^μ and $X^\mu + dX^\mu$ is $ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu, \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$.

Linear transformations of the coordinates which preserve the form of the interval form the *Poincaré group* of symmetries. These transformations are of the form

$X^{\mu} = X_0^{\mu} + \Lambda^{\mu}{}_{\nu}\,\tilde{X}^{\nu}$; X_0^{μ} are translations; $\Lambda^{\mu}{}_{\nu}$ are matrices of Lorentz rotations that form the Lorentz group. For infinitely small Lorentz transformation one has $\Lambda^{\alpha}{}_{\mu} = \delta^{\alpha}{}_{\mu} + \zeta^{\alpha}{}_{\mu}$.

Show that $\zeta_{\mu\nu}$ are antisymmetric matrices and total number of parameters of the Poincare group is 10.

Consider now how spacetime points are moved under the symmetry transformation. For infinitely small transformation one has

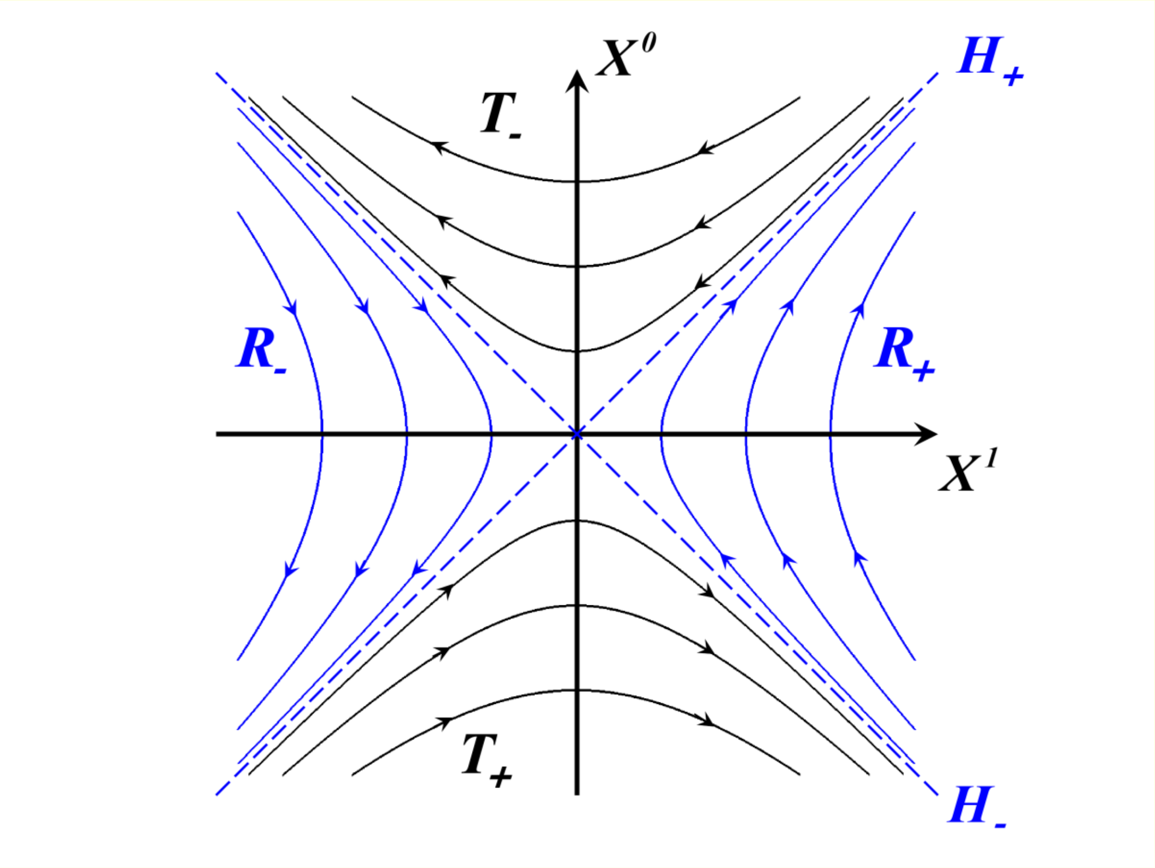
$$X^{\mu} \to X^{\mu} + \xi^{\mu}, \quad \xi^{\mu} = \varepsilon^{\mu} + \eta^{\mu\alpha} \zeta_{\alpha\nu} X^{\nu}.$$

Vectors ξ^{μ} generating these transformations are called Killing vectors. They obey equations $\xi_{(\mu,\nu)} = 0$.

Integral lines of the Killing vectors are determined be the equations $\dfrac{dX^{\mu}}{d\lambda} = \xi^{\mu}(X).$

Boost transformations in $(T - X)$-plane: $\xi^{\mathrm{T}} = X, \xi^{\mathrm{X}} = T.$

Integral lines: $\dfrac{dT}{d\lambda} = X, \quad \dfrac{dX}{d\lambda} = T \quad \Rightarrow T^2 - X^2 = const$



Consider a world line in the Minkowski spacetime given by the following equation

$$X^{\mu}(\tau) = (a^{-1}\sinh(a\tau), a^{-1}\cosh(a\tau), 0, 0)$$

Simple calculations give the velocity u^{μ} and acceleration w^{μ} for such a motion

$$u^{\mu} = \frac{dX^{\mu}}{d\tau} = \left(\cosh(a\tau), \sinh(a\tau), 0, 0\right),$$

$$w^{\mu} = \frac{du^{\mu}}{d\tau} = \left(a\sinh(a\tau), a\cosh(a\tau), 0, 0\right),$$

$$u^2 = -1, \quad w^2 = a^2, \quad \ell = a^{-1} \quad (c = 1)$$

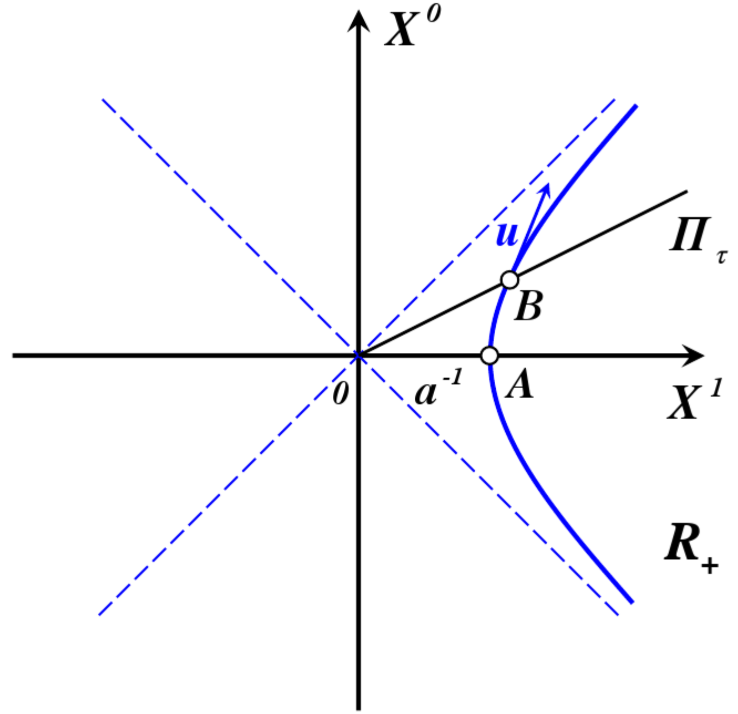
Observations:

(i) $X^{\mu}(\tau)$ coincides with an integral line of the boost generator on $(T-X)$-plane;

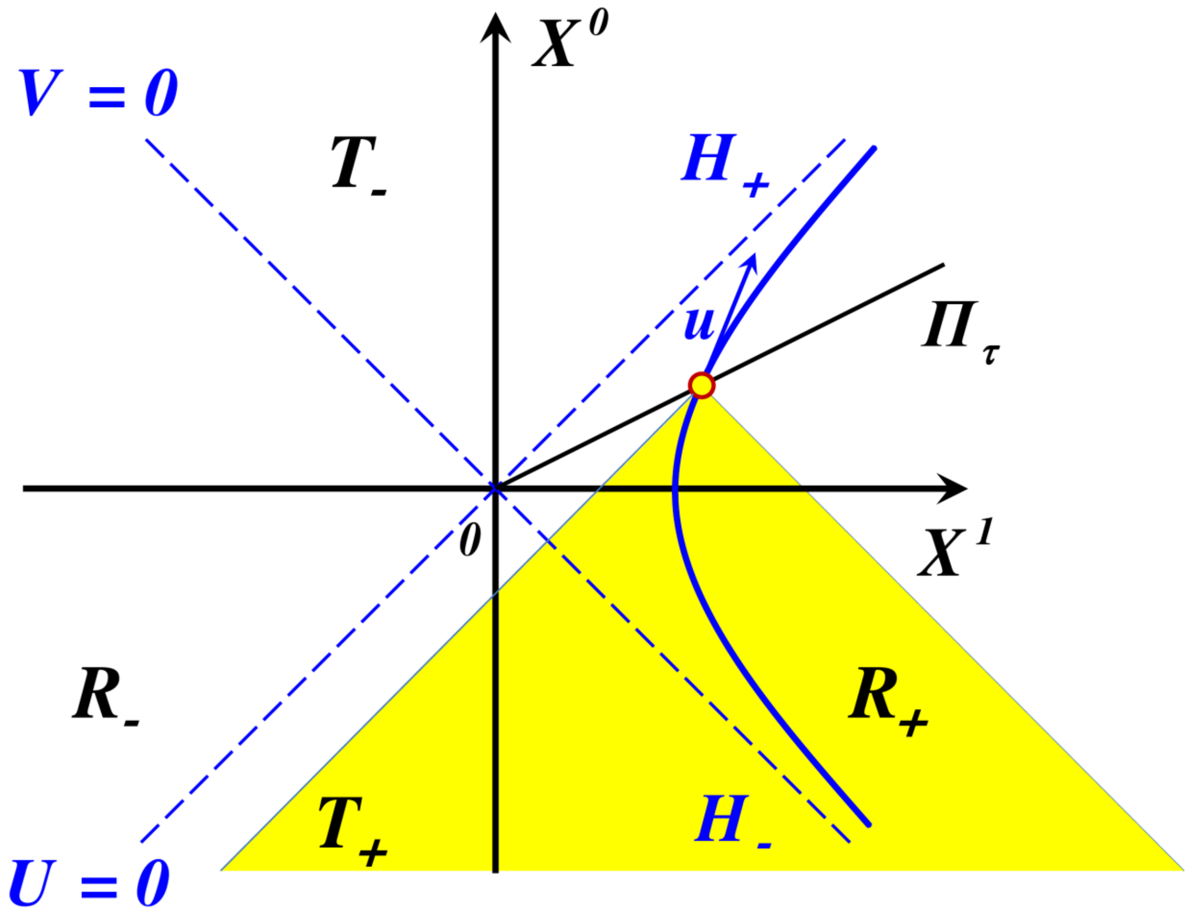
(ii) τ is the proper time parameter;

(iii) $X^{\mu}(\tau)$ is a world line of a uniformly accelerated object.

Consider a charged (with charge e) particle of mass m which is initially at rest. Suppose at the moment of time $t=0$ one switches on a constant electric field $F_{\mu\nu}=-2E\,\delta^0_{[\mu}\delta^1_{\nu]}$. Prove that at $t>0$ the particle moves with a constant acceleration $a=eE/m$.



Π_τ is a set of events simultaneous with $X^{\mu}(\tau)$ in a reference frame of the accelerated observer. Its distance to the origin O remains constant: $(X^{\mu})^2 = \ell^2 = a^{-2}$.



X^0
V = 0
T-
H+
u
Πτ
0
X^1
R-
R+
T+
H-
U = 0

Introduce new (so called Rindler) coordinates in R_+

$X^\mu = (\rho\cosh(a\tau), \rho\sinh(a\tau), y, z)$. Minkowski metric in these coordinates takes the form

$$ds^2 = -a^2\rho^2 d\tau^2 + d\rho^2 + dy^2 + dz^2.$$

This is a metric of a homogeneous gravitational field.

An inertial particle, $X^\mu = (\gamma T, b + v\gamma T, 0, 0), \quad [\gamma = (1-v^2)^{-1/2}]$

in Rindler coordinates has the following equation of motion

$$\rho = \frac{b\cosh(a\tau_0)}{\cosh[a(\tau-\tau_0)]}, \quad \tanh(a\tau_0) = v.$$

In the Rindler frame it takes infinite time τ to reach the horizon ρ=0.

Consider now a massless scalar field $\Box\varphi = 0$ propagating in the Rindler space. Its solution in the Cartesian coordinates can be decomposed into monohromatic plane wave modes

$$\varphi = \varphi_0 \exp(i\Phi), \quad \Phi = k_\mu X^\mu = -\omega T + \vec{k}\vec{X}.$$

For a plane wave in X-direction: $\Phi = -\omega(T - X)$.

In the Rindler frame $\Phi = \frac{\omega}{a}\exp(-a\tau)$, and its frequency is

$$\varpi = -\frac{d\Phi}{d\tau} = \omega\exp(-a\tau).$$

Lessons

(i) Rindler coordinates cover only `quarter' of the total Minkowski ST;

(ii) To `cover' all the ST it is required 4 of them;

(iii) Rindler observer sees only `half' of ST;

(iv) `Visible' and `invisible' domains are separated by the null surface of the event horizon;

(v) It takes infinite Rindler time for a particle to reach the event horizon, while the proper time is finite;

(vi) Light emitted by a falling particle is red-shifted. Red-shift infinitely grows near the horizon.