BLACK HOLE PHYSICS

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9. ROTATING BLACK HOLES

Kerr-NUT-(A)dS metric `derivation’

4D Kerr-NUT-(A)dS
Derivation in 3 Simple Steps

Step 1: Write flat ST metric in ellipsoidal coordinates

$$dS^2 = -dt^2 + dX^2 + dY^2 + dZ^2$$

$$X = \sqrt{r^2 + a^2}\,\sin\theta\cos\phi, \quad Y = \sqrt{r^2 + a^2}\,\sin\theta\sin\phi,$$

$$Z = r\cos\theta, \quad \frac{X^2+Y^2}{r^2+a^2} + \frac{Z^2}{r^2} = 1$$

$$dS^2 = -dt^2 + (r^2 + a^2\cos^2\theta)\left(\frac{dr^2}{r^2 + a^2} + d\theta^2\right) +$$

$$+ (r^2 + a^2)\sin^2\theta\, d\phi^2$$

Step 2: Rewrite this metric in `algebraic' form

$$y = a\cos\theta, \tau = t - a\phi, \quad \psi = a^{-1}\phi$$

$$ds^2 = -\frac{R(r)}{r^2+y^2}(d\tau + y^2 d\psi)^2 + \frac{Y(y)}{r^2+y^2}(d\tau - r^2 d\psi)^2$$

$$+(r^2+y^2)[\frac{dr^2}{R(r)} + \frac{dy^2}{Y(r)}],$$

$$R = r^2 + a^2, \quad Y = a^2 - y^2$$

(i) Coefficients are rational functions;
(ii) `Almost symmetric' form

Step 3: Use this form of the metric to solve Einstein equations

$$R_{\mu\nu} = -3\lambda g_{\mu\nu} \quad (\lambda = -\Lambda/3)$$

$$ds^2 = -\frac{R(r)}{r^2+y^2}(d\tau + y^2 d\psi)^2 + \frac{Y(y)}{r^2+y^2}(d\tau - r^2 d\psi)^2$$

$$+ (r^2+y^2)[\frac{dr^2}{R(r)} + \frac{dy^2}{Y(r)}]$$

[Carter 1968]

Practical hints

GRTensor program (by Kayl Lake et.al.
(for Maple and Mathematica)
grtensor.phy.queensu.ca/

```
> restart:
> grtw();
```

GRTensorII Version 1.79 (R4)

6 February 2001

Developed by Peter Musgrave, Denis Pollney and Kayll Lake

Latest version available from: http://grtensor.phy.queensu.ca/

/home/frolov/grii/metrics

```
> makeg(sss);

Makeg 2.0: GRTensor metric/basis entry utility

To quit makeg, type 'exit' at any prompt.

Do you wish to enter a 1) metric [g(dn,dn)],
                       2) line element [ds],
                       3) non-holonomic basis [e(1)...e(n)], or
                       4) NP tetrad [l,n,m,mbar]?

makeg>2;
Enter coordinates as a LIST (eg. [t,r,theta,phi]):
makeg>[r,y,tau,psi];

Enter the line element using d[coord] to indicate differentials.
(for example,  r^2*(d[theta]^2 + sin(theta)^2*d[phi]^2)
[Type 'exit' to quit makeg]
 ds^2 =
makeg>-(RR(r)/(r^2+y^2))*(d[tau]+y^2*d[psi])^2+(Y(y)/(r^2+y^2))*(d[t
      au]-r^2*d[psi])^2+(r^2+y^2)*(d[r]^2/RR(r)+d[y]^2/Y(y)):

If there are any complex valued coordinates, constants or functions
for this spacetime, please enter them as a SET ( eg. { z, psi } ).

Complex quantities [default={}]:
makeg>:
{}
```

The values you have entered are:

Coordinates $= [r, y, \tau, \psi]$

Metric:

$$g_{r\ r} = \frac{r^2}{\mathrm{RR}(r)} + \frac{y^2}{\mathrm{RR}(r)}$$

$$g_{r\ y} = 0$$

$$g_{r\ \tau} = 0$$

$$g_{r\ \psi} = 0$$

$$g_{y\ r} = 0$$

$$g_{y\ y} = \frac{r^2}{\mathrm{Y}(y)} + \frac{y^2}{\mathrm{Y}(y)}$$

$$g_{y\ \tau} = 0$$

$$g_{y\ \psi} = 0$$

$$g_{\tau\ r} = 0$$

$$g_{\tau\ y} = 0$$

$$g_{\tau\ \tau} = -\frac{\mathrm{RR}(r)}{r^2+y^2} + \frac{\mathrm{Y}(y)}{r^2+y^2}$$

$$g_{\tau\ \psi} = -\frac{\mathrm{RR}(r)\, y^2}{r^2+y^2} - \frac{\mathrm{Y}(y)\, r^2}{r^2+y^2}$$

$$g_{\psi\ r} = 0$$

$$g_{\psi\ y} = 0$$

$$g_{\psi\ \tau} = -\frac{\mathrm{RR}(r)\, y^2}{r^2+y^2} - \frac{\mathrm{Y}(y)\, r^2}{r^2+y^2}$$

$$g_{\psi\ \psi} = -\frac{\mathrm{RR}(r)\, y^4}{r^2+y^2} + \frac{\mathrm{Y}(y)\, r^4}{r^2+y^2}$$

```
You may choose to 0) Use the metric WITHOUT saving it,
                  1) Save the metric as it is,
                  2) Correct an element of the metric,
                  3) Re-enter the metric,
                  4) Add/change constraint equations,
                  5) Add a text description, or
                  6) Abandon this metric and return to Maple.
```

```
makeg>0;
Calculated ds for sss (0.004000 sec.)
```

Default spacetime = sss

For the sss spacetime:

Coordinates

x(*up*)

$$x^a = [r, y, \tau, \psi]$$

Line element

$$ds^2 = \left(\frac{r^2}{\mathrm{RR}(r)} + \frac{y^2}{\mathrm{RR}(r)}\right) d\,r^2 + \left(\frac{r^2}{\mathrm{Y}(y)} + \frac{y^2}{\mathrm{Y}(y)}\right) d\,y^2 + \left(-\frac{\mathrm{RR}(r)}{r^2+y^2} + \frac{\mathrm{Y}(y)}{r^2+y^2}\right) d\,\tau^2$$
$$+ 2\left(-\frac{\mathrm{RR}(r)\, y^2}{r^2+y^2} - \frac{\mathrm{Y}(y)\, r^2}{r^2+y^2}\right) d\,\tau\ d\ \psi + \left(-\frac{\mathrm{RR}(r)\, y^4}{r^2+y^2} + \frac{\mathrm{Y}(y)\, r^4}{r^2+y^2}\right) d\,\psi^2$$

```
makeg() completed.
```

```
> grcalc(R(dn,dn),Ricciscalar):
  gralter(R(dn,dn),Ricciscalar,1):
```

```
  grdisplay(Ricciscalar);
Calculated g(dn,dn,pdn) for sss (0.000000 sec.)
Calculated Chr(dn,dn,dn) for sss (0.004000 sec.)
Calculated detg for sss (0.000000 sec.)
Calculated g(up,up) for sss (0.004000 sec.)
Calculated Chr(dn,dn,up) for sss (0.004000 sec.)
Calculated R(dn,dn) for sss (0.028000 sec.)
Calculated Ricciscalar for sss (0.000000 sec.)
```

$$CPU\ Time\ = 0.048$$

```
Component simplification of a GRTensorII object:

Applying routine simplify to object R(dn,dn)
Applying routine simplify to object Ricciscalar
```

$$CPU\ Time\ = 0.024$$

For the sss spacetime:

Ricci scalar

$$R = -\frac{\left(\frac{d^2}{dr^2}\mathrm{RR}(r)\right)+\left(\frac{d^2}{dy^2}\mathrm{Y}(y)\right)}{r^2+y^2}$$

```
> RA:=r->(r^2+a^2)*(1+lambda*r^2)-2*M*r;
  YA:=y->(a^2-y^2)*(1-lambda*y^2)+2*N*y;
```

$$RA := r \to (r^2+a^2)(1+\lambda r^2) - 2\,M\,r$$

$$YA := y \to (a^2-y^2)(1-\lambda y^2) + 2\,N\,y$$

```
> grdef('W{p q}:=R{p q}+3*lambda*g{p q}');
  grcalcalter(W(dn,dn),1);
  grmap(W(dn,dn),subs,RR=RA,Y=YA,'x');
Created definition for W(dn,dn)

Simplification will be applied during calculation.

Applying routine simplify to object W(dn,dn)
Calculated W(dn,dn) for sss (0.028000 sec.)
```

$$CPU\ Time\ = 0.028$$

```
Applying routine subs to W(dn,dn)
```

```
> grcalc(W(dn,dn));
  gralter(W(dn,dn),1);
  grdisplay(_);
```

$$CPU\ Time\ = 0.$$

```
Component simplification of a GRTensorII object:

Applying routine simplify to object W(dn,dn)
```

$$CPU\ Time\ = 0.008$$

For the sss spacetime:

W(dn,dn)

W(*dn*, *dn*)

$$\mathrm{W}_{a\ \ b} = \textit{All components are zero}$$

'Trace equation' gives:

$$\partial_r^2 R + \partial_y^2 Y = 12\lambda(r^2 + y^2),$$

$$\partial_r^2 R - 12\lambda r^2 = c, \quad \partial_y^2 Y - 12\lambda y^2 = -c,$$

$$R = \lambda r^4 + \frac{c}{2} r^2 + \alpha r + \beta,$$

$$Y = \lambda y^4 - \frac{c}{2} y^2 + \gamma r + \delta$$

The other equations fix some constants and we have

$$R = (r^2 + a^2)(1 + \lambda r^2) - 2Mr,$$

$$Y = (a^2 - y^2)(1 - \lambda y^2) + 2Ny$$

$a -$ rotation parameter, $M -$ mass,

$N -$ `NUT' parameter, $\lambda -$ `cosmological term'

In what follows we put $\lambda = N = 0$ and coordinates `back'

$$y = a\cos\theta, \tau = t - a\phi, \quad \psi = a^{-1}\phi.$$

After this we obtain Kerr solution in standard Boyer-Lindquist coordinates (t, r, θ, ϕ).

Kerr metric

$$ds^2 = -\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mra\sin^2\theta}{\Sigma}dt\,d\phi + \frac{A\sin^2\theta}{\Sigma}d\phi^2$$

$$+\frac{\Sigma}{\Delta}dr^2 + \Sigma\,d\theta^2,$$

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2,$$

$$A = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta.$$

At far distances Kerr metric takes the form

$$ds^2 \approx -\left(1-\frac{2M}{r}\right)dt^2 - \frac{4Ma\sin^2\theta}{r}dt\,d\phi + dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2).$$

From this asymptotic form one can conclude that ***M*** is the mass and ***J=Ma*** is the angular momentum of the black hole.

The parameter ***a*** in the Kerr metric is called the **rotation parameter**. Like the mass ***M*** it has a dimensionality of length. Their ratio is a dimensionless parameter

$$\alpha = a/M$$

which is called the ***rotation rapidity***.

Infinite redshift surface

From the form of the Kerr metric it is easy to conclude that

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mra\sin^2\theta}{\Sigma} dt\, d\phi + \frac{A\sin^2\theta}{\Sigma} d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2 ,$$

$$-\xi_t^2 = 1 - \frac{2Mr}{\Sigma}.$$

$$\Sigma - 2Mr = r^2 - 2Mr + a^2 \cos^2\theta = 0,$$

$$r = r_0(\theta), \;\; r_0(\theta) \equiv M + \sqrt{M^2 - a^2\cos^2\theta}.$$

Surface determined by this condition

for $a \neq 0$ is timelike (not null), and hence

thisis not a horizon.

This surface is also called an *ergo surface*

How to find the horizon surface? Suppose its equation is $S(r,\theta)=0$. The surface is null when its gradient vector $S_{,\mu}$ is null

$$g^{\mu\nu}S_{,\mu}S_{,\nu}=0 \Rightarrow \Delta\, S_{,r}^2+S_{,\theta}^2=0.$$

This equation is satisfied when $S=\Delta=0$.

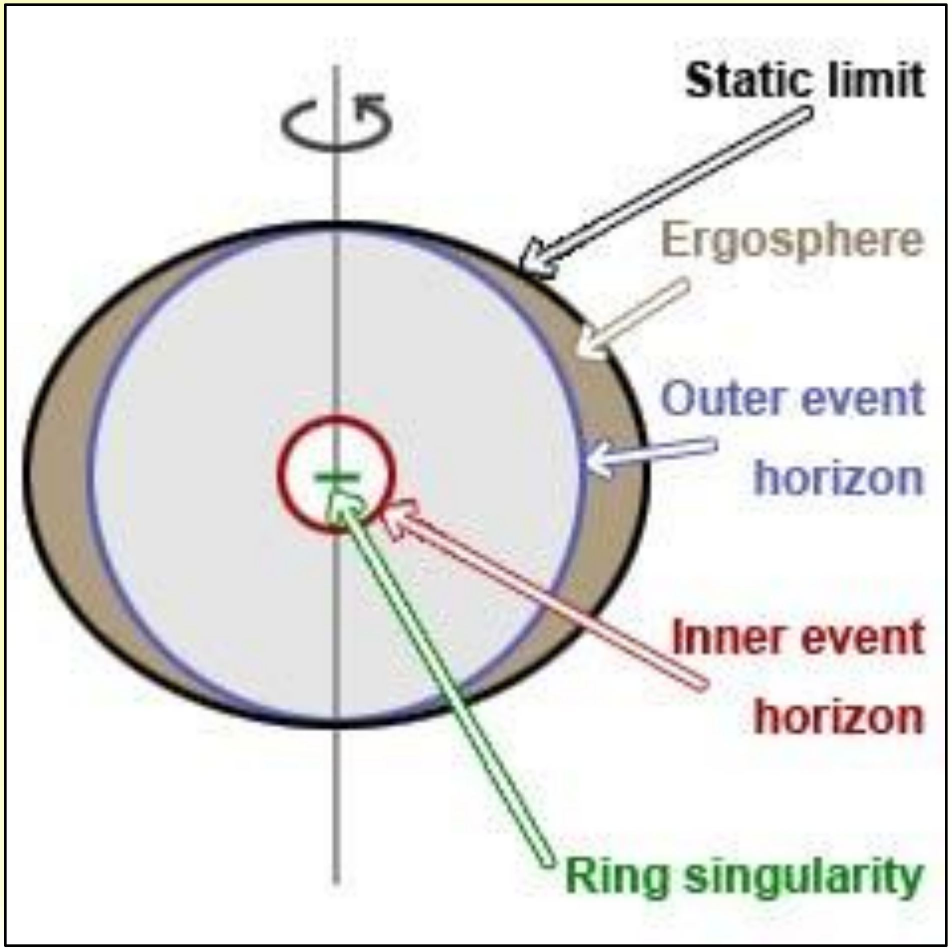
The event horizon is located at $r = r_+ \,, \qquad r_\pm = M \pm \sqrt{M^2 - a^2}\,.$
Which are the roots of the equation $\Delta(r) = 0$

The event horizon of the Kerr spacetime is a null 3-dimensional surface. Its spatial slices have the geometry of a 2-dimensional distorted sphere.

The rotating black holes exist only for

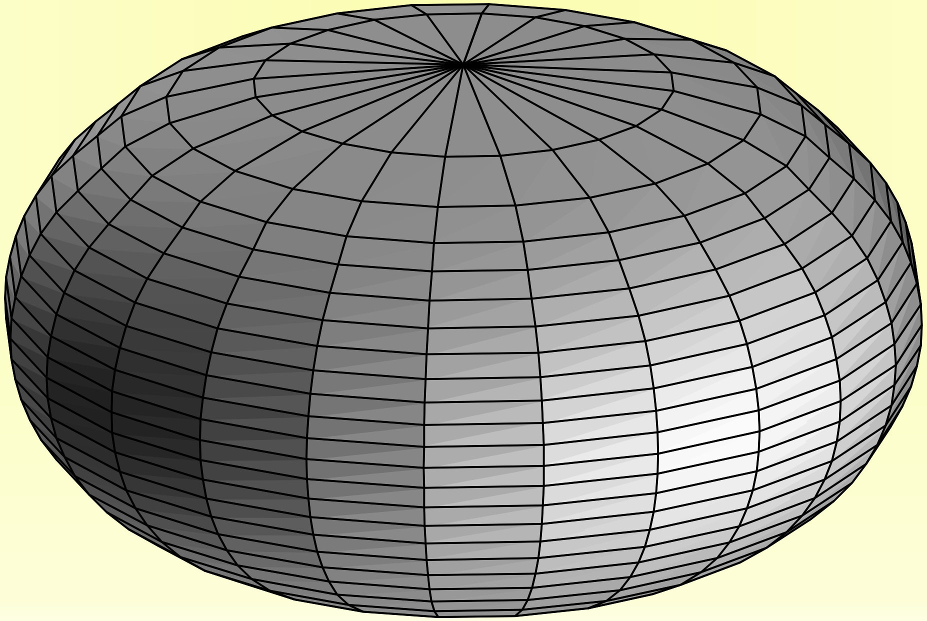
$$a \le M$$

For $a > M$ the Kerr solution does not have a horizon and it describes a naked singularity. It is generally believed that such a singularity does not arise in real physical processes.



The embedding diagram of a two-dimensional section of the event horizon of the Kerr black hole. The diagram is for the critical value of the rotation parameter $a = M\frac{\sqrt{3}}{2}$ **so that the Gaussian curvature vanishes at the poles.**

For higher values of a the internal metric of the horizon can not be obtained by embedding it into a flat 3d space.



Consider (non-geodesic) circular orbits $u^\mu \sim \eta^\mu = \xi_t^\mu + \omega \xi_\phi^\mu$.

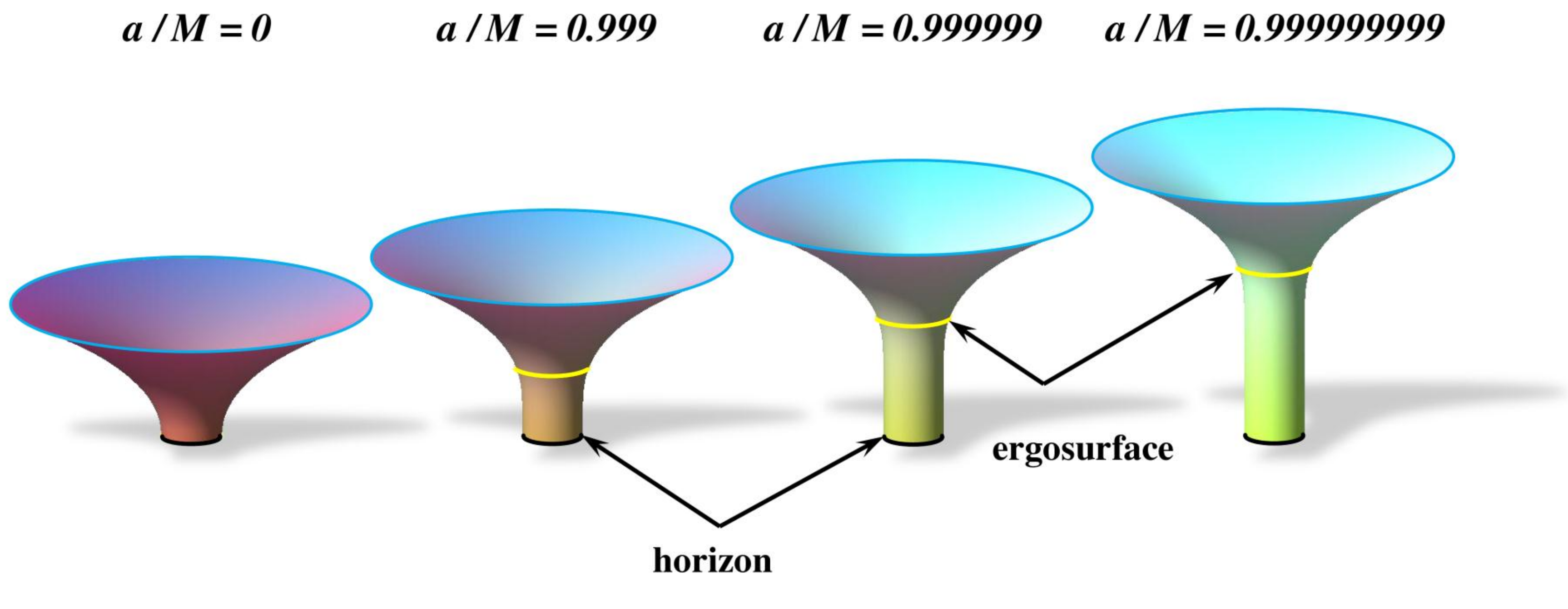
A condition that u^μ null implies

$$\omega_\pm = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}}{g_{\phi\phi}}, \quad g_{t\phi}^2 - g_{tt} g_{\phi\phi} = \Delta \sin^2 \theta.$$

Real solutions exist only when $\Delta \geq 0$. Vector u^μ is timelike for $\omega_- < \omega < \omega_+$. Vector u^μ becomes null at the horizon and one has $\omega_- = \omega_+ = \Omega$. This is angular velocity of the black hole. Inside ergosurface, where $g_{tt} > 0$, both $\omega_\pm$ are positive. Particle are always co-rotating with the black hole.

Analysis of the Keplerian circular motion of particles in the equatorial plane shows that radius of ISCO can be arbitrary close to r_+.

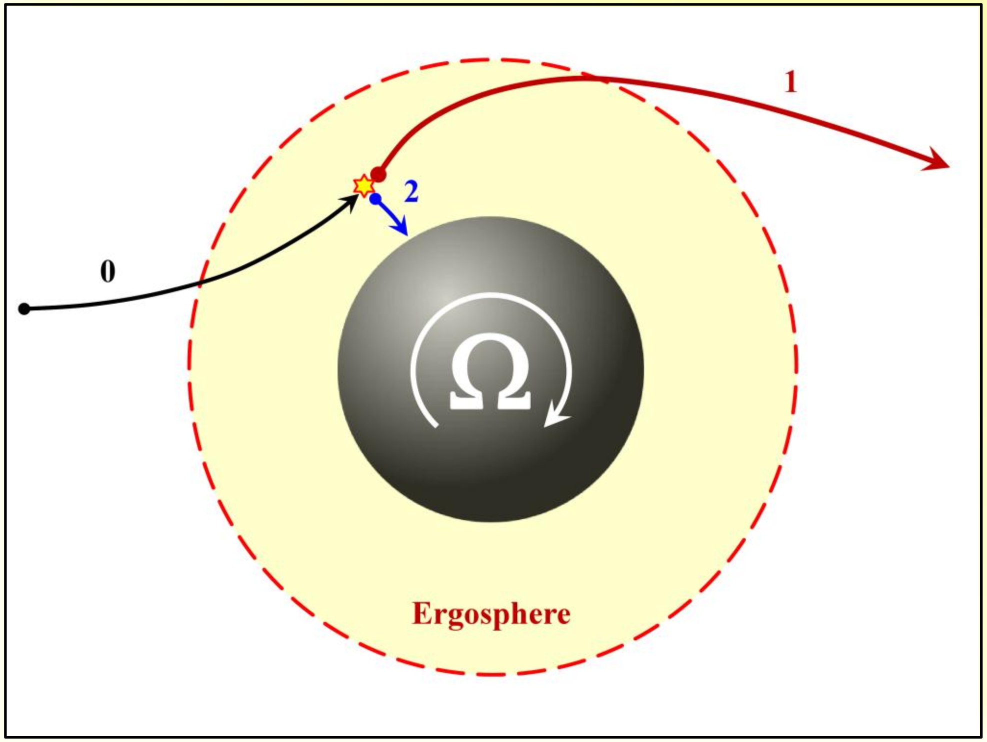
Einstein-Rosen bridges for Kerr spacetime



Negative energy orbits

Then the energy of a particle or a photon may be negative $E < 0$ This happens only when they move inside the ergosphere. This property is related to the fact that $\xi^{\mu}_{(t)}$ is not timelike inside ergosphere. Since the corresponding momentum p_{μ} is a future directed timelike or null vector, the energy of a particle inside the ergosphere **may be both positive and negative**.

$$E = -p_{\mu}\,\xi^{\mu}_{(t)}\,,$$



Penrose process. A particle 0 enters the ergosphere and decays there into two particles, 1 and 2. One of them with a negative energy (2) falls into the black hole. The other one (1) escapes the ergosphere with the energy exceeding the energy of the original particle.

Particle and light propagation in Kerr ST :

Integrals of motion.

Let ξ^μ be a Killing vector and u^μ is 4-velocity of the particle (photon). Then $P = \xi_\nu u^\nu$ is conserved.

$$\frac{dP}{d\tau} = \frac{dx^\mu}{d\tau} P_{;\mu} = u^\mu u^\nu \xi_{\nu;\mu} + \xi_\nu u^\mu u^\nu_{;\mu} = 0.$$

Let $K_{\mu\nu}$ be a symmetric tensor obeying the property $K_{(\mu\nu;\alpha)} = 0$. Then $P = K_{\mu\nu} u^\mu u^\nu$ is conserved.

$$\frac{dP}{d\tau} = \frac{dx^\alpha}{d\tau} P_{;\alpha} = u^\mu u^\nu u^\alpha K_{\nu\mu;\alpha} + 2K_{\mu\nu} u^\mu u^\alpha u^\nu_{;\alpha} = 0.$$

Such an object $K_{\mu\nu}$ is called a Killing tensor.

Metric $g_{\mu\nu}$ is a trivial example of the Killing tensor.

Kerr metric has 2 Killing vectors ξ_{t}^{μ} and ξ_{ϕ}^{μ}.

Carter (1968) found that it also has a Killing tensor.

This gives 4 integral of motion and make particle anf light equationscompletely integrable.

Energy extraction from rotating black holes

Consider Penrose process in more details. Local conservation low requires $p_0^\mu = p_1^\mu + p_2^\mu$. For each of the particles we define their energy and angular momentum $\varepsilon_{\rm i} = -p_i^\mu \xi_{t,\mu}$, $j_{\rm i} = p_i^\mu \xi_{\phi,\mu}$. Denote by $l^\mu = \xi_t^\mu + \Omega \xi_\phi^\mu$ a null generator of the horizon. Here Ω is the angular velocity of the black hole $\Omega = \dfrac{a}{r_+^2 + a^2}$.

Assume that particle 2 has negative energy. Then

$0 \geq l^{\mu} p_{2\mu} = -\varepsilon_2 + \Omega\, j_2 \Rightarrow j_2 \leq \varepsilon_2 / \Omega \Rightarrow$

$j_1 - j_0 = -j_2 \geq -\varepsilon_2 / \Omega.$

This means that the extraction of the energy is always accompanied by the extraction of the angular momentum.

Parameters of the black holes change

$\delta M = \varepsilon_2, \delta J = j_2 \Rightarrow \delta M - \Omega \delta J \geq 0.$

For the most economic (`reversable') process $\delta M = \Omega \delta J$.

$$A = 4\pi(r_+^2 + a^2) = 8\pi(M^2 + \sqrt{M^4 - J^2})$$

$$\delta A = 16\pi(M + \frac{M^3}{\sqrt{M^4 - J^2}})[\delta M - \Omega\delta J] \geq 0.$$

If the process is not reversable, the surface area of the black hole grows.

This is a special case of the general theorem proved by Hawking: In the absence of a singularity the surface area of the black hole is a non-decreasing function of time, provided the weak energy condition is satisfied.

Irreducible mass

$$M_{ir} = (A/16\pi)^{1/2}, \quad M_{ir}^2 = \frac{1}{2}(M^2 + \sqrt{M^4 - J^2})$$

$$M^2 = M_{ir}^2 + \frac{J^2}{4M_{ir}^2} \geq M_{ir}^2$$

Consider a black hole with parameters M_0, J_0.

After energy extraction it has mass $M_1 \geq M_{ir}(M_0, J_0)$.

If J_1=0 then in the most efficient (reversable) process $\Delta M = M_0 - M_{ir}(M_0, J_0)$. For extremal black hole $J_0 = M_0^2 \Rightarrow M_{ir} = M_0/\sqrt{2}$. So that

$\Delta M = (1 - 1/\sqrt{2})M_0 = 0.29 M_0$. For a stellar mass BH with $M_0 = 10 M_\odot$ this gives the energy 0.6×10^{55} erg.

Superradiance

$$\Phi \sim f(r,\theta)\exp(-i\omega t + im\phi)$$

$$\varepsilon = \hbar\omega, \quad j = \hbar m,$$

$$\delta M - \Omega\delta J \geq 0, \quad \delta J / \delta M = m / \omega,$$

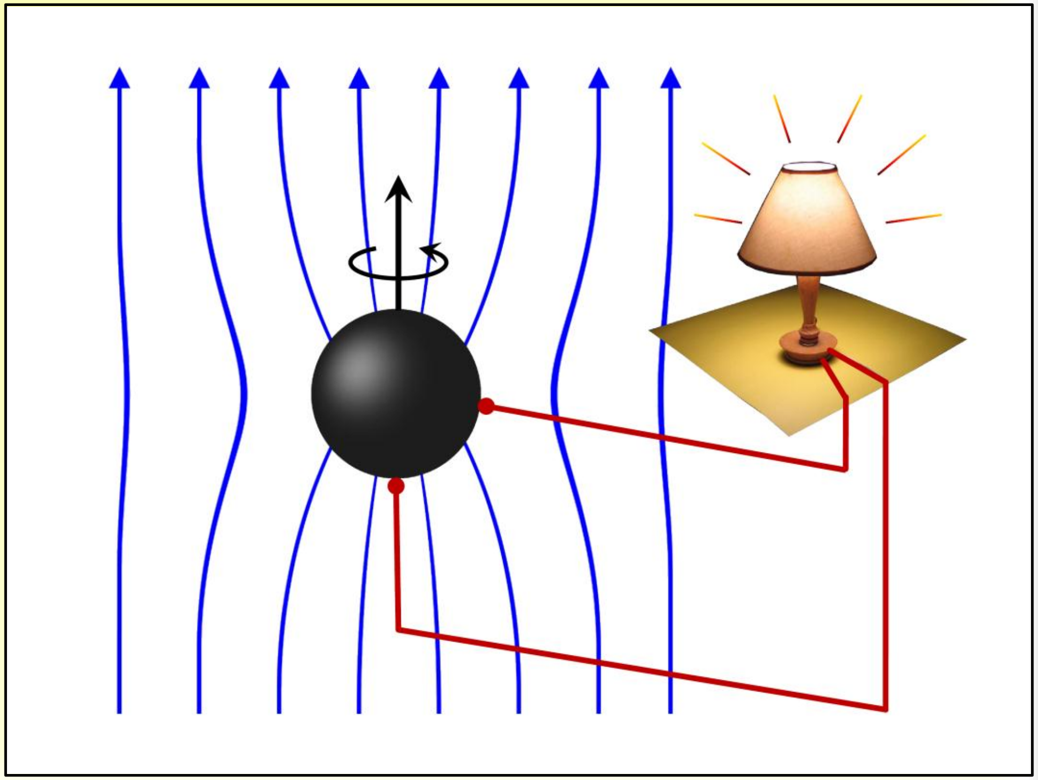
$$\frac{\delta M}{\hbar\omega}[\hbar\omega - \hbar m\Omega] \geq 0.$$

Energy is extracted when $\omega \leq m\Omega$.

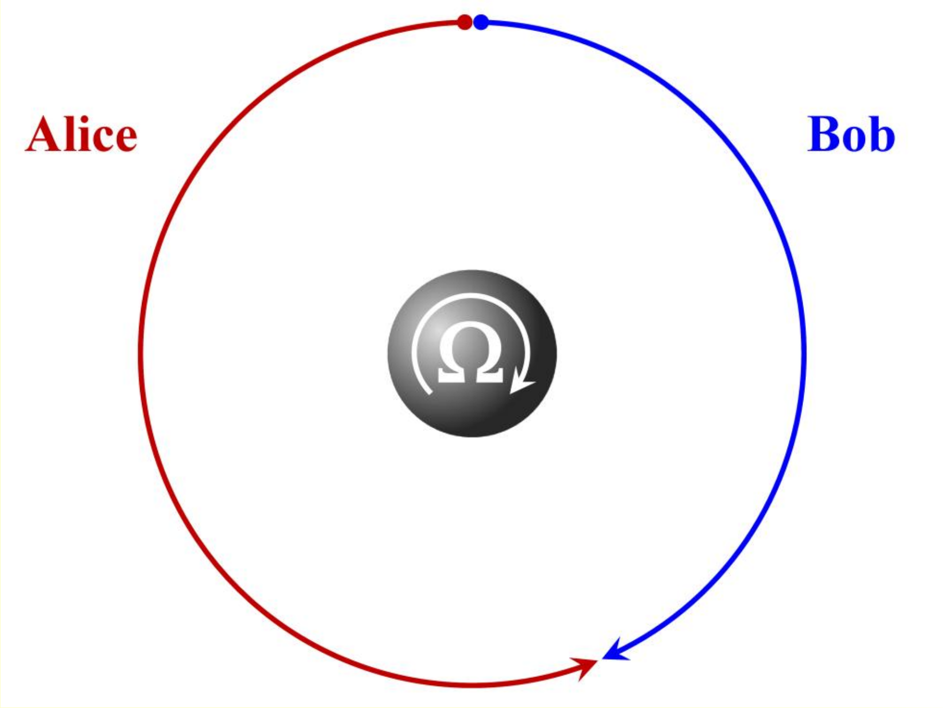
Black hole in magnetic field

The *black hole electrodynamics* is quite non-trivial if we deal not with an isolated black hole, but with a black hole surrounded by matter. In a generic case a black hole is surrounded by accreting matter, which forms an *accretion disk*. This matter is usually hot enough and is in the form of plasma.

The plasma is an ideal conductor and the magnetic field (if it is present) is frozen into it. In the presence of the rotating plasma the well known *dynamo mechanism* can amplify the magnetic field. As a result of this mechanism magnetic field in the black hole vicinity is generated, which can produce observable effects. (Blandford-Znajek mechanism)



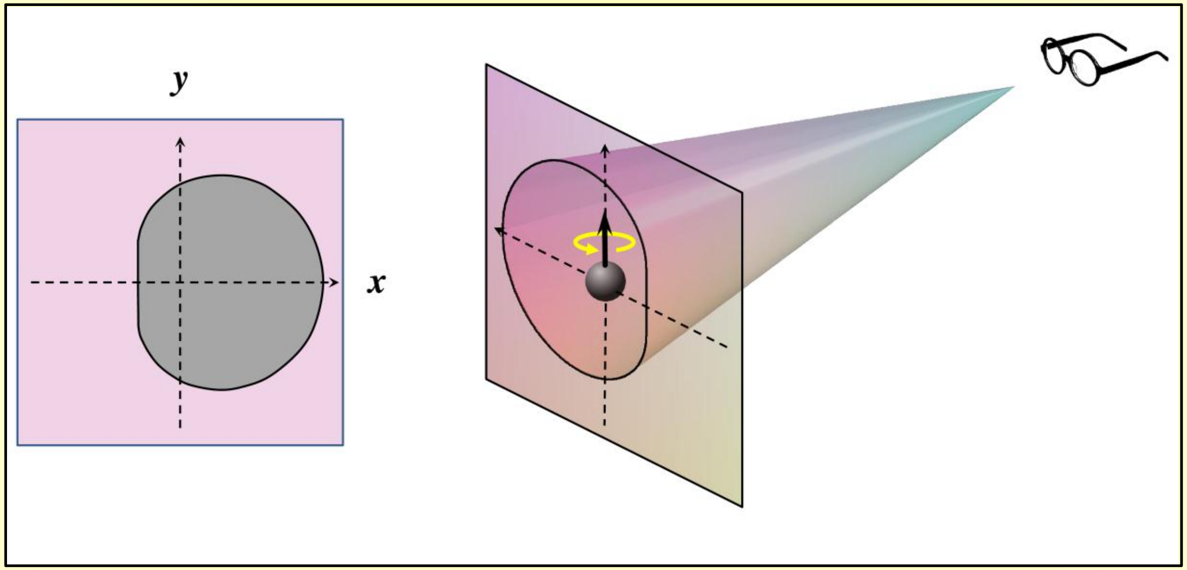
Twin paradox in Kerr spacetime



A free motion of objects along a circular trajectory gives an interesting example of the twin paradox in the General Relativity. Consider two observers, Alice and Bob, moving in the opposite directions along a circular orbit of the same radius r in the equatorial plane of the Kerr black hole. Let us assume that Bob is moving in the direction of rotation of the black hole and Alice is counter-rotating. Because the black hole drags into rotation the space around it, Alice should move faster than Bob to stay at the same circular orbit. Therefore Alice covers more distance than Bob after their first encounter till the next one. Her higher speed in combination with other relativistic effects leads to the slower proper time pace for Alice. So Bob grows old faster than Alice, in spite of the fact that both move geodesically along the same orbit. The rotation of the black hole makes a big difference.

Black hole "shadow"

Imagine that behind the rotating black hole there is a source of light. Let its angular size be much larger than the angular size of the black hole. Then a distant observer will see a dark spot on the *(x-y)-* plane, which is an apparent image of the black hole. This is what is called the black hole shadow.



The rim of this shadow corresponds to photons which are marginally trapped by the black hole. They revolve around the black hole many times before nally reach the distant observer. The size and shape of the rim depends on the black hole parameters (*M* and *a*) as well as on the inclination angle 0 between the direction to the distant observer and the axis of symmetry.

The observer is looking in the direction of the black hole. On the background of the sky the black hole will be visible as a black spot ("shadow") with the shape of a deformed disk which is slightly shifted in the direction of rotation of the black hole (to the right). For an extremal black hole *a=M* , when observed from the equatorial plane, the left edge of the shadow is a vertical straight line between the points $(-2M, -\sqrt{3}M)$ **and** $(-2M, \sqrt{3}M)$ **. The right edge is at *x = 7M*.**