BLACK HOLE PHYSICS

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10. Higher dimensional Black Holes

Why Large Extra Dimensions?

Motivations :

(1) Extra-dimensions and string theory

(2) Brane-world models

(3) Black holes as probes of extra dimensions

(4) Micro BHs production in colliders?

(5) Generic and non-generic properties of BHs

Higher dimensional gravity is stronger at small scales and weaker at large scales than the 4D one: $F \sim G^{(D)} M_1 M_2 / r^{(D-2)}$

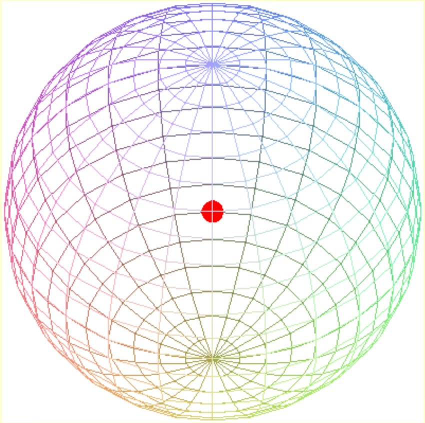
If extra dimensions are compact with size L, then at the distance larger than L one has standard 4D gravity with $G = G^{(D)} / L^{(D-4)}$.

For $L \sim 0.1$ mm higher dim. gravity may be as strong as other interactions.

Properties of HD black holes with $r_g \leq L$ are determined by D-dim Einstein equations, so that testing their properties in our 4D experiments one probes extra dimensions (e.g. micro BH at colliders?)

Gravity in Higher Dim. Spacetime

$$\Delta^{(4+k)}\Phi = -a_k G^{(k)} M \delta^{3+k}(\vec{x})$$



$$\vec{F} = -\nabla\Phi$$

$$F \sim \frac{G^{(k)}M}{r^{2+k}}, \quad k = D-4$$

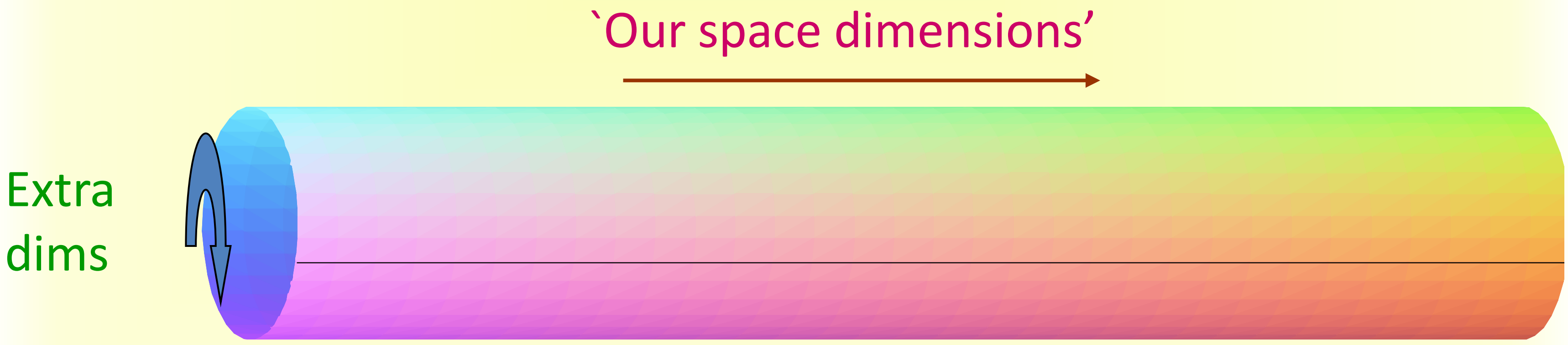
No bounded orbits (prove!)

Gravity at small scales is stronger than in 4D

4D Newton law is confirmed for r>l.

Q.: How to make gravity strong (HD) at small scales without modifying it at large scales?

A.: Compactification of extra dimensions

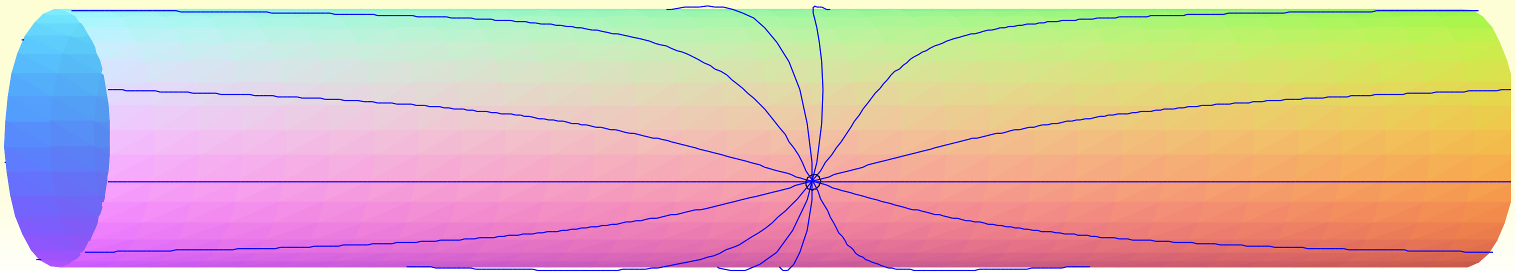


Gravity in ST with Compact Dims

Example: $$\Delta^{(4)}\Phi = 0, \quad \mathrm{M}^4 = R^3 \times S^1$$

$$\Phi \sim G^* M \sum_{n=-\infty}^{\infty} \frac{1}{\vec{r}^{\,2} + (z+nl)^2} = \frac{G^* M \pi}{lr} \frac{\sinh(\pi r/l)\cosh(\pi r/l)}{\cosh^2(\pi r/l) - \cos^2(\pi z/l)}$$

$$\Phi(\vec{r}, z=0) \sim \frac{G^* M \pi}{lr} \coth(\pi r/l), \quad G = G^* \pi / l$$



In a general case: # STdimensions is D,

non-compactified dimensionsis $D-4$: $G = \frac{G^D}{l^{D-4}}$.

For $r \ll l \Rightarrow F \sim \frac{G l^{D-4} M_1 M_2}{r^{D-2}}$; for $r \gg l \Rightarrow F \sim \frac{G M_1 M_2}{r^2}$.

One can introduce a function $G(r)$ such that

for $r \ll l \Rightarrow G(r) \sim \frac{G l^{D-4}}{r^{D-4}}$ and for $r \gg l \Rightarrow G(r) = G$.

Then $F \sim \frac{G(r) M_1 M_2}{r^2} \Rightarrow$ Running gr.coupling `constant'.

At small scales, that is at large energy, the running coupling `constant' becomes large and gravitational interaction becomes as strong as other physical interaction.

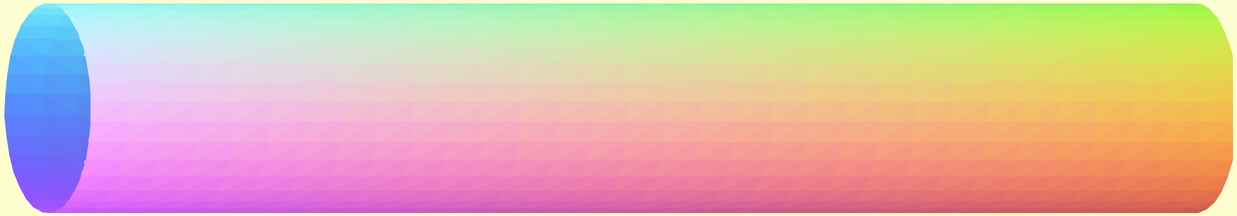
`Solution' of the hierarchy problem

$$\left(\frac{Gm_p^2}{e^2}\right) \sim 10^{-36}.$$

$$\frac{G(r_*)m_p^2}{r_*^2} = \frac{e^2}{r_*^2} \quad \Rightarrow \quad r_* = \left(\frac{Gm_p^2}{e^2}\right)^{1/k} l$$

$$For\ l \sim 0.01cm\ and\ D-4=k=2 \quad r_* \sim 10^{-20}cm$$

Kaluza-Klein tower



$$\Phi(x^\mu, y) = \Phi(x^\mu, y + 2\pi L) \quad \Rightarrow \quad \Phi(x^\mu, y) = \sum_n \Phi_n(x^\mu)\, e^{iny/L}$$

$$W = -\tfrac{1}{2} \int dx^4\, dy\, [\, \nabla\Phi\nabla\Phi^* + m_0^2\, \Phi\Phi^* \,]$$

$$W = -\tfrac{1}{2} \sum_n \int dx^4\, [\, \nabla_\mu \Phi_n \nabla^\mu \Phi^* + (m_0^2 + \frac{n^2}{L^2})\Phi\Phi^* \,]$$

KK tower with the mass spectrum $m^2 = m_0^2 + n^2 / L^2$

Brane World Models

Bosons, fermions and gauge fields are localized within the 4D brane

Gravity is not localized and `lives' in D-dim bulk space

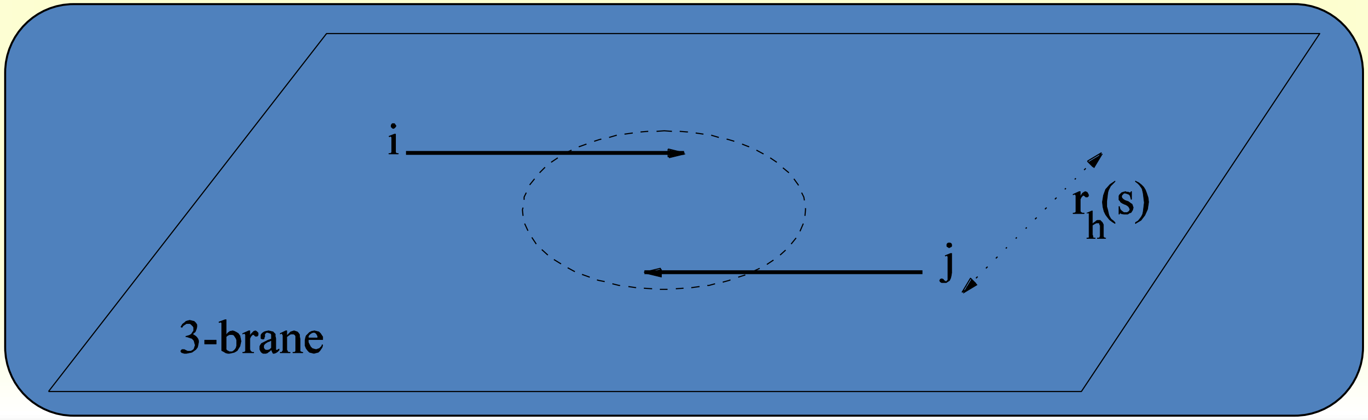
Fundamental scale of order of TeV. Large extra dimensions generate Planckian scales in 4D space

Black Holes as Probes of Extra Dims

We consider black holes in the mass range

$$10^{-21}\,g \ll M \ll 10^{27}\,g$$

Mini BHs creation in colliders



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STUDY OF POTENTIALLY DANGEROUS EVENTS DURING HEAVY-ION COLLISIONS AT THE LHC: REPORT OF THE LHC SAFETY STUDY GROUP

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GENEVA
2003

BH formation

Bolding Phase

Thermal (Hawking) decay

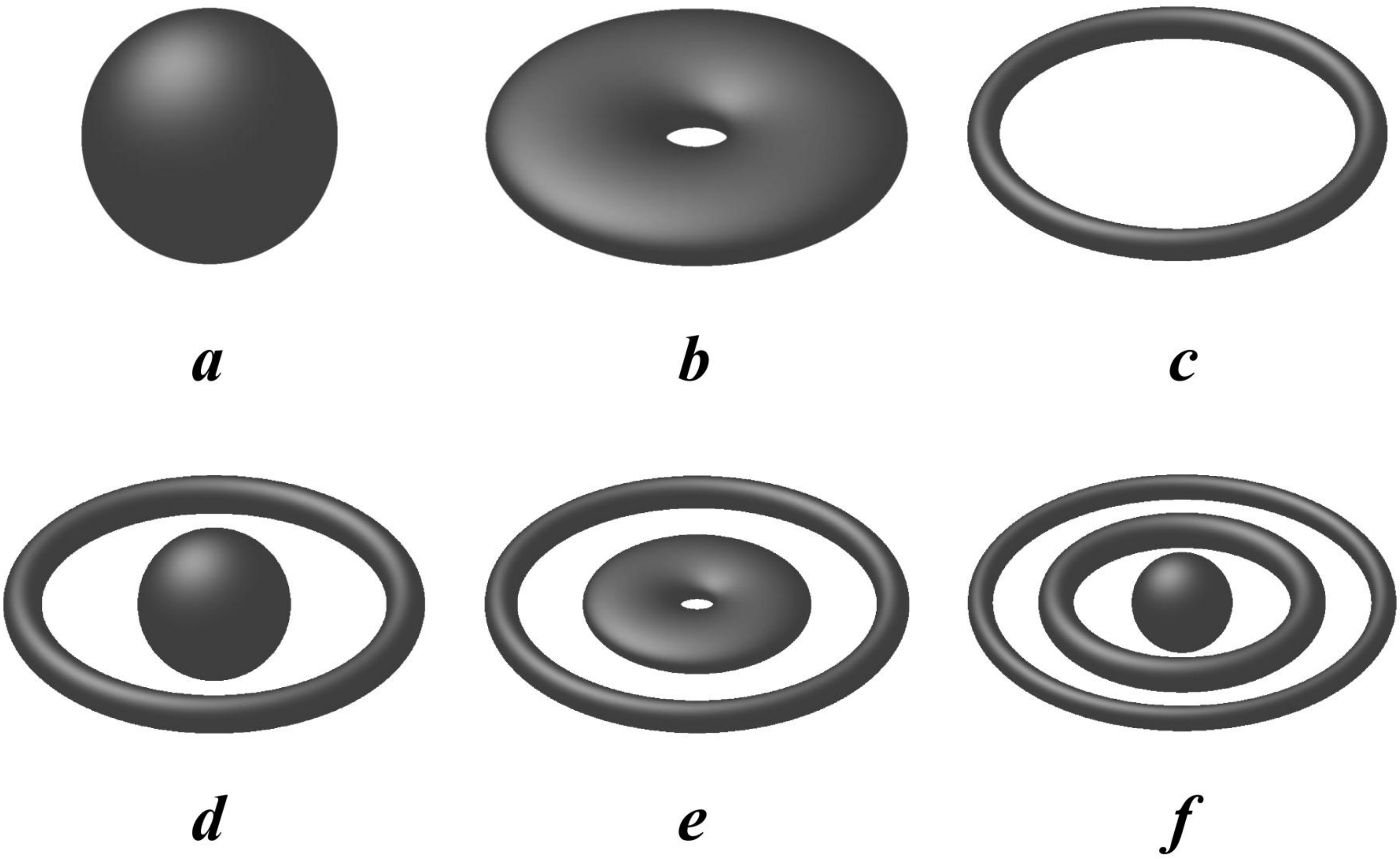
Higher dimensional BHs

Angular momentum in Higher Dimensions

$$(D = 2n + \varepsilon)$$

$$\mathbf{J} = \begin{pmatrix} J_1 & 0 & \ldots & 0 \\ 0 & J_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & J_{n-1+\varepsilon} \end{pmatrix}, \quad J_i = \begin{pmatrix} 0 & j_i \\ -j_i & 0 \end{pmatrix}$$

5D vac. stationary black holes



(1) No Uniquness Theorem: For given M

and J more than one BH solution

(2) Complete integrability and separation

of variables are generic properties

of HD analogues of Kerr BH

(*hidden symmetry generators* $V.F$ & $D.K.$ 07)

(3) Stability of HD BHs ?

We focus on higher dimensional rotating black holes with the spherical topology of horizon in a ST which asympotically is either flat or (Anti)DeSitter:

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

Main Results

In the most general (Kerr – NUT – (A)dS) higher – dimensional black hole spacetime:

(1) Geodesic motion is completely integrable.

(2) Hamilton – Jacobi and Klein – Gordon equa – tions allow the complete separation of variables

Motivations

Separation of variables allows one to reduce a physical problem to a simpler one in which physical quantities depend on less number of variables. In case of complete separability original partial differential equations reduce to a set of ordinary differential equations

Separation of variables in the Kerr metric
is used for study:
(1) Black hole stability
(2) Particle and field propagation
(3) Quasinormal modes
(4) Hawking radiation

Main lesson: Properties of higher dimensional rotating BHs and the 4D Kerr metric are very similar.

HDBHs give a new wide class of completely integrable systems

Higher Dimensional Black Holes

Tangherlini '63 metric (HD Schw.analogue)

....

Myers&Perry '86 metric (HD Kerr analogue)

....

Kerr-NUT-AdS '06 (Chen, Lu, and Pope; The most general HD BH solution with spherical topology of the horizon)

"General Kerr-NUT-AdS metrics in all dimensions", Chen, Lü and Pope, Class. Quant. Grav. 23 , 5323 (2006).

$$n = \left[D/2 \right], \;\; D = 2n + \varepsilon$$

$$R_{\mu\nu} = (D-1)\lambda g_{\mu\nu}$$

λ, $M - mass$, $a_k - (n-1+\varepsilon)$ *rotation parameters*,

$M_\alpha - (n-1-\varepsilon)$ *`NUT' parameters*

$Total\ \#\ of\ \ parameters\ is\ D-\varepsilon$

Differential forms

Antisymmetric tensor $\Leftrightarrow$ differential form

$$A_{[\alpha_1 \ldots \alpha_p]} \quad \Leftrightarrow A^p = A_{[\alpha_1 \ldots \alpha_p]} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_p}$$

$$A_{[[\alpha_1 \ldots \alpha_p]} B_{[\alpha_1 \ldots \alpha_q]]} \Leftrightarrow A^p \wedge B^q$$

$$A_{[\alpha_1 \ldots \alpha_p, \alpha_{p+1}]} \Leftrightarrow dA^p$$

$$\mathrm{B}_{[\alpha_{p+1} \ldots \alpha_D]} = e_{\alpha_{p+1} \ldots \alpha_D}{}^{\alpha_1 \ldots \alpha_p} A_{[\alpha_1 \ldots \alpha_p]} \Leftrightarrow B^{D-p} = *A^p$$

Generator of Symmetries

PRINCIPLE CONFORMAL KY TENSOR

2-form $h_{[\mu\nu]}$ with the following properties:

(i) Non-degenerate (maximal matrix rank)

(ii) Closed $dh = 0$

(iii) Conformal KY tensor

$$\nabla_c h_{ab} = g_{ca}\xi_b - g_{cb}\xi_a, \qquad \xi_a = \tfrac{1}{D-1}\nabla^b h_{ba}$$

ξ_a is a primary Killing vector

All Kerr-NUT-AdS metrics possess a PRINCIPLE CONFORMAL KY TENSOR

(V.F.&Kubiznak '07)

Uniqueness Theorem

A solution of Einstein equations with the cosmological constant which possess a PRINCIPLE CONFORMAL KY TENSOR is a Kerr-NUT-AdS metric

(Houri,Oota&Yasui ’07 ’09; Krtous, V.F. .&Kubiznak ’08;)

4D Kerr metric example

Principal CCKY tensor $h = db$

$$b = \frac{1}{2}[(y^2 - r^2)d\tau - r^2 y^2 d\psi]$$

$$H^{\nu}{}_{\mu} = h^{\nu\lambda} h_{\mu\lambda}; \quad \det(H^{\nu}{}_{\mu} - \kappa\delta^{\nu}{}_{\mu}) = 0,$$

$$\kappa_+ = y^2, \quad \kappa_- = -r^2$$

Lesson: Essential coordinates (r, y),

Killing coordinates (τ, ψ)

All of this is valid for arbitrary functions

$R(r)$ and $Y(y)$ (`off shel') :

$$ds^2 = -\frac{R(r)}{r^2+y^2}(d\tau + y^2 d\psi)^2 + \frac{Y(y)}{r^2+y^2}(d\tau - r^2 d\psi)^2$$

$$+(r^2+y^2)[\frac{dr^2}{R(r)} + \frac{dy^2}{Y(r)}].$$

For Kerr metric: $R = r^2 + a^2, \quad Y = a^2 - y^2$

BRIEF REMARKS ON COMPLETE INTEGRABILITY

Dynamical System

Phase space: $\{M^{2m}, \Omega, H\}$; Ω_{AB} is a non-degenerate 2-form (symplectic structure); H is a scalar function on M^{2m} called a Hamiltonian. $\eta^A = H_{,B}\Omega^{BA}$ is a generator of the Hamiltonian flow $(\Omega_{AB}\Omega^{BC} = \delta_A^C)$.

Equation of motion is $\dot{z}^A = \eta^A$. Its solutions determine evolution of the system.

Poisson brackets $\{F, G\} = \Omega^{AB} F_{,A} G_{,B}$. One has $\dot{F} = \{H, F\}$.

If $\{H, F\} = 0$ F is called an integral of motion.

One can always introduce such (canonical) coordinates in which $z^A = (q_1, \ldots, q_m, p^1, \ldots, p^m)$ and $\Omega = \sum_{i=1}^{m} dq_i \wedge dp^i$.

Liouville theorem: Dynamical equations in 2m dimensional phase space are completely integrable if there exist m independent commuting integrals of motion. In such a case a solution can be found by using algebraic relations and integrals.

$F(p,q)$ and $Q(p,q)$ commute if their Poisson brackets vanish, $\{F,Q\}=0$

These integrals of motion F_i can be used as coordinatesi on the phase space. Moreover, there exist canonical coordinates $(I_j(F_i),\psi_i)$, called action-angle variables, in which the equations of motion are trivial: $\dot{I}_i=0,\quad \dot{\psi}_i=\partial H/\partial I_i=const$

Relativistic Particle as a Dynamical System

Preferable coordinates in the phase space:

$$(x^\mu,\ p_\mu = g_{\mu\nu}\dot{x}^\nu),\quad H(p,x) = \tfrac{1}{2} g^{\mu\nu} p_\mu p_\nu .$$

Monomial in momenta integrals of motion

$\mathcal{K} = K^{\mu\ldots\nu} p_\mu \ldots p_\nu$ imply that $K_{\mu\ldots\nu}$ is Killing tensor:

$$K_{(\mu\ldots\nu;\alpha)} = 0$$

$$\{\mathcal{K}_1, \mathcal{K}_2\} = 0 \Leftrightarrow \quad [K_1, K_2] = 0.$$

Motion of particle in D-dimensional ST is completely integrable if there exist D independent commuting Killing tensors (vectors)

Metric is a best known example of rank 2 Killing tensor

If $\mathcal{K}_{(n)}$ and $\mathcal{K}_{(m)}$ are 2 monomial integrals of order n and m, then $\mathcal{K}_{(n)} \bullet \mathcal{K}_{(m)}$ is a monomial integrals of order $n+m$. The corresponding Killing tensor is called reducible.

Denote $D = 2n + \varepsilon$.

D-dimensional Kerr-NUT-AdS metric has $n + \varepsilon$ Killing vectors. For complete integrability of geodesic equations one needs n more integrals of motion.

In 4D: $n = 2, \varepsilon = 0$, 2 Killing vectors ξ_t, ξ_ϕ

GENERAL SCHEME

Killing - Yano (KY) tensor

$$\nabla_{(\mu_1} k_{\mu_2)\mu_3 \ldots \mu_{p+1}} = 0$$

$$K_{\alpha\beta} = k_{\alpha\mu_2\ldots\mu_p} k_{\beta}{}^{\mu_2\ldots\mu_p} \text{ is a Killing tensor}$$

$$(K \;\; = k \bullet k \;\;) \Rightarrow K_{(\alpha\beta;\gamma)} = 0$$

CKY=Conformal Killing-Yano tensor

$$h_{\mu_1\mu_2\ldots\mu_p} = h_{[\mu_1\mu_2\ldots\mu_p]}, \quad \tilde{h}_{\mu_2\ldots\mu_p} = \nabla^{\mu_1} h_{\mu_1\mu_2\ldots\mu_p}$$

$$\nabla_{(\mu_1} h_{\mu_2)\mu_3\ldots\mu_{p+1}} = g_{\mu_1\mu_2} \tilde{h}_{\mu_3\ldots\mu_{p+1}} - (p-1) g_{[\mu_3(\mu_1} \tilde{h}_{\mu_2)\ldots\mu_{p+1}]}$$

$$\tilde{h}_{\mu_2\ldots\mu_{p+1}} = \frac{1}{D-p+1} \nabla^{\mu_1} h_{\mu_1\mu_2\ldots\mu_p}$$

Closed conformal Killing - Yano tensor (CCKY)

This is a CKY tensor h with an additional property:

$dh = 0.$

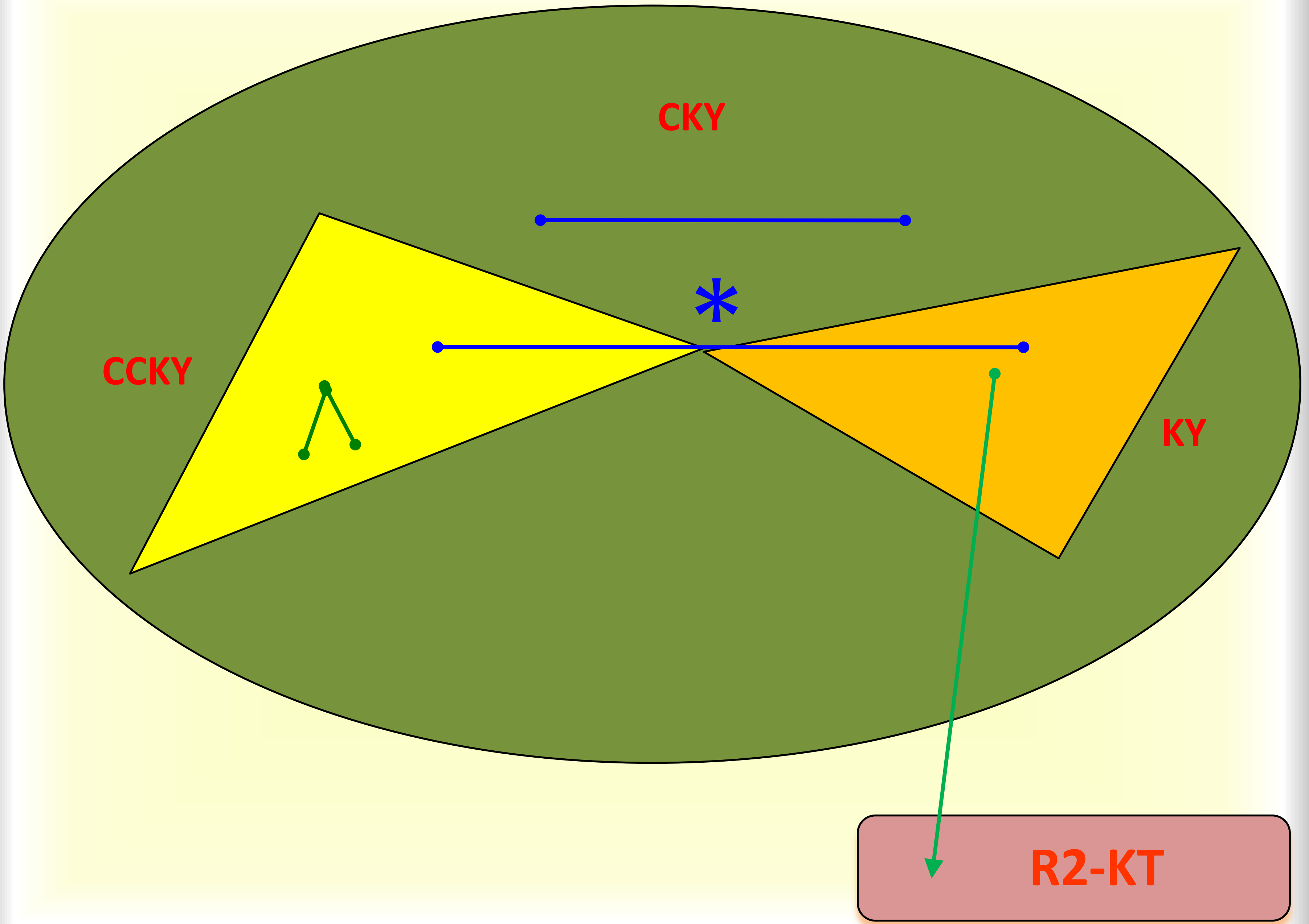
Properties of CKY tensor

Hodge dual of CKY tensor is CKY tensor

Hodge dual of closed CKY tensor is KY tensor:
*CCKY=KY and *KY=CCKY

External product of two closed CKY tensors
is a closed CKY tensor

(Krtous,Kubiznak,Page&V.F. '07; V.F. '07)


CKY
CCKY
KY
R2-KT

Killing-Yano Tower

Killing-Yano Tower

$CCKY: \;\; h \;\Rightarrow\;\; h^{\wedge(j)} = \underbrace{h \wedge h \wedge \ldots \wedge h}_{j\ times}$

KY tensors: $k_{(j)} = {*h^{\wedge(j)}}$

Killing tensors: $K^{(j)} = k_{(j)} \bullet k_{(j)}$

Primary Killing vector: $\xi_a = \frac{1}{D-1} \nabla^b h_{ba}$

Secondary Killing vectors: $\xi_j = K_j \bullet \xi$

Total number of conserved quantities:

$$(n+\varepsilon)+(n-1)+1=2n+\varepsilon=D$$

KV KT g

The integrals of motion are functionally independent and in involution. The geodesic equations in the Kerr-NUT-AdS ST are completely integrable.

Separation of variables in HD Black Holes

Separation of variables in HJ and KG equations in 5D ST (V.F. Stojkovic ’03)

Separability depends on the choice of coordinates

In $D = 2n + \varepsilon$ dimensional ST the Principal CKY tensor (as operator) has n 2D eigenspaces with eigenvalues x^a. They can be used as coordinates.

D canonical coordinates: n essential coordinates x^a and $n + \varepsilon$ Killing coordinates ψ_j.

Complete separability takes place in these canonical coordinates.

Hamilton-Jacobi $(\nabla S)^2 = m^2$

$\Updownarrow$ WKB $\Phi \sim \exp(iS)$

Klein-Gordon $(\Box - m^2)\Phi = 0$

$\Updownarrow$ "Dirac eqn=$\sqrt{\text{KG eqn}}$"

Dirac equation $(\gamma^\mu \nabla_\mu + m)\Psi = 0$

Complete separation of variables in KG and HJ eqns in Kerr-NUT-AdS ST (V.F.,Krtous&Kubiznak '07)

Separation constants KG and HJ eqns and integrals of motion (Sergyeyev&Krtous '08)

Separation of variables in Dirac eqns in Kerr-NUT-AdS metric (Oota&Yasui '08, Cariglia, Krtous&Kubiznak '11)

FURTHER DEVELOPMENTS

Separability of gravitational perturbations in Kerr-NUT-(A)dS Spacetime (Oota andYasui '10)

Charged particle motion in a weakly charged HD black holes (Krtous&V.F. '11)

Metrics admitting a principal Killing-Yano tensor with torsion (Houri, Kubiznak, Warnick and Yasui '12)

SUMMARY OF THIS PART

The most general spacetime admitting PCKY tensor is Kerr-NUT-(A)dS. It has the following properties:

- It is of the algebraic type D

- It allows a separation of variables for the Hamilton-Jacoby, Klein-Gordon, Dirac, tensorial gravitational perturbations.

- The geodesic motion in such a spacetime is completely integrable.

Possible generalizations to degenerate PCKY tensor and non-vacuum STs

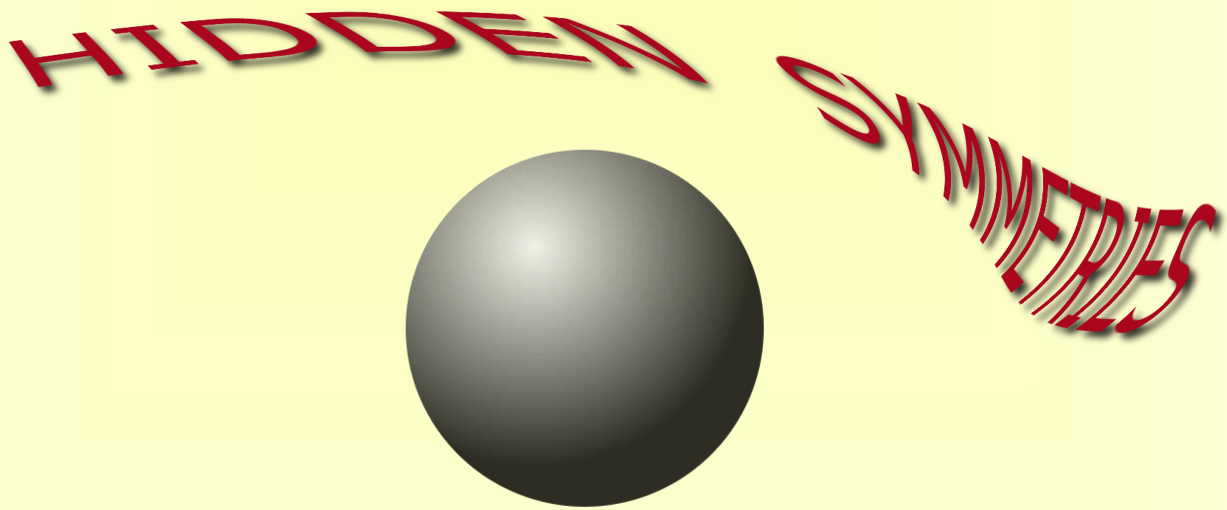
Review Articles

"Higher-Dimensional Black Holes: Hidden Symmetries and Separation of Variables" (V.F. and Kubiznak '08)

"Hidden Symmetries and Integrability in Higher Dimensional Rotating Black Hole Spacetimes", (Cariglia, Krtous and Kubiznak '11)

"Hidden Symmetry and Exact Solutions in Einstein Gravity" (Yasui and Houri '11)

A Problem



BLACK HOLES HIDE THEIR SYMMETRIES. WHY AT ALL HAVE THEY SOMETHING TO HIDE?