BLACK HOLE PHYSICS

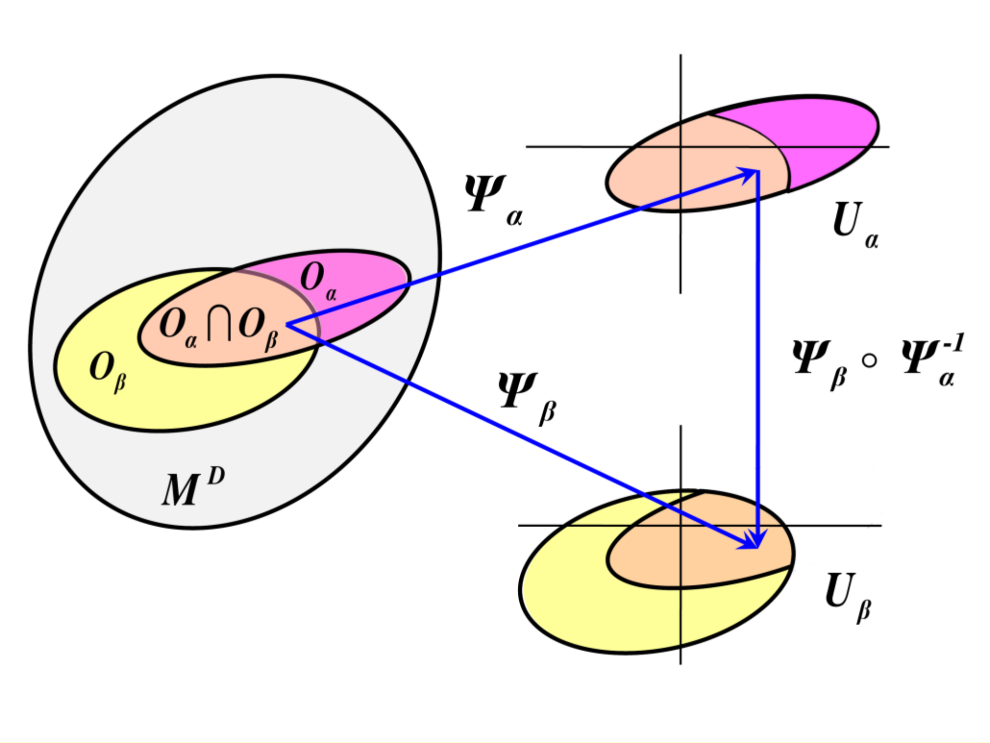
Valeri Frolov
University of Alberta, Edmonton

Brief course of lectures at 18th APCTP Winter School on Fundamental Physics

Pohang, January 20 -- January 28, 2014

7. EINSTEIN EQUATIONS

Spacetime model in General Relativity is a differential manifold. Its points represent events. In the vicinity of each point there exist a domain covered by a local coordinate system. These coordinate maps are consistent and cover all the manifold.



Gravitational field is described by a metric $g_{\mu\nu}$ on the ST manifold.

Basic elements and facts of the Riemannian geometry:

(1) By a coordinate transformation metric at a given point p can be put in the form $g_{\mu\nu}(p) = \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$;

(2) These coordinates can be choosen so that $\partial_\lambda g_{\mu\nu}(p) = 0$;

(3) A scalar product of two vectors is $(A, B) = g_{\mu\nu} A^\mu B^\nu$;

(4) A partial derivative ∂_μ can be `upgraded' to a covariant derivative ∇_μ. Its action on a tensor is again tensor.

(5) Covariant derivative do not commute. Their commutator is proportional to the curvature tensor: $(\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu) A_\beta = A_\alpha R^\alpha{}_{\beta\mu\nu}$

(6) Parallel transport along a curve: Let $x^{\mu}(\lambda)$ be a curve and

$u^{\mu} = dx^{\mu}(\lambda)/d\lambda$ be a tangent vector to it. Then A^{μ} is paralell

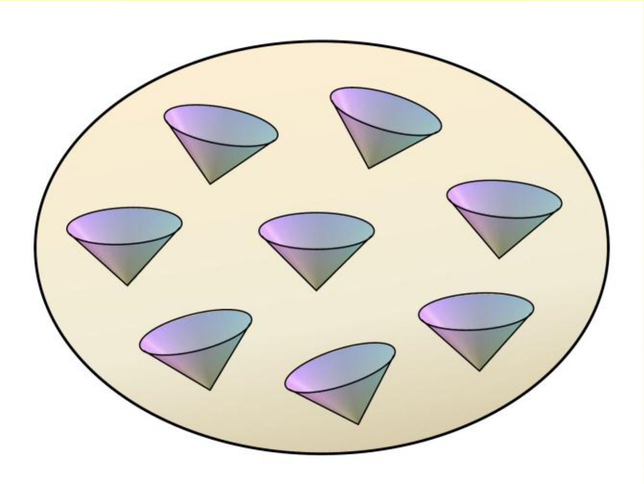
transported if $u^{\nu}\nabla_{\nu}A^{\mu} = 0,\ (\nabla_{u}A^{\mu} = u^{\nu}A^{\mu}{}_{;\nu} = 0)$;

(7) Geodesic line (particle world line) $u^{\nu}u^{\mu}{}_{;\nu} = 0$;

(8) Interval between two close events x^{μ} and $x^{\mu} + dx^{\mu}$ is

$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}$;

(9) Causal structure. Local null cones: $ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = 0$.



(10) $\xi^{\mu}(x)$ is a generator of 1-parameter diffeo:

$dx^{\mu} / d\lambda = \xi^{\mu}(x)$;

(11) Symmetry is diffeo preserving the metric. It is generated by Killing vector fields $\xi_{(\mu;\nu)} = 0$;

(12) Integrals of motion $P_{\xi} = \xi_{\mu} u^{\mu}$

Einstein-Hilbert action

$$S[g] = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}\,(R-2\Lambda)$$

In the presence of matter the Einstein-Hilbert action must be modified by adding the matter action which we write in the form

$$S_m[\Phi, g] = \frac{1}{c}\int d^4x\sqrt{-g}\,L_m(\Phi, g)$$

The variation of the *Einstein-Hilbert action* is

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}} = -\frac{c^3}{16\pi G}(G^{\alpha\beta}+\Lambda g^{\alpha\beta})$$

Here $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ **is the Einstein tensor.**

The variation of the matter action over the metric gives the symmetric tensor of rank two

$$\frac{1}{c}T^{\alpha\beta} = \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\alpha\beta}}$$

This tensor is called the (metric) stress-energy tensor or energy-momentum tensor of the matter.

Einstein equations

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta}$$

The terms on the left-hand side depend only on the spacetime geometry, while on the right-hand side we have the stress-energy tensor of the matter fields.

8. SPHERICALY SYMMETRIC BLACK HOLES

Spherically symmetric ST

A spacetime is called spherically symmetric if there exist coordinates in which its metric takes the form

$$ds^2 = \gamma_{AB}\, dx^A dx^B + r^2 d\omega^2, \gamma_{AB} = \gamma_{AB}(x), r = r(x),$$

$$d\omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2, \;\; A, B = 0,1;$$

This metric admits 3 Killing vectors

$$\xi_1 = -\cos\phi\,\partial_\theta + \cot\theta \sin\phi,$$

$$\xi_2 = \sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi, \xi_3 = \partial_\phi.$$

Prove that the commutators of these vector fields are of the form $[\xi_i, \xi_j] = \varepsilon_{ijk}\, \xi_k$; ε_{ijk} is a 3D Levi-Civita symbol.

Dimensional reduction

$$S_{sph}[\gamma, r] \equiv 14\int_{M^2} d^2x\sqrt{-\gamma}\left[R\,r^2 + 2\,r^{:A}\,r_{:A} + 2 - 2\Lambda r^2\right].$$

One may consider $\varphi \sim \ln r$ as a new scalar field, which together with γ_{AB} determines 2D dilaton gravity.

Variation of the reduced action gives equations

$$r_{:AB} = -(r/2)\gamma_{AB}\left(-\tfrac{1}{r^2} + \frac{r^{:C}\,r_{:C}}{r^2} + 2\,\frac{r_{:C}^{:C}}{r} + \Lambda\right),$$

$$\frac{r_{:C}^{:C}}{r} - \tfrac{1}{2}R + \Lambda = 0;$$

$\xi^A = e^{AB}r_{;B}$ is a new Killing vector (Birkhoff theorem).

A spherically symmetric vacuum solution of Einstein equations with a cosmological constant is determined by one essential constant (mass *M*) and can be written in the form

$$ds^2 = -g\,dt^2 + \frac{dr^2}{g} + r^2\,d\omega^2,$$

$$g = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2.$$

Schwarzschild metric

$$ds^2 = -g\,dt^2 + \frac{dr^2}{g} + r^2\,d\omega^2,$$

$$g = 1 - \frac{2M}{r}.$$

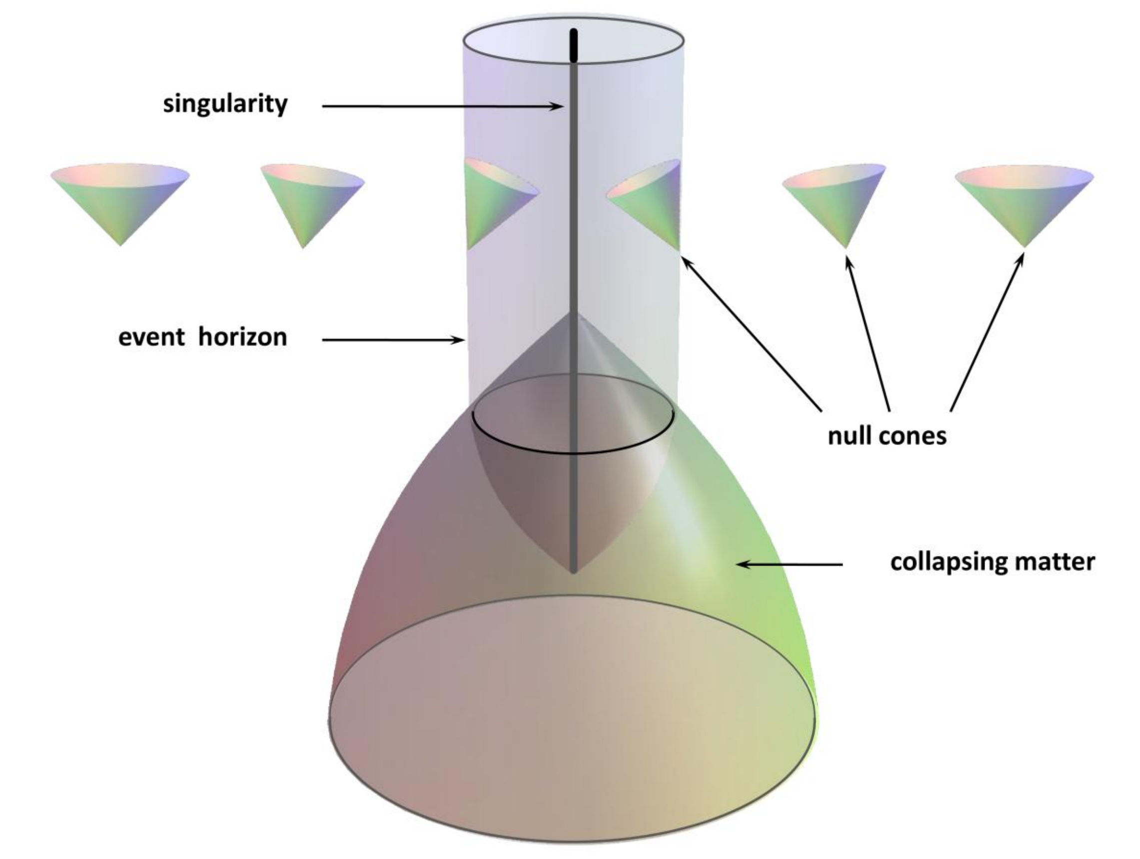
The radius =2*M* is known as the gravitational radius or the Schwarzschild radius. In physical units, after restoring *G* and c constants, it has the value

$$r_S = \frac{2GM}{c^2}$$

Schwarzschild metric

$$ds^2 = -\left(1-\frac{r_S}{r}\right)dt^2 + \left(1-\frac{r_S}{r}\right)^{-1}dr^2 + r^2\,d\omega^2$$

describes the gravitational field in vacuum, outside a spherical distribution of matter. This matter may be either static or have radial motion preserving the spherical symmetry. According to Birkhoff's theorem, the external metric does not depend on such motion. In the absence of matter, the metric describes an exterior of a spherically symmetric static black hole. In this case r_S is the radius of its event horizon.

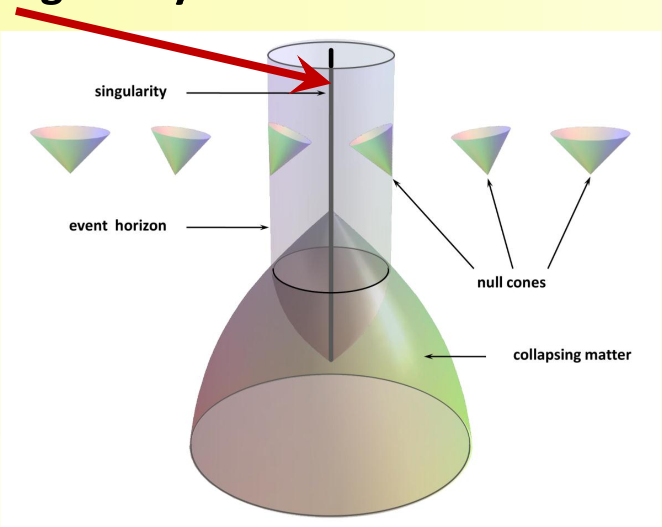


Schematic picture illustrating a black hole formation. Time goes in the vertical direction. Collapse of the matter results in the creation of the event horizon. It is formed some time before the surface of the collapsing body crosses the gravitational radius. Soon after the formation the horizon becomes stationary. Future directed local null cones are shown. Inside the horizon these cones are strongly tilted, so that the motion with the velocity less or equal to the speed of light brings a particle closer to the singularity.

The null surface separating the black hole exterior and interior is, in fact, a regular surface of the spacetime manifold. This can be tested by calculating curvature invariants. For example, the so called Kretschman invariant for the Schwarzschild metric is finite at the gravitational radius.

$$R^2 \equiv R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} = \frac{12 r_S^2}{r^6}$$

In the other coordinate systems the horizon is perfectly regular. At the center of the black hole at *$r = 0$* the curvature is infinite. Near this singularity the tidal forces infinitely grow. This is a physical singularity. It cannot be removed by coordinate transformation.



Schwarzschild metric in the vicinity of the horizon

Consider the (t-r)-sector of the Schwarzschild metric

$$ds^2 = \underbrace{-g\,dt^2 + \frac{dr^2}{g}}_{d\gamma^2} + r^2\,d\omega^2\,,$$

$$g = 1 - \frac{2M}{r}\,.$$

In the vicinity of the horizon: $r = r_S(1+y), \quad y \ll 1.$

the proper length distance from the horizon $\rho = \int_{r_S}^{r} \frac{dr}{\sqrt{g}} \simeq 2r_S\sqrt{y}\,,$

$$d\gamma^2 = -\kappa^2 \rho^2 dt^2 + d\rho^2$$

$\kappa = 1/(2r_S) = 1/(4M)$ **is the surface gravity of the black hole.**

$$ds^2 \approx -\kappa^2 \rho^2 dt^2 + d\rho^2 + r_S^2 d\omega^2$$

If we consider region, which in the transverse direction has the size much smaller than r_S , then this sphere can be approximated by a 2D plane. In such a near horizon region this approximation gives the 4D Rindler metric.

Main facts of the near-horizon physics:

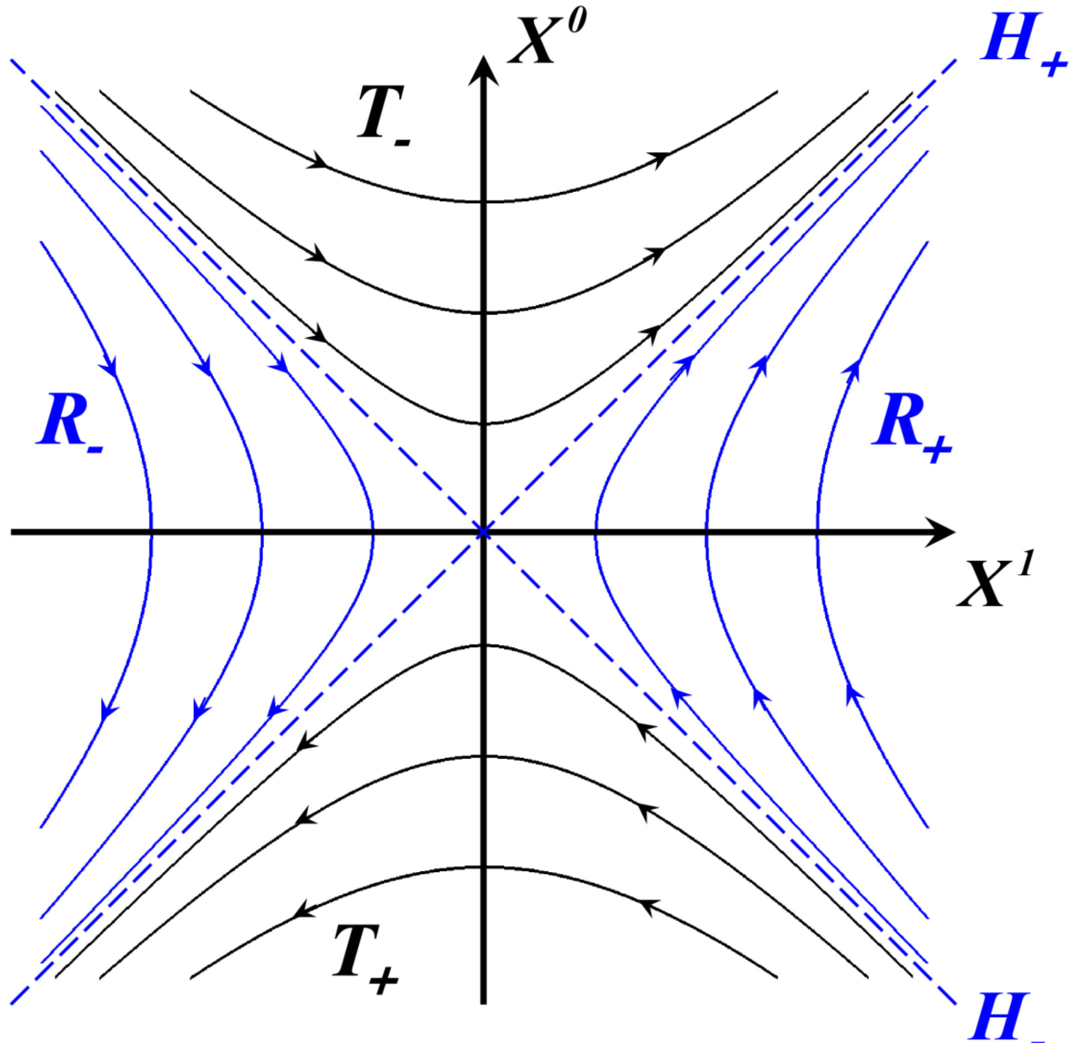
Many properties of the black hole in the near horizon region directly follow from the analysis in the Rindler.

1. For a particle (light ray) falling into a black hole it takes a finite proper time (afine parameter) to reach the event horizon;
2. The time t measured by an external observer for the same process is infinitely large;
3. The redshift factor for the light emitted by an object freely-falling into the black hole as measured by a distant observer, is $\sim e^{-\kappa t}$ **, where** κ **is the surface gravity of the black hole;**
4. Infinite redshift surface $(\xi_{(t)}^2 = 0)$ **coincides with the event horizon.**

The spacetime described in the Schwarzschild coordinates is geodesically incomplete. The Rindler approach allows one to conclude that beyond the Schwarzschild horizon there exists a continuation of the geometry. In particular, one can expect that there must exist regions where the Killing vector becomes spacelike.

Rindler space

$$ds^2 \approx -\kappa^2 \rho^2 dt^2 + d\rho^2 + r_S^2 d\omega^2$$



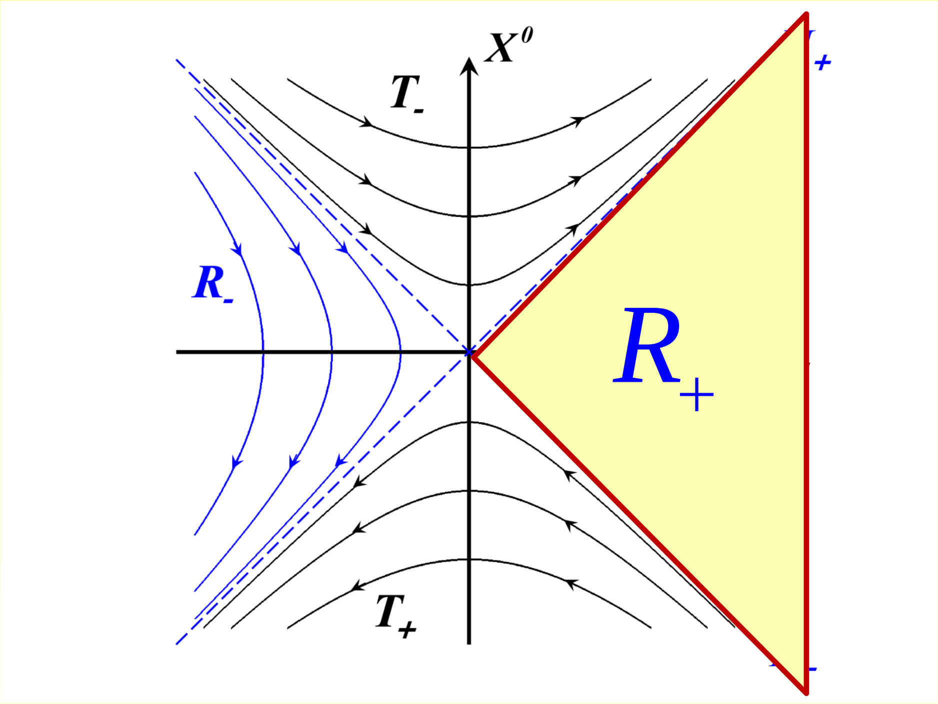
Many faces of the Schwarzschild Black hole

The Schwarzschild metric does not cover a complete spacetime. In this sense, the Schwarzschild coordinates (t; r) are similar to the Rindler coordinates acting in the R_+ domain

In Schwarzschild geometry one can use an analogue of null Rindler coordinates

$$U = (X^0 - X^1)/\sqrt{2},$$

$$V = (X^0 + X^1)/\sqrt{2}$$



Kruskal metric

In Kruskal coordinates, $-UV = \left(\frac{r}{2M} - 1\right)\exp\left(\frac{r}{2M} - 1\right),$

the Schwarzschild metric reads $ds^2 = 2\,B\,dU\,dV + r^2\,d\omega^2,$

$$B = -\frac{16M^3}{r}\exp\left[-\left(\frac{r}{2M} - 1\right)\right].$$

Synge (1950); Fronsdal (1959); Kruskal (1960); Szekeres (1960); Novikov (1963,1964)

Instead of the null coordinates *U* and *V* it is possible to introduce timelike and spacelike coordinates

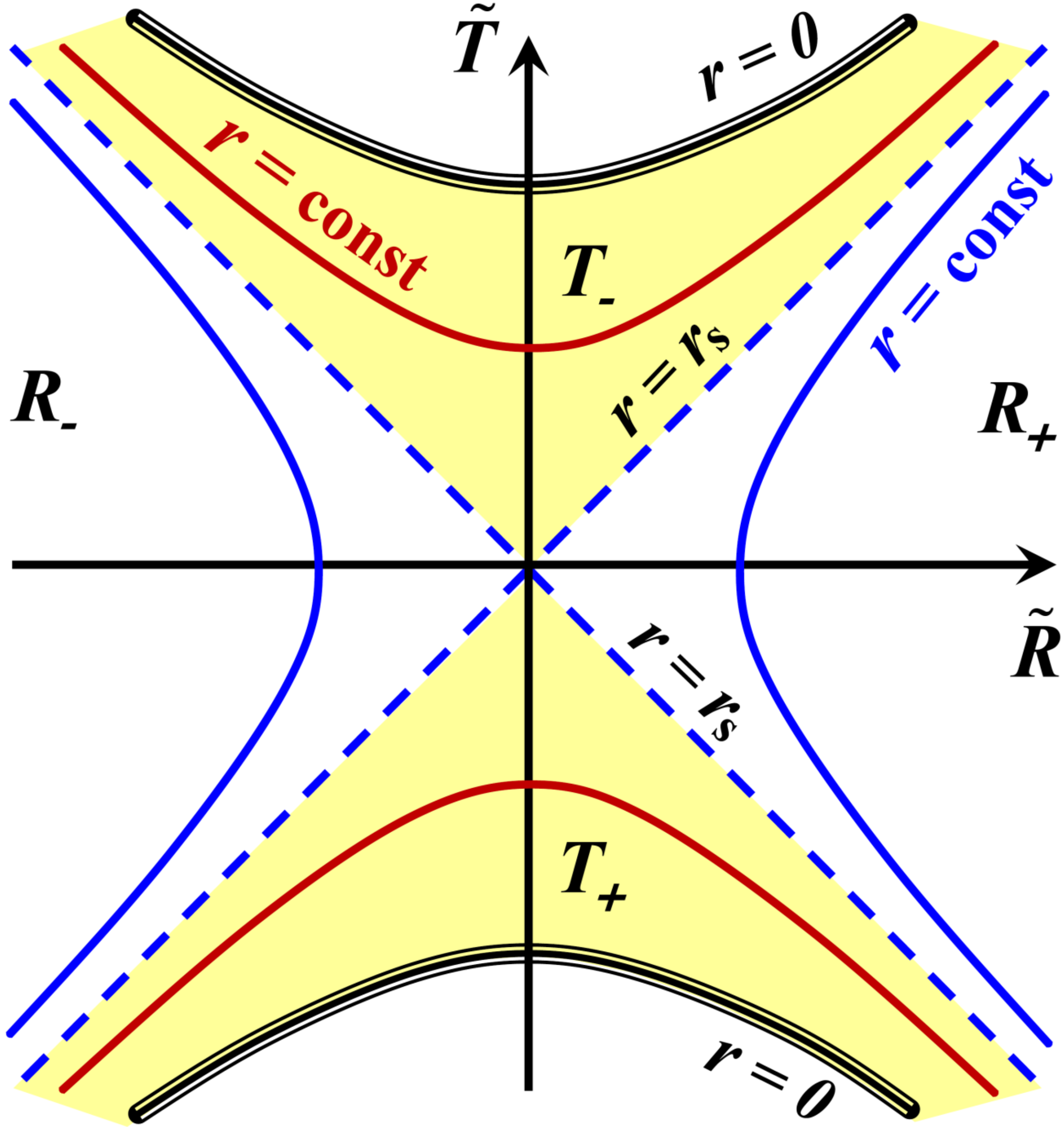
$$\tilde{T} = (V + U)/\sqrt{2}, \qquad \tilde{R} = (V - U)/\sqrt{2}.$$

$$ds^2 = \frac{2r_S^3}{r} e^{-(r/r_S - 1)} (-d\tilde{T}^2 + d\tilde{R}^2) + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

where r is a function of $\tilde{T}$ **and** $\tilde{R}$

$$2\left(\frac{r}{r_S} - 1\right) e^{(r/r_S - 1)} = \tilde{R}^2 - \tilde{T}^2$$

The global structure of the *Kruskal spacetime* is illustrated in Figure



$\tilde{T}$
r = 0
r = const
T_-
r = const
$r = r_s$
R_-
R_+
$\tilde{R}$
$r = r_s$
T_+
r = 0

Carter-Penrose diagram

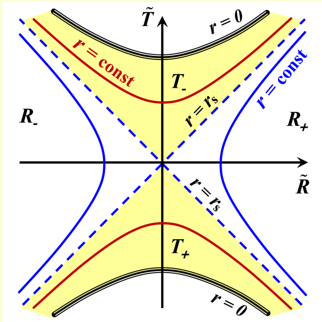
There is a special modification of the spacetime diagrams, similar to Kruskal one, which makes the global causal structure of the spacetime, including its properties at infinity, more profound. Let us denote

$$\mathcal{U} = \arctan U \,,$$

$$\mathcal{V} = \arctan V \,.$$

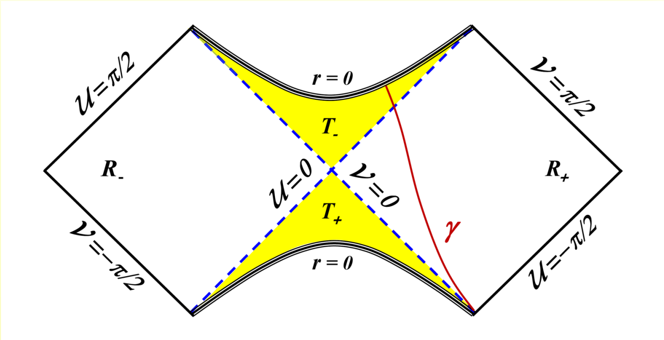
The spacetime diagrams where the infinity is brought to finite coordinate distance are known as *Carter-Penrose conformal diagrams*.

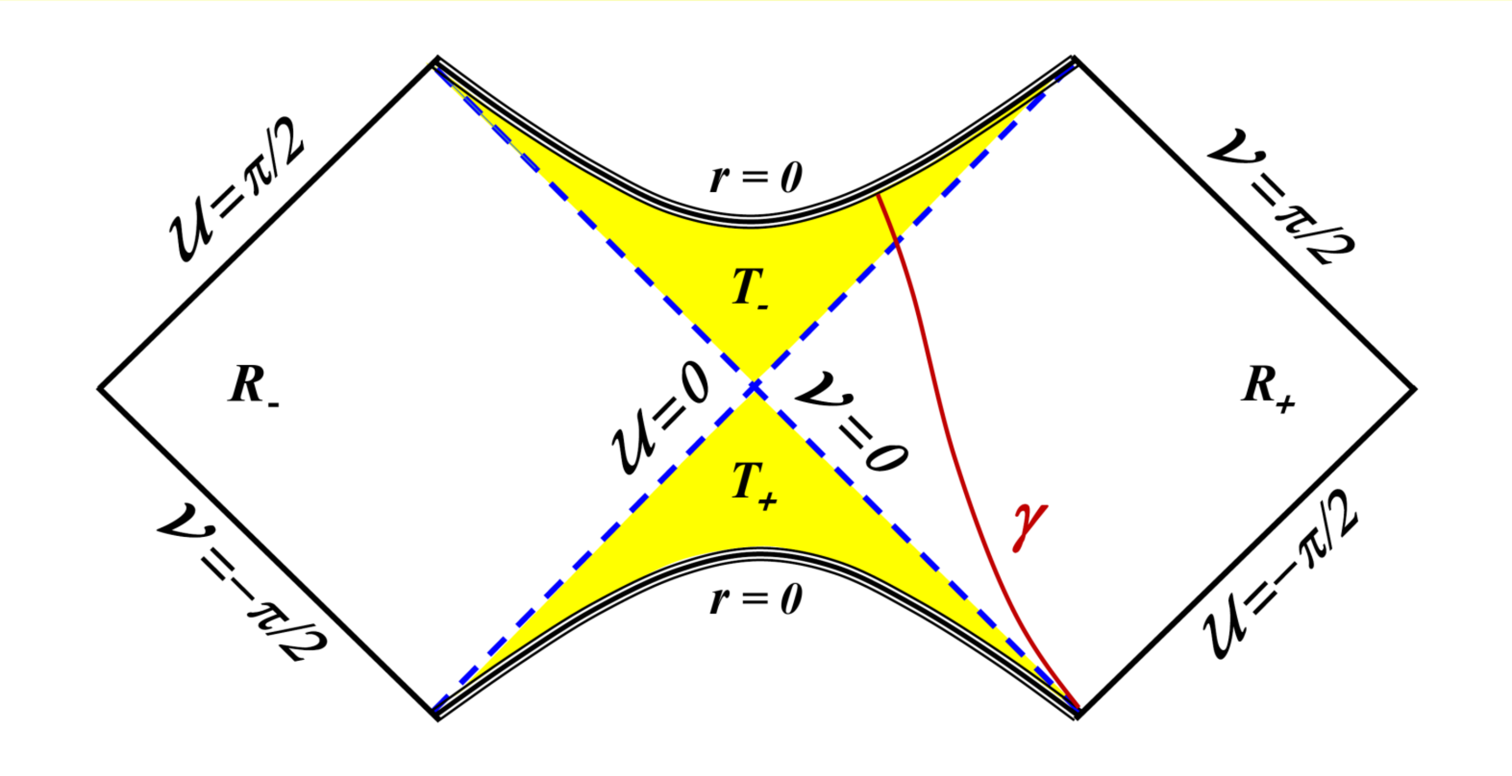
$$ds^2 = \frac{2B\; d\mathcal{U}\, d\mathcal{V}}{\cos^2\mathcal{U}\,\cos^2\mathcal{V}} + r^2\, d\omega^2 \,.$$



$$\mathcal{U} = \arctan U\,,$$

$$\mathcal{V} = \arctan V\,.$$





Einstein-Rosen bridge

The equation *$U = -V$* determines a three-dimensional spacelike slice of the Kruskal spacetime. This slice passes through the bifurcation surface of the horizons. It has two branches. This subspace is called the Einstein-Rosen bridge. Its internal geometry

$$dl^2 = \frac{dr^2}{1-2M/r} + r^2\, d\omega^2 = \Omega^4\, dl_0^2\,,$$

$$\Omega = 1 + \frac{M}{2\rho},$$

$$r = \rho(1 + M/2\rho)^2\,.$$

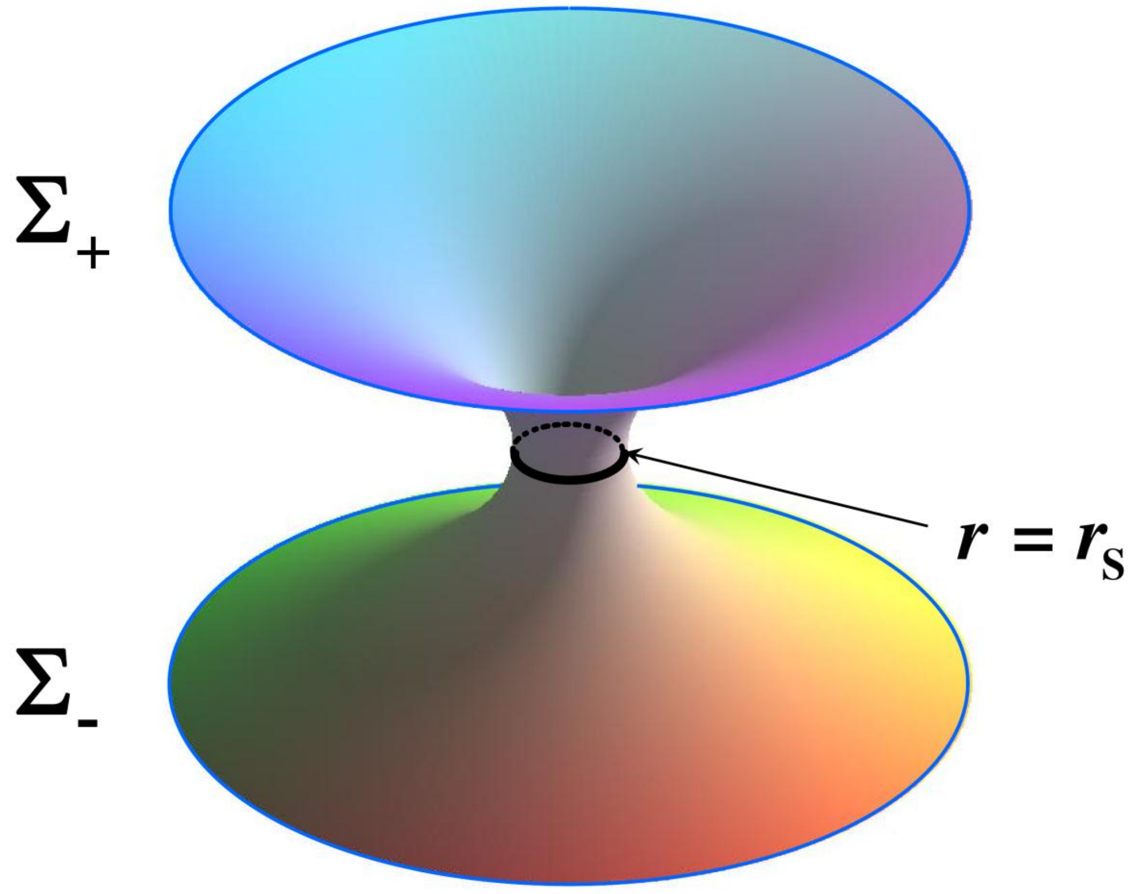
$$dl_0^2 = d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,.$$

The geometry of a two-dimensional section $\theta = \pi/2$ of the metric can be embedded in a at three-dimensional space as a revolution surface

$$dL^2 = dz^2 + dr^2 + r^2\, d\phi^2 = dr^2 (1 + z'^2) + r^2\, d\phi^2 \,,$$

where $z = \pm 2\sqrt{2M(r - 2M)}$

Einstein-Rosen bridge



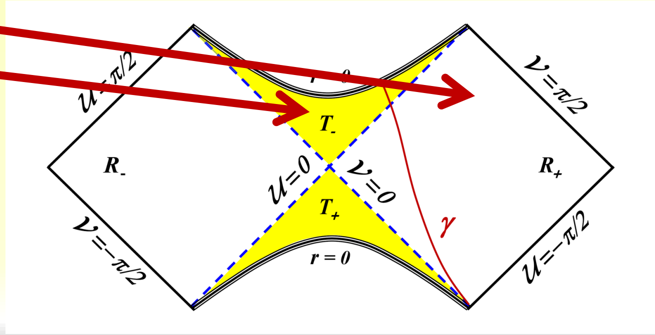
Painleve-Gullstrand metric

The Schwarzschild spacetime can be represented in many different coordinate systems, each being convenient for different applications. In the acoustic analogue models of gravity the Painleve-Gullstrand metric appears naturally. The Schwarzschild geometry in the Painleve-Gullstrand coordinates is

$$ds^2 = -g\,d\tilde{t}^2 \pm 2\sqrt{1-g}\,d\tilde{t}dr + dr^2 + r^2\,d\omega^2,$$

$$\tilde{t} = t \pm \int \frac{\sqrt{1-g(r)}}{g(r)}\,dr. \qquad g = 1 - \frac{r_S}{r} - \frac{\Lambda}{3}r^2,$$

Here the sign + correspond to the coordinate patch covering the usual Schwarzschild domain T_- and R_+



Eddington-Finkelstein coordinates

Another useful representation of the Schwarzschild black hole is related to coordinates associated with the free falling photons. The geodesics of radially moving photons are described by the equation.

$$t = const \pm \int \frac{dr}{g},$$

It is convenient to introduce a *tortoise coordinate*

$$r_* \equiv \int \frac{dr}{g} = \int \frac{dr}{1-\frac{r_S}{r}} = r + r_S \ln\left|\frac{r}{r_S} - 1\right|$$

And the null coordinate $v = t + r_*$

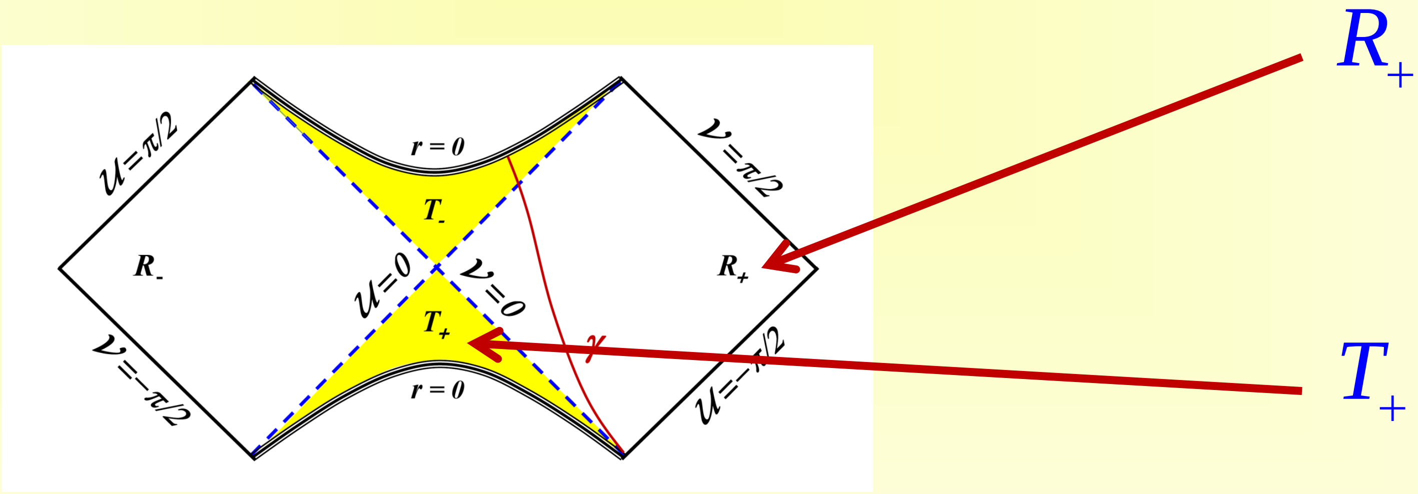
$$ds^2 = -\left(1 - \frac{r_S}{r}\right) dv^2 + 2\,dvdr + r^2\,d\omega^2\,, d\omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2\,.$$

Sometimes it is convenient to use the *retarded time* coordinate

$$u = t - r_*$$

$$ds^2 = -\left(1 - \frac{r_S}{r}\right) du^2 - 2\, du dr + r^2\, d\omega^2 \,.$$

These coordinates are called the outgoing Eddington-Finkelstein coordinates



Charged black holes

If an electrically charged particle falls into the Schwarzschild black hole it becomes charged. To describe such a charged black hole one has to solve the Einstein-Maxwell equations and take into account the stress-energy tensor of the electromagnetic field. The spherically symmetric solution of the problem can be found in a similar way as the Schwarzschild solution. It is easy to check that for the spherically symmetric electric field the condition is satisfied, so the the generalized Birkhoff's theorem is valid and the metric is of the form

$$ds^2 = -g\,dt^2 + g^{-1}\,dr^2 + r^2\,d\omega^2\,.$$

$$g = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2\,,$$

$$A_\mu = -\delta^t_\mu \frac{Q}{r}\,.$$

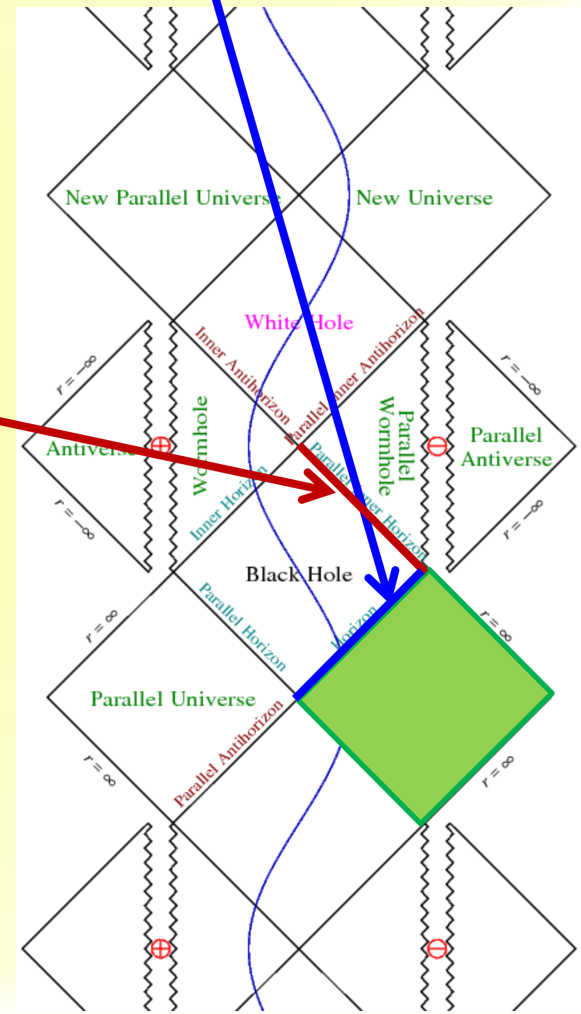
For $\Lambda = 0$ this solution is known as the ***Reissner-Nordström spacetime*****.**

The radius of the horizon of the charged black hole is $r_+ = M + \sqrt{M^2 - Q^2}$.

For the charged black hole the function *g(r)* has two roots

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \; .$$

r_- is an *inner* or *Cauchy horizon*, located inside the black hole.



Note that in astrophysical applications the electric charge is usually negligibly small. This is because the electromagnetic coupling constant is many orders of magnitude stronger than the gravitational one. For two electrons the electromagnetic interaction is proportional to

$$\alpha = e^2/\hbar c = 1/137.036$$

while for their gravitational attraction

$$G m_e^2/\hbar c = 1.75 \cdot 10^{-45}$$

Because of this huge disparity electrically charged black holes in the interstellar medium will attract charges of the opposite sign and repel charges of the same sign. Eventually they become almost neutral. The charge of such a black hole of mass obeys an inequality

$$\frac{Q}{M} \leq \frac{m_e}{e} \sim 0.5 \times 10^{-21} \,.$$

If there were magnetic monopoles in nature then black holes could acquire a magnetic charge. The metric of the magnetically charged black hole coincides with that of the electrically charged one

$$ds^2 = -g\, dt^2 + g^{-1}\, dr^2 + r^2\, d\omega^2 \, .$$

$$g = 1 - \frac{2M}{r} + \frac{N^2}{r^2} - \frac{\Lambda}{3} r^2 \, ,$$

$$A_\mu = -\delta_\mu^\phi \, N \cos\theta.$$

Euclidean black hole

In the `physical' spacetime the signature of the metric is (- ; + ; + ; +). The Schwarzschild metric is static and it allows an analytical continuation to the Euclidean one (+ ; + ; + ; +) . This continuation can be obtained by making the Wick's rotation $t = it_E$. The corresponding space, called a Euclidean black hole, has interesting mathematical properties and has important physical applications.

The two-dimensional part of the Euclidean metric $$d\gamma_E^2 = g\, dt_E^2 + \frac{dr^2}{g}$$

Near the Euclidean horizon is has the form $$d\gamma_E^2 \approx \kappa^2 \rho^2\, dt_E^2 + d\rho^2$$

In a general case this metric has a conical singularity. This singularity vanishes if t_E is a periodic coordinate with the period

$$t_E \in \left(0, \frac{2\pi}{\kappa}\right)$$

Euclidean Schwarzschild metric in the vicinity of the horizon

Consider the (t-r)-sector of the Schwarzschild metric

$$ds^2 = g\,dt^2 + \frac{dr^2}{g} + r^2\,d\omega^2\,,$$

$$g = 1 - \frac{2M}{r}\,.$$

$d\gamma^2$

In the vicinity of the horizon: $r = r_S(1+y),\quad y \ll 1.$

the proper length distance from the horizon $\rho = \int_{r_S}^{r} \frac{dr}{\sqrt{g}} \simeq 2r_S\sqrt{y}\,,$

$$d\gamma^2 = \kappa^2 \rho^2 dt^2 + d\rho^2$$

$\kappa = 1/(2r_S) = 1/(4M)$ **is the surface gravity of the black hole.**

$$ds^2 \approx \kappa^2 \rho^2 dt^2 + d\rho^2 + r_S^2 d\omega^2$$

Regularity at ρ=0 requires t_E is periodic with a period $2\pi/\kappa$.

This condition determines Hawking themperature

$$\Theta_H = \frac{\hbar c^3}{8\pi G k_B M}.$$

$$\Theta_H = \frac{\hbar c^3}{8\pi G k_B M}.$$

Why $\hbar$ enters the expression for the temperature?

We keep $\hbar$ in the relations, but put $c = G = 1$

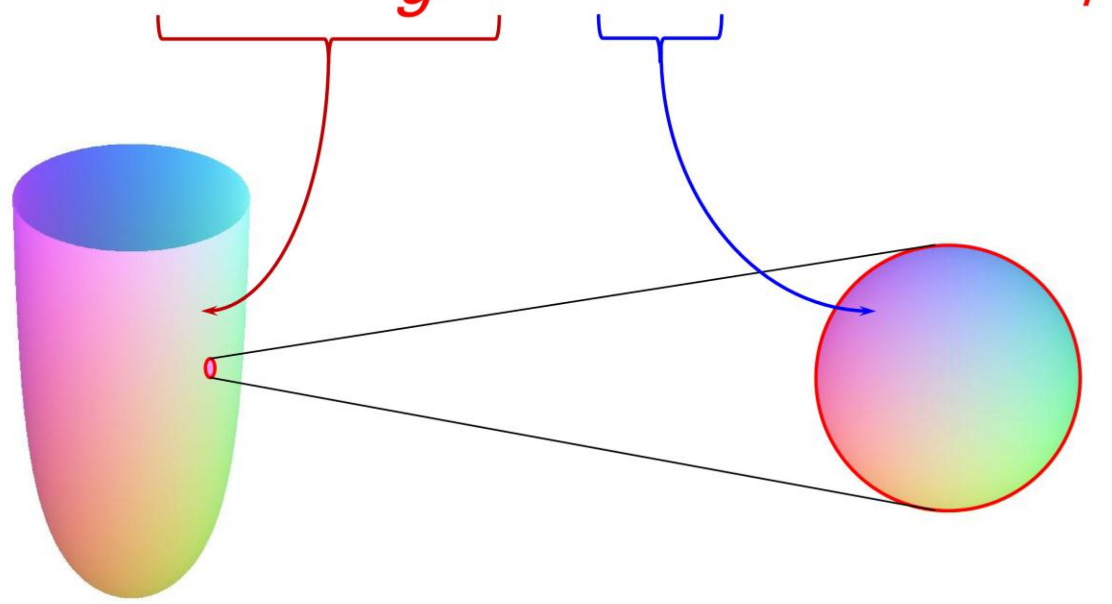
$$\Theta \sim E \sim \hbar\omega \sim \frac{\hbar}{\lambda} \sim \frac{\hbar\kappa}{2\pi} \sim \frac{\hbar}{8\pi M}$$

We used here that λ is equal to the period of the Euclidean time.

Gibbons-Hawking instanton

This regular four dimensional Euclidean space is called the Euclidean black hole or the Gibbons-Hawking instanton. The inverse period in Euclidean time is equal

$$ds^2 = g\,d\tau^2 + \frac{dr^2}{g} + r^2\,d\omega^2, \qquad g = 1 - \frac{r_s}{r}$$



$$\Theta_H = \frac{\kappa}{2\pi}$$

is called the Hawking temperature of the black hole.

Higher Dimensional Spherical Black Holes

$$ds^2 = -g\,dt^2 + \frac{dr^2}{g} + r^2\,d\omega_{D-2}^2\,,$$

(D-2)-dimensional unit sphere

If the matter distribution has the property $T_{AB} \sim \gamma_{AB}$ **one can prove that the generalized Birkhoff's theorem is valid.**
Birkhoff's theorem states that any spherically symmetric solution of the vacuum field equations must be static and asymptotically flat.

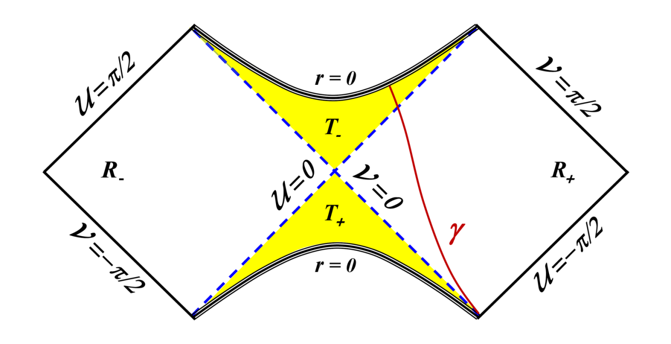
$$g = 1 - \left(\frac{r_S}{r}\right)^{D-3} - \frac{2\Lambda}{(D-1)(D-2)}r^2\,.$$

For $\Lambda = 0$

$$ds^2 = -g\, dt^2 + \frac{dr^2}{g} + r^2\, d\omega_{D-2}^2\,,$$

$$g = 1 - \left(\frac{r_S}{r}\right)^{D-3}. \qquad r_S^{D-3} = \frac{8\Gamma\left(\frac{D-1}{2}\right) G^{(D)} M}{(D-2)\,\pi^{(D-3)/2}},$$

This vacuum spherically symmetric solution is known as the *Tangherlini metric*. It describes higher dimensional spherically symmetric black hole in an asymptotically flat spacetime.



The global structure of the complete spacetime for this metric is similar to the Kruskal 4D solution.

Euclidean black holes in higher dimensions

$$ds_E^2 = g\,dt_E^2 + \frac{dr^2}{g} + r^2\,d\omega_{D-2}^2 . \qquad t_E \in (0, 2\pi/\kappa)$$

It describes the *Euclidean black hole* which is regular at the *Euclidean horizon*

$$\kappa = \frac{D-3}{2\,r_S} . \qquad \Theta_H = \frac{\kappa}{2\pi} .$$

In the case of the Schwarzschild-(anti) de Sitter metric

$$\kappa = \frac{1}{2}\left|\frac{\partial g}{\partial r}\right|_{r=r_+} = \left|\frac{D-3}{2\,r_+} - \frac{\Lambda\, r_+}{D-2}\right| .$$

Particle and light motion near Schwarzschild black hole

$$ds^2 = -g\,dt^2 + \frac{dr^2}{g} + r^2\,d\phi^2\,, \qquad g = 1 - \frac{r_S}{r}.$$

Geodesic equation implies

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}_{\beta} = 0.$$

Motion is planar.

Using rigid rotation one can put

$$\theta_0 = \pi/2, \quad \dot{\theta}_0 = 0.$$

$$\ddot{\theta}_0 + \Gamma^{\theta}_{ab}\dot{x}^{a}\dot{x}_{b}\Big|_0 = 0; \quad a, b \neq \theta$$

$\Gamma^{\theta}_{ab} \sim \partial_{\theta} g_{ab} = 0$ (on the plane $\theta_0 = \pi/2$).

Hence $\ddot{\theta}_0 = 0,$ and $\theta_0 = \pi/2$ is a solution.

$$x^\mu(\tau),\ u^\mu(\tau) = (\dot t, \dot r, \dot\theta, \dot\phi),$$

$$u^2 = -1 \Rightarrow -g\dot t^2 + \dot r^2/g + r^2\dot\phi^2 = -1$$

$$E \equiv -\xi^\mu_{(t)} u_\mu = g\,\dot t,$$

$$L \equiv \xi^\mu_{(\phi)} u_\mu = r^2\dot\phi.$$

$$\dot r^2 = E^2 - g\,(1 + L^2/r^2),$$

$$\dot\phi = L/r^2,$$

$$\dot t = g^{-1}E.$$

$$\dot{r}^2 = E^2 - U, \quad \zeta = r_S / r,$$

$$U = (1 - r_S / r)(1 + L^2/r^2) =$$

$$(1-\zeta)(1+\ell^2\zeta^2);$$

$$\ell = L / r_S$$

Maximum and minimum of U conside at a point where

$U_{,\zeta\zeta} = 2\ell^2(1-3\zeta) = 0 \Rightarrow \zeta_{ISCO} = 1/3,$

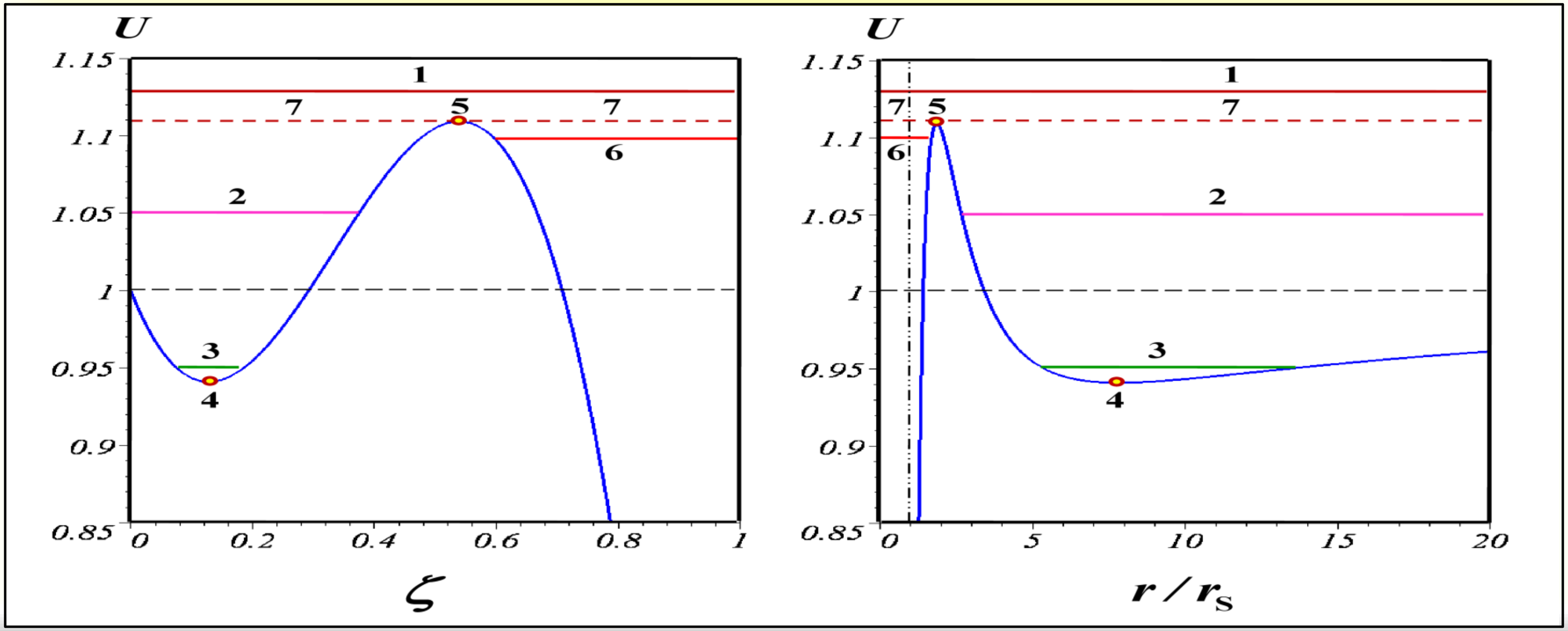
$r_{ISCO} = 3r_S = 6M.$

For acircular orbit with this radius we also have

$U_{,\zeta} = 0 \Rightarrow \ell = \sqrt{3}; \quad E_{ISCO} = \sqrt{U_{ISCO}} = \sqrt{8/9}$

Types of trajectories

1. gravitational capture;
2. hyperbolic motion (scattering;
3. bounded orbits;
4. stable circular orbits;
5. unstable circular orbits
6. near horizon trapped motion;
7. marginal outer and inner orbits;

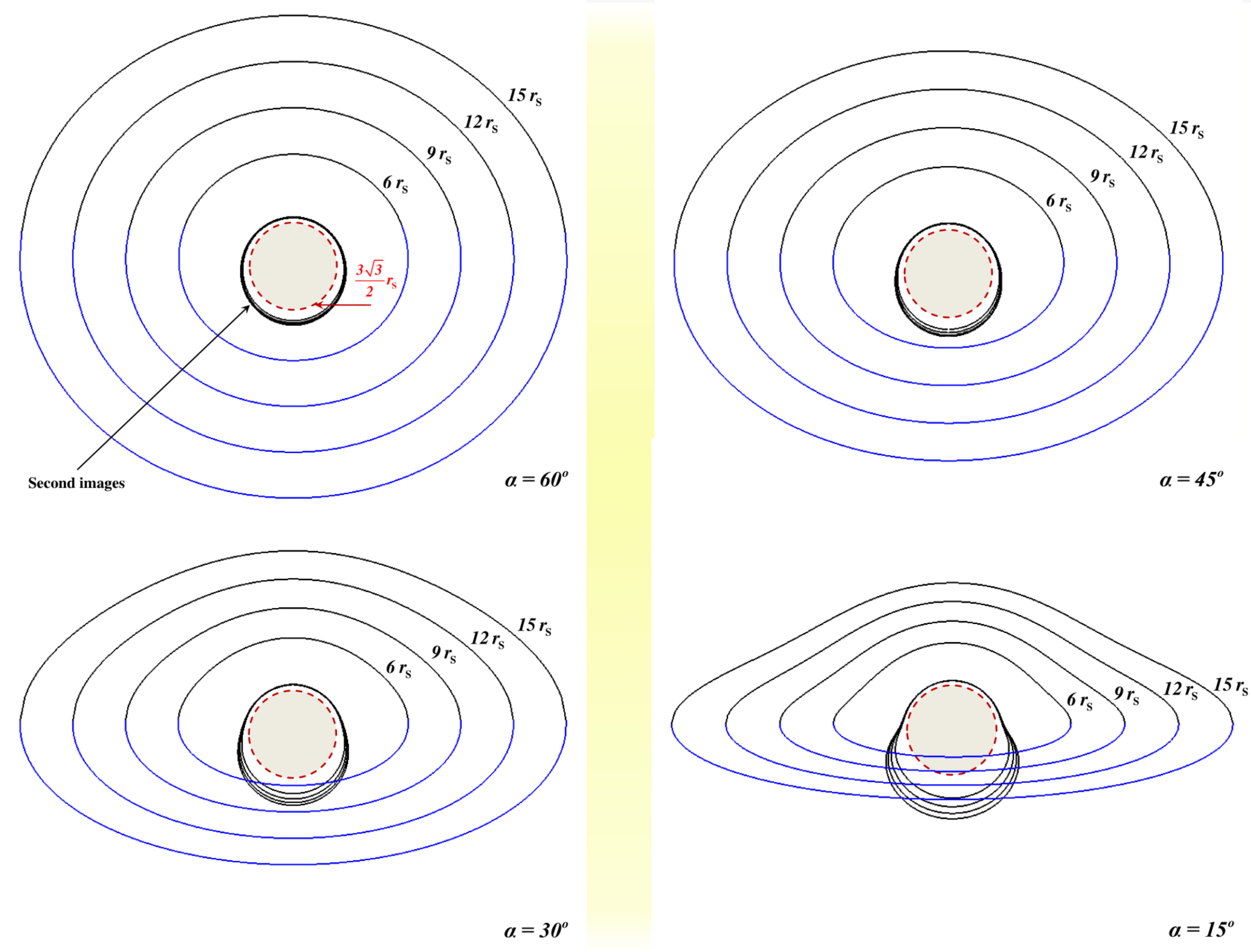


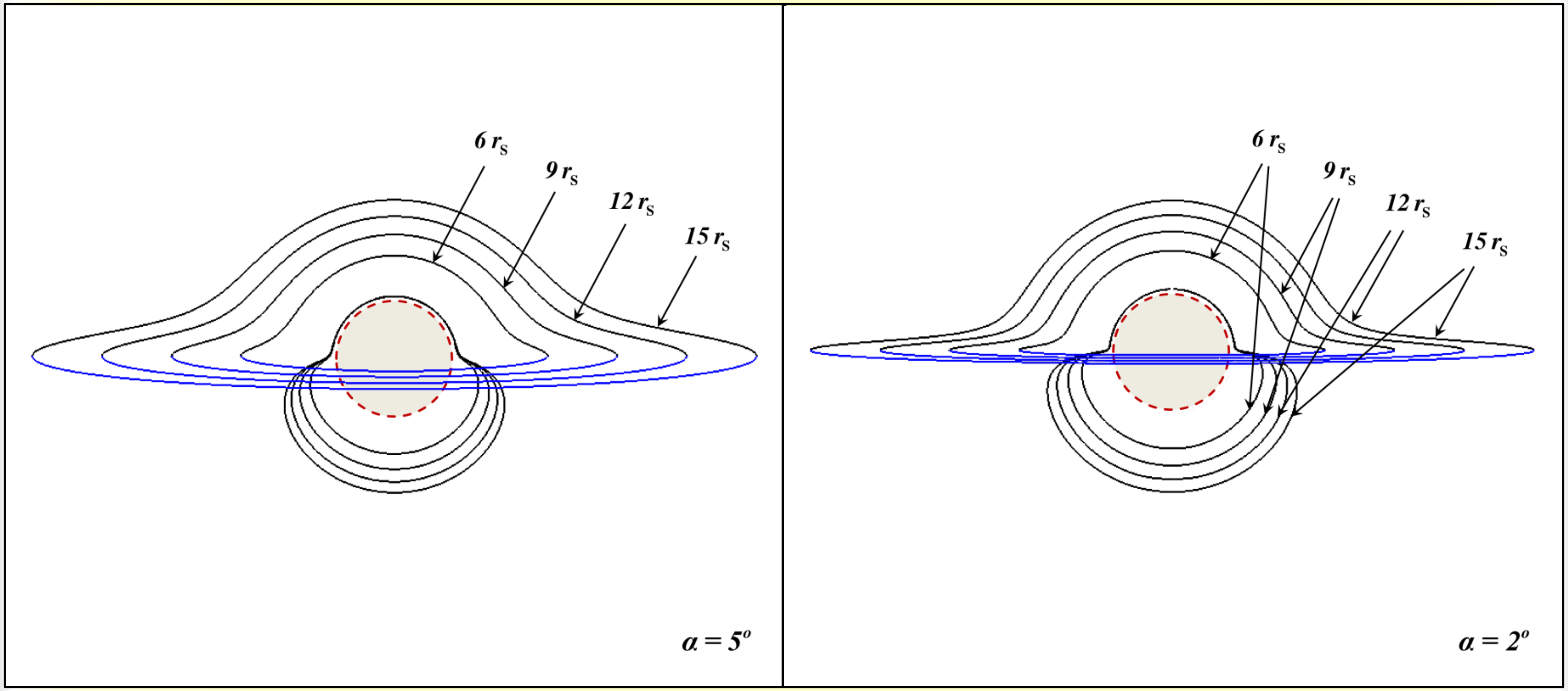
In astrophysical black holes matter falling onto a black hole usually forms an *accretion disk*. Particles are moving approximately along circular (Keplerian) orbits. As a result of loss of energy and angular momentum the radius of their orbits slowly decreases. They move closer and closer to the black hole. A particle may loose up to

$$(1-E)mc^2 \approx 0.057\,mc^2$$

of its proper rest-mass energy until it reaches the *innermost stable circular orbit* (ISCO). After this, it falls into the black hole.

This scenario gives the estimation for the efficiency of a static black hole. Namely this part of the total energy can be extracted from the accreting matter.


15 r_s
12 r_s
9 r_s
6 r_s
$\frac{3\sqrt{3}}{2} r_s$
Second images
$\alpha = 60^o$
15 r_s
12 r_s
9 r_s
6 r_s
$\alpha = 45^o$
15 r_s
12 r_s
9 r_s
6 r_s
$\alpha = 30^o$
6 r_s
9 r_s
12 r_s
15 r_s
$\alpha = 15^o$



6 r_s
9 r_s
12 r_s
15 r_s
$\alpha = 5^o$
6 r_s
9 r_s
12 r_s
15 r_s
$\alpha = 2^o$

Additional material to this section

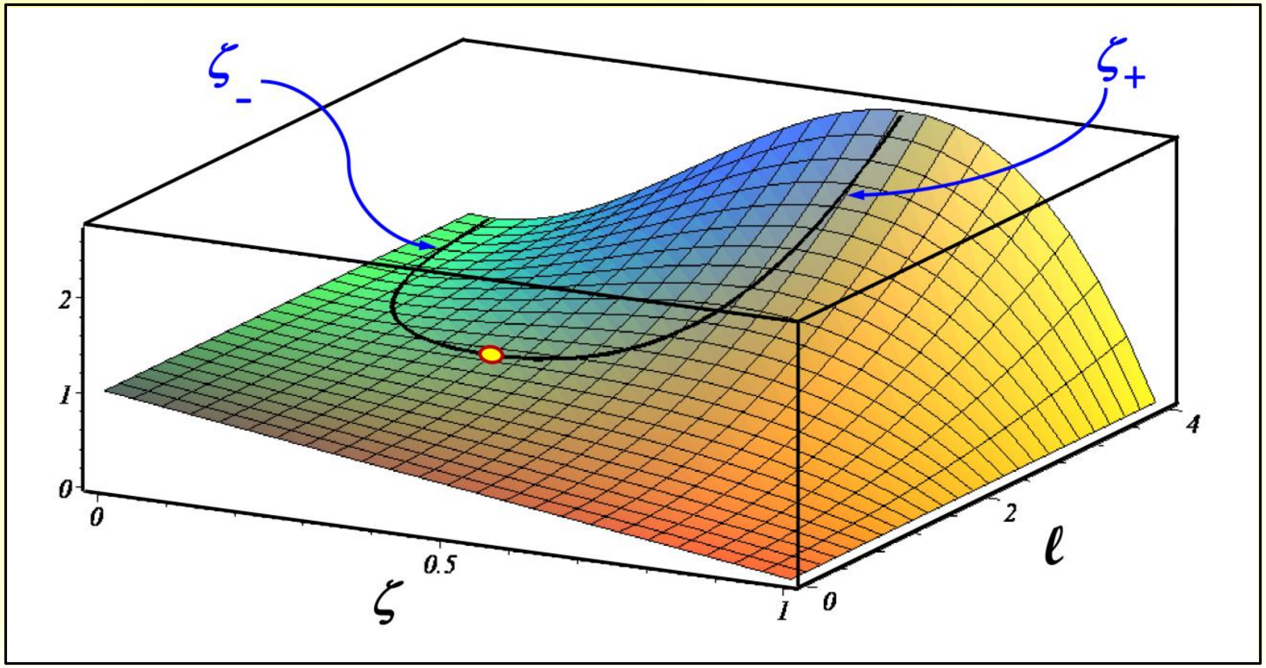
Types of trajectories

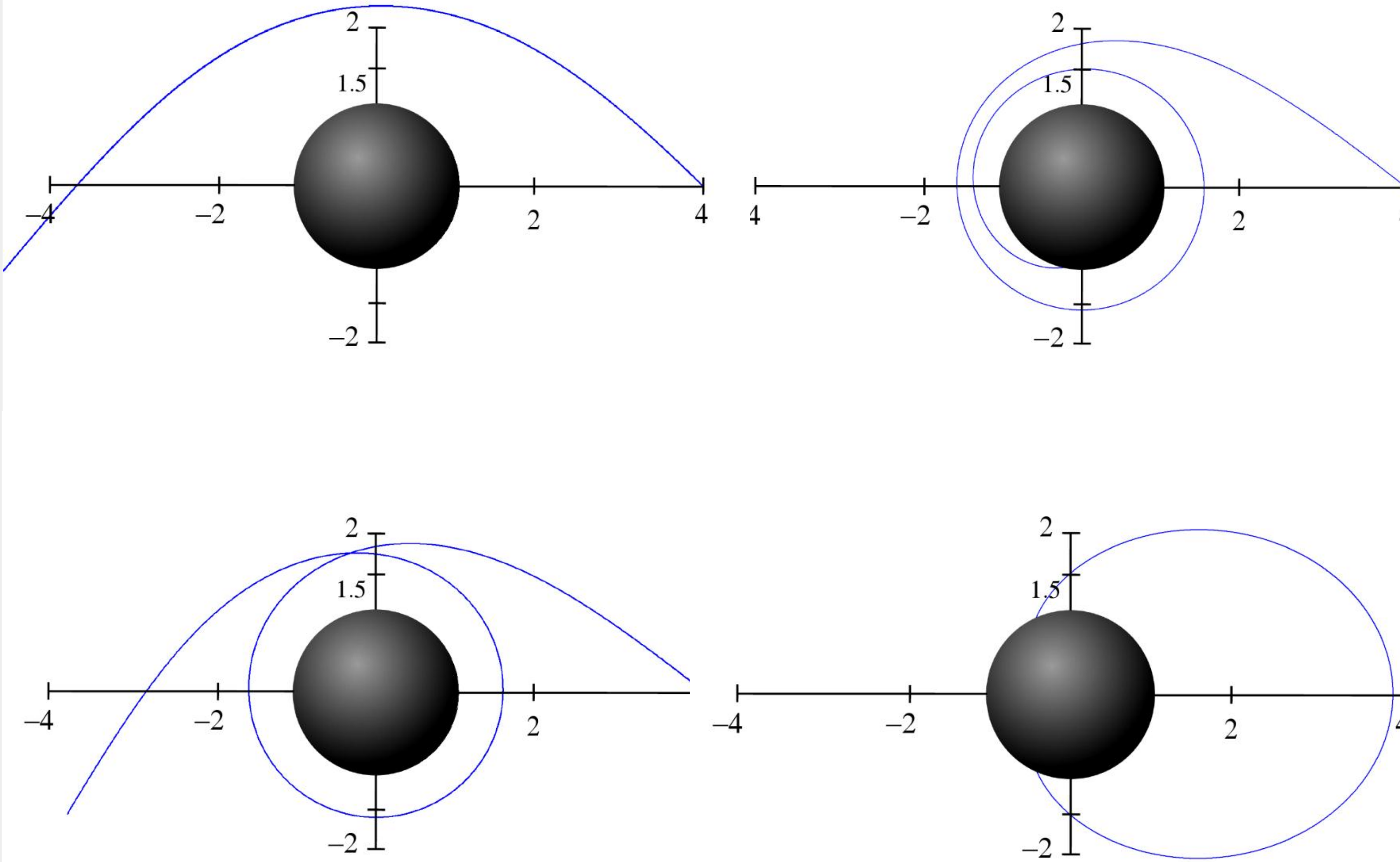
$$\left(\frac{dr}{d\tau}\right)^2 = \mathcal{E}^2 - U, \qquad U = g\left(1 + \frac{\mathcal{L}^2}{r^2}\right).$$

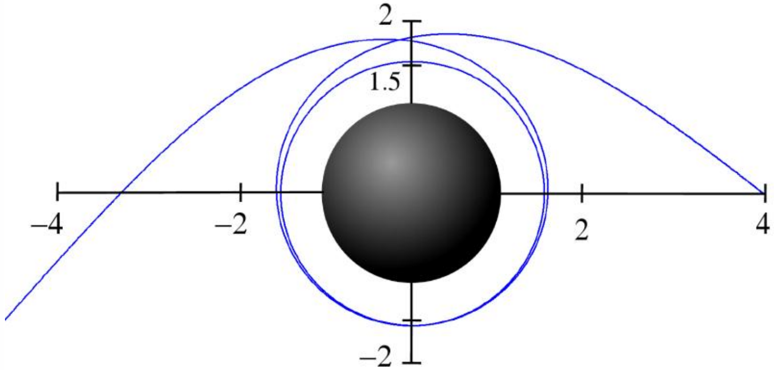
Effective potential

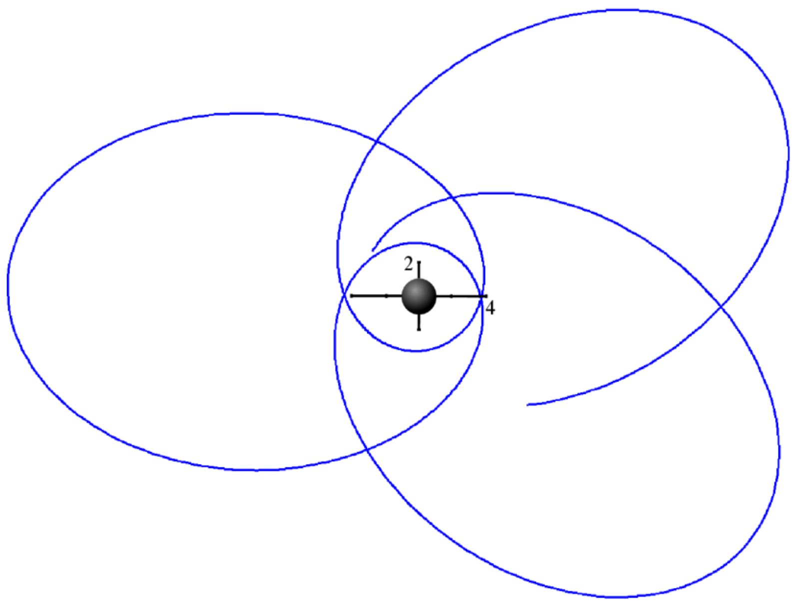
$$\mathcal{E} = E/m, \qquad \mathcal{L} = L/m$$

$$g = 1 - \varsigma,$$
$$\varsigma = r_S / r,$$
$$\ell = \mathcal{L} / r_S$$




2
1.5
−4
−2
2
4
−2
2
1.5
4
−2
2
4
−2
2
1.5
−4
−2
2
−2
2
1.5
−4
−2
2
4
−2


2
1.5
−4
−2
2
4
−2


2
4

Gravitational capture

For a given ℓ the gravitational capture occurs when $E^2 > U_+$. Let us calculate the capture cross-section

$$\sigma(v) = \pi b^2(v)$$

for a particle which has the velocity v at infinity. b is an impact parameter.

$E^2 = p^2 + m^2$ and the capture threshold condition $E^2 = U_+$ imply

$$p = m\sqrt{U_+ - 1}. \qquad p = mv/\sqrt{1-v^2}$$

$$U_+(\ell) \equiv \frac{2[\ell(\ell^2+9)+(\ell^2-3)^{3/2}]}{27\,\ell} = \frac{1}{1-v^2}. \quad \Rightarrow \quad \ell(v)$$

$$b = r_S \frac{\ell}{\sqrt{U_+(\ell)-1}} = r_S \ell(v) \frac{\sqrt{1-v^2}}{v}.$$

Combining these equations we obtain the cross-section

$$\sigma(v) = \pi r_S^2 \frac{\ell^2(v)(1-v^2)}{v^2}.$$

For a non-relativistic motion $v \ll 1$ and $U_+ \approx 1$, $\ell \approx 2$

$$b \approx \frac{2r_S}{v}, \qquad \sigma_{\text{non-rel}}(v) \approx \frac{4\pi r_0^2}{v^2}.$$

For the ultrarelativistic motion $v \to 1, \quad \ell \to \infty, \quad U_+ \approx 4\ell^2/27$

$$b \approx \frac{r_S\sqrt{27}}{2},$$

$$\sigma_{\text{ultra-rel}}(v) \approx \frac{27}{4}\pi r_S^2 .$$

Circular and marginal orbits

Circular motion around a black hole is an important special case of motion of a particle at constant radius. The point at the minimum of potential corresponds to stable motion; and that at the maximum, to unstable one. The latter motion has no analogue in Newtonian theory.

The maximum and minimum appear on the U curve when $\ell > \sqrt{3}$
If $\ell < \sqrt{3}$ the U curve is monotonic.

When $\ell > \sqrt{3}$ the minima of the curves correspond to $r > 3r_S$

Thus, stable circular orbits exist only for $r > 3r_S$

At smaller distances there are only unstable circular orbits corresponding to the maxima of the potential.

The critical circular orbit which separates stable trajectories from unstable ones corresponds to

$$r = 3r_S$$

This orbit is called an *innermost stable* or *marginally stable* *orbit (ISCO)*. Particles move along it at the velocity $v = c/2$

$$E_{\min} = \sqrt{U_-}\,|_{\ell=\sqrt{3}} = \sqrt{8/9} \approx 0.943.$$

This is the motion with the maximum possible

$$\Delta E = (1 - E_{\min})mc^2 \approx 0.057\,mc^2\,.$$

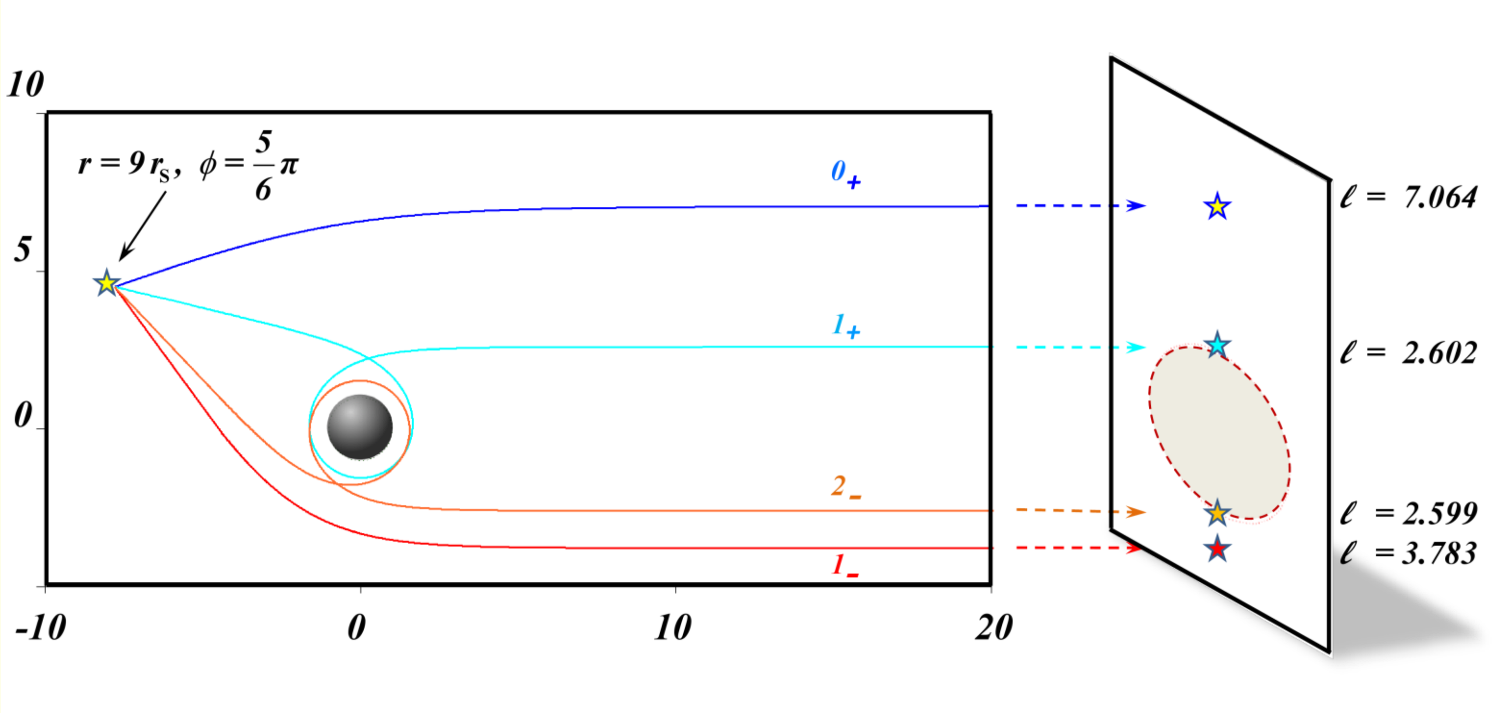
The velocity of motion on (unstable) orbits with $r \leq 3r_S$ grows from c/2 to c on the last circular orbit with $r = 1.5\,r_S$

Light propagation

There is a variety of effects connected with action of the gravitational field of a black hole on the light propagation in its vicinity:

1. Light rays are bended, so that images of bright objects are distorted;
2. A point-like source may have many images;
3. A cross-section of a beam of light rays after its passing near the black hole may shrunk;
4. Beam's shape is distorted;
5. Visible brightness of images depends on their position;
6. Time of a light signal arrival is delayed when it passes near the black hole;
7. Registered frequency of radiation, emitted by an object moving near the black hole, is shifted.

Multiple images



The turning points for the radial motion can be found from geodesic equations for null rays. One can show that the there is a minimum angular momentum parameter

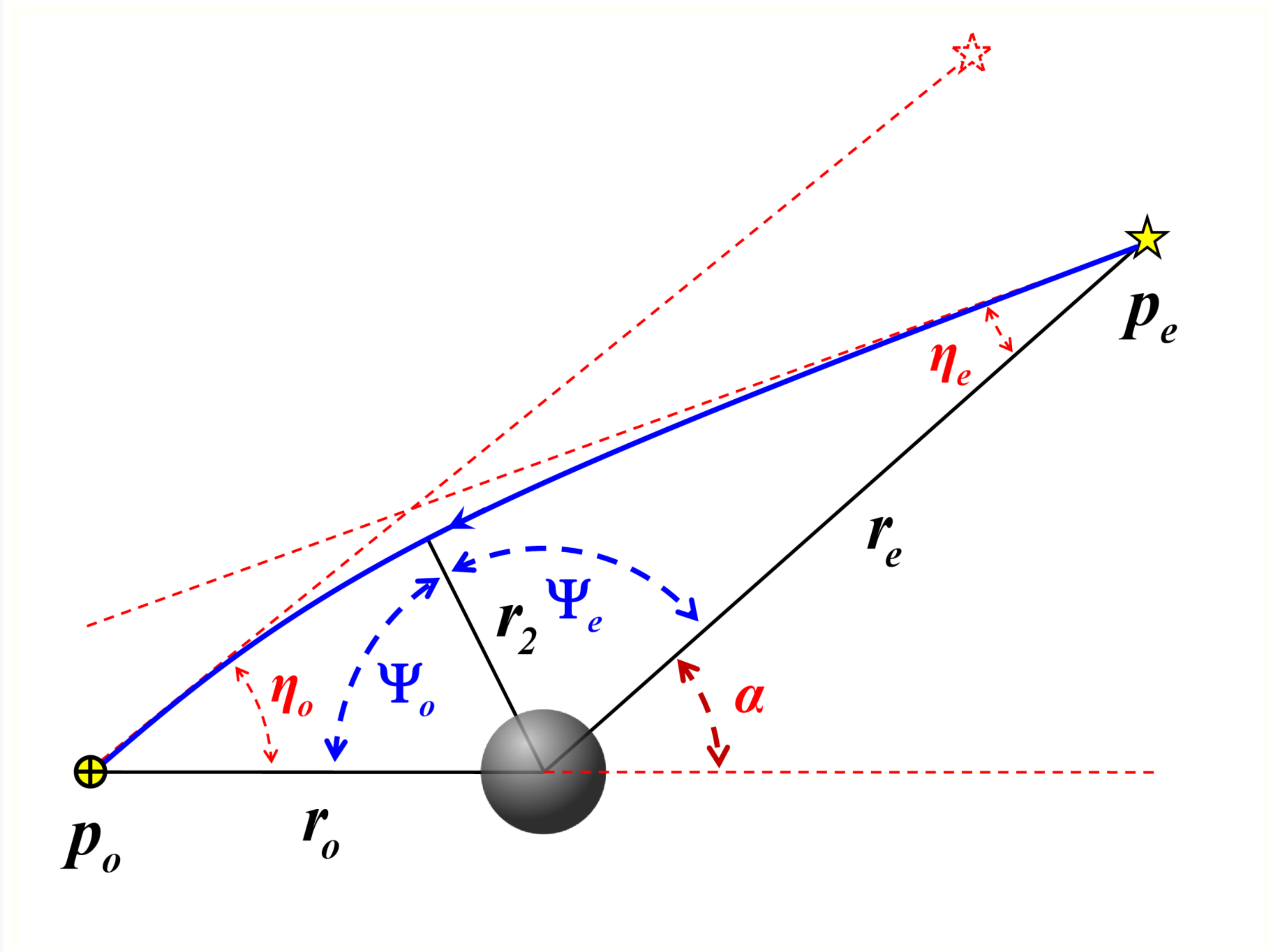
$$\ell_{\text{min}} = 3\sqrt{3}/2$$

corresponding to the turning point at $r = 3/2\, r_{\text{S}}$

For $\ell < \ell_{\text{min}}$ a null ray propagates from infinity to the horizon. Such a null ray is captured by the black hole. Thus the capture cross-section for photons is

$$\sigma_{\text{photons}} = \pi \ell_{\text{min}}^2 r_{\text{S}}^2 = \frac{27}{4} \pi r_{\text{S}}^2 \,.$$

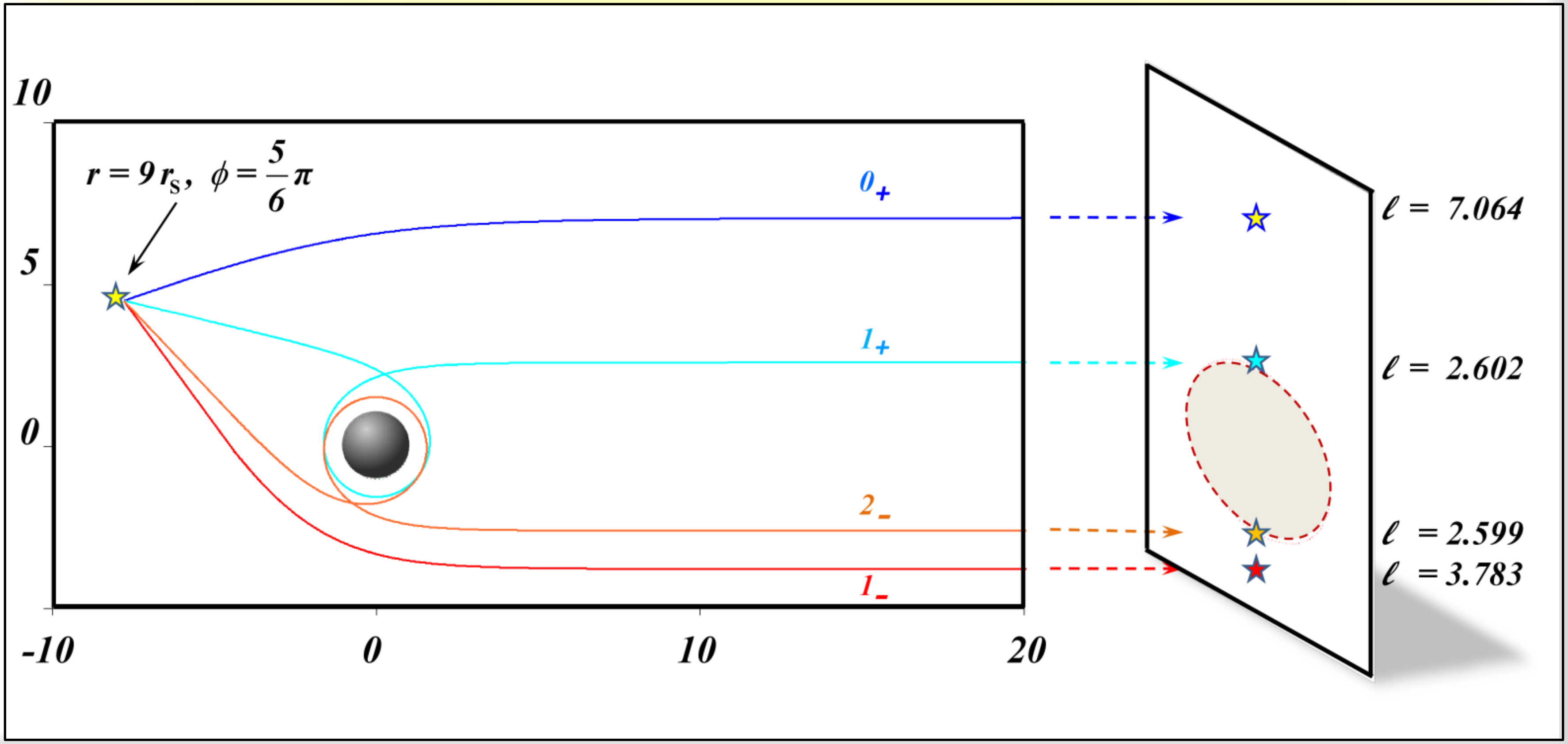
For $\ell > \ell_{\text{min}}$ a null ray has a radial turning point.



p_e
η_e
r_e
r_2
Ψ_e
η_o
Ψ_o
α
p_o
r_o

Gravitational lens

A black hole as any other gravitating body acts on light as an optical lens with distortion. But a black hole is a very special gravitating object. Its main feature is a strong gravitational field.



Images of circular orbits

Let us consider a visible shape of the circular orbit near the black hole as seen by a distant observer. If the orbit is very far away from the horizon and we look at it from some finite inclination angle to the orbit plane then, evidently, the orbit will look like an ellipse because null rays emitted from the objects on the orbit and observed by us never enter the region of the strong gravitational field. Though, if we look in the direction of the black hole we nd out also many images of this orbit, concentrated at angles close to the angle corresponding to the critical impact parameter

$$\ell_{\min} = 3\sqrt{3}/2$$

The picture is much more interesting when the orbit is located closer to the black hole and the observer is looking from inclination angles $\iota \sim \pi/2$