BLACK HOLE PHYSICS

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Brief course of lectures at 18th APCTP Winter School on Fundamental Physics

Pohang, January 20 -- January 28, 2014

11. Classical and Quantum Fields near Black Holes

Hawking radiation

In 1974 Hawking theoretically discovered that the vacuum in the presence of a black hole is unstable. The quantum vacuum decay generates particles. Part of the particle created by the black hole reaches an infinity observer and form the Hawking radiation. This radiation has thermal distribution over energies and the corresponding temperature (as measured at infinity) is known as the Hawking temperature. For a black hole of mass *M* it is

$$\Theta_{\mathrm{H}} = \frac{\hbar c^3}{8\pi G k M}$$

Estimation of probability of quantum creation of particles from vacuum by astatic external field: Denote by q `charge' of the particle of mass m and by Γ the field strength

(probability)=$P \sim \exp(-l/\lambda_m)\,\delta(q\Gamma l - 2mc^2)$,

$\lambda_m = \hbar/mc$ is the Compton length and l is a separation of particles when they become `real'.

$$P \sim \exp(-\frac{2m^2c^3}{q\Gamma\hbar})$$

Static electric field: $q = e, \Gamma = E \Rightarrow P \sim \exp(-\frac{\pi m^2 c^3}{eE\hbar})$

(Schwinger, 1951)

In gravity: $q = m, \Gamma = \kappa = \dfrac{c^4}{4GM},$

$$P \sim \exp(-\frac{2mc^3}{\kappa\hbar}) = \exp(-\frac{mc^2}{\Theta_H}),$$

$$\Theta_{\mathrm{H}} = \frac{\hbar c^3}{8\pi GkM}$$

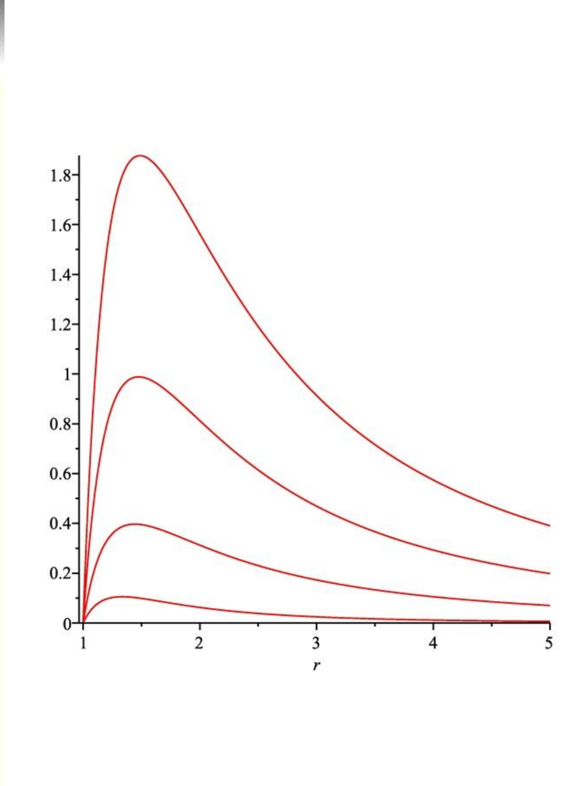
Hawking radiation (for pedestrians)

In a domain where ξ_t is timelike energy E of any particle is positive $\Rightarrow$ energy conservation forbids a process of creation of a pair if both of them are in this domain.

Consider a non-rotating BH. In this case 2 types of processes are allowed: (1) creation of one particle outside BH and the other inside it; (2) creation of two particles inside the black hole. For an external observer the option (2) is not interesting.

For the case (1) there are two options: (i) a particle created outside the horizon penetrates through the potential barrier, and (ii) it is reflected by the potential barrier and fall into BH.

Particles of type (1i) form Hawking radiation.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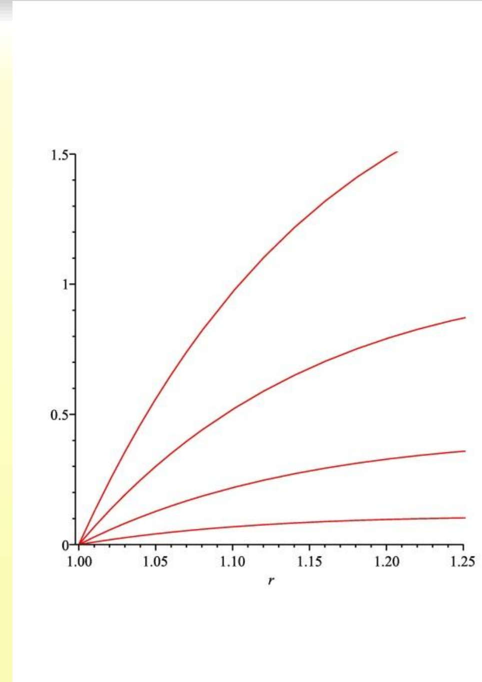
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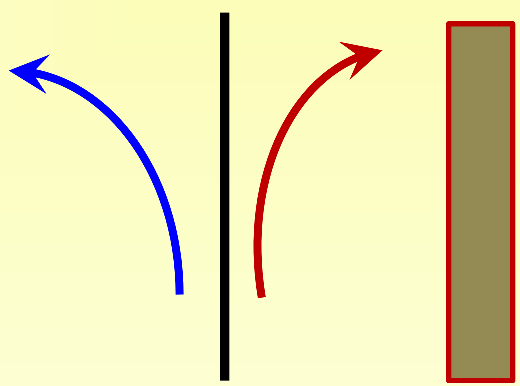


Order of magnitude estimation

$$[\dot{E}] \sim \frac{[m][L]^2}{[T]^3}, \quad [\hbar] \sim \frac{[m][L]^2}{[T]},$$

$\dot{E} \sim \dfrac{\hbar c^2}{r_S^2}$. This relation is valid in any number of dimensions.

As a result of the Hawking radiation the black hole looses its mass. The rate of the mass loss is

$$\dot{M} = -C(m_{Pl}/M)^2$$

Here *C* is dimensionless coefficient which depends of the number and properties of the emitted (massless) fields. Negative energy flux through the horizon (quantum effect)!

The life time of the black hole with respect to the Hawking evaporation is

$$T \sim t_{\text{Pl}}(M/m_{\text{Pl}})^3$$

This effect might be important for the small mass primordial black holes.

$$T_{evap} \sim 10^{10}\,yr\left(\frac{M}{10^{15}\,g}\right)^3$$

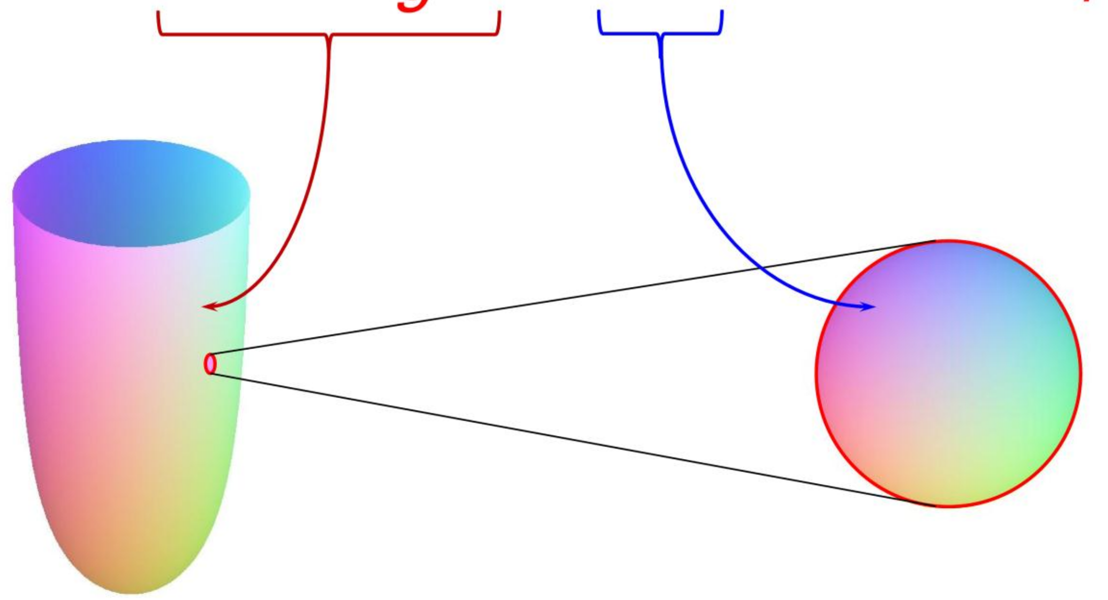
`Ground state'- no-boundary wave function

Barvinsky, V.F., Zelnikov, “Wavefunction of a Black Hole and the Dynamical Origin of Entropy”, Phys.Rev. D51, 1741 (1995)

Gibbons-Hawking instanton

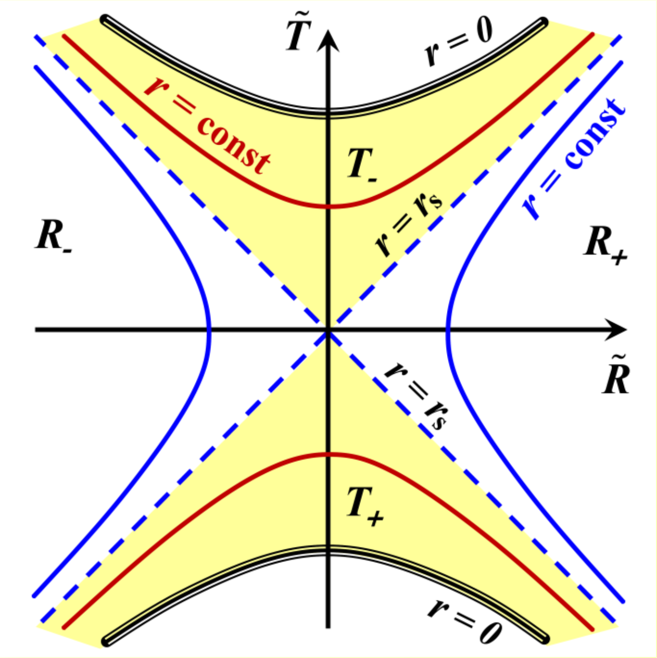
This regular four dimensional Euclidean space is called the Euclidean black hole or the Gibbons-Hawking instanton. The inverse period in Euclidean time is equal

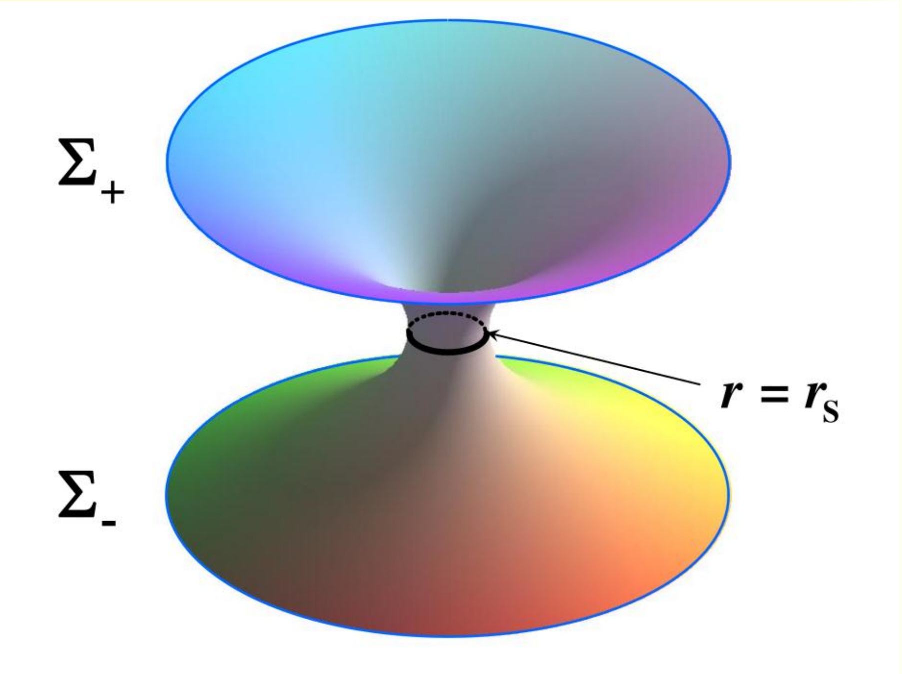
$$ds^2 = g\,d\tau^2 + \frac{dr^2}{g} + r^2\,d\omega^2, \qquad g = 1 - \frac{r_s}{r}$$



$$\Theta_H = \frac{\kappa}{2\pi}$$

is called the Hawking temperature of the black hole.





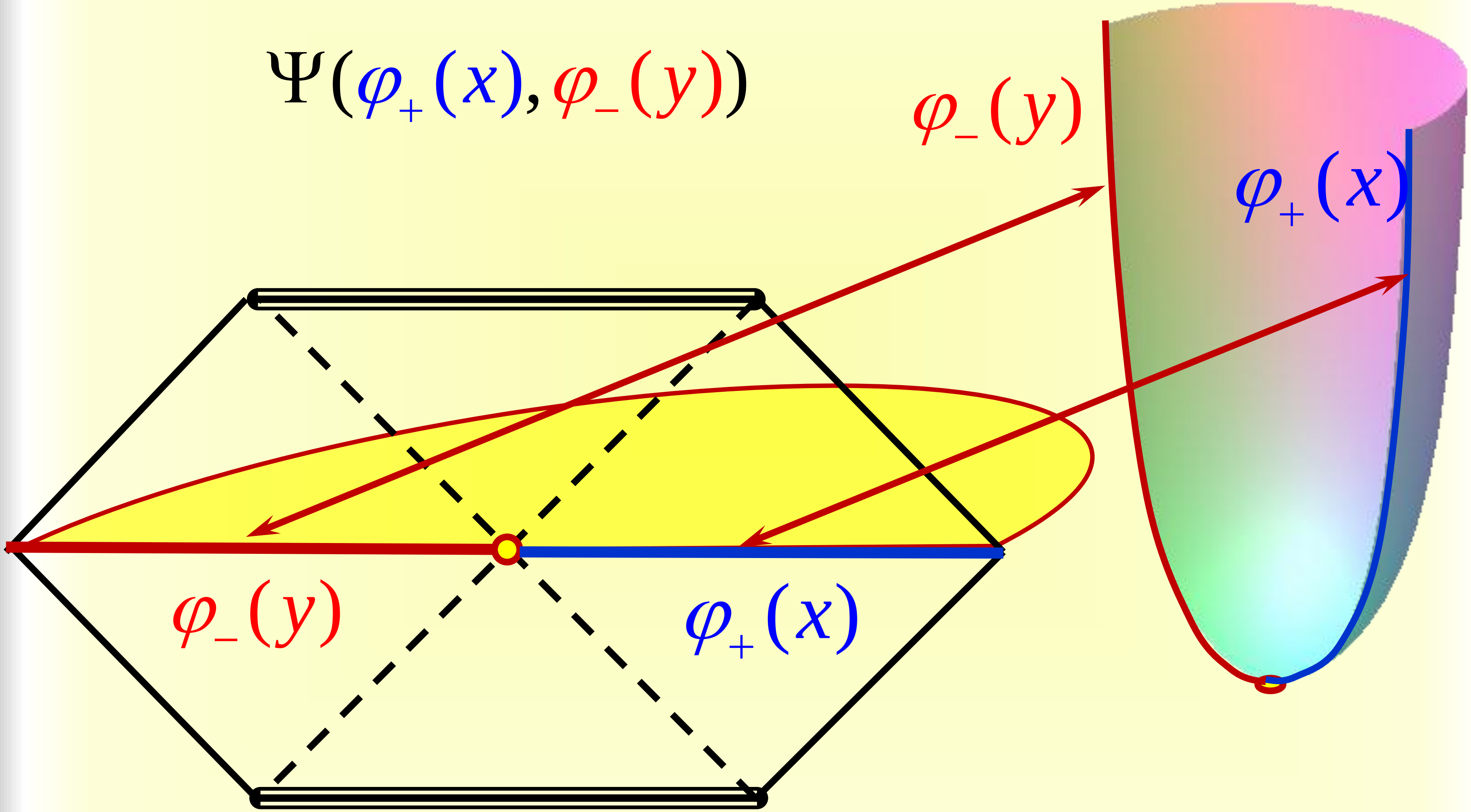
$|\Psi\rangle$ is a wavefunction of (free) quantum field $\hat{\Phi}$.

$$\varphi(\vec{x}) = \Phi(t=0,\vec{x}) \Rightarrow \hat{\Phi}(t=0,\vec{x})\left|\varphi(\vec{x})\right\rangle = \varphi(\vec{x})\left|\varphi(\vec{x})\right\rangle$$

$$\left\langle \varphi(\vec{x}) \middle| \Psi \right\rangle = \Psi(\varphi(\vec{x}))$$

$$\varphi(\vec{x}) = \varphi_+(x) + \varphi_-(y)$$

$$\Psi(\varphi_+(x), \varphi_-(y))$$



No-boundary wave function

$$\Psi(\varphi_+(x), \varphi_-(y)) = Z^{-1} \int_{BC} [D\Phi(\tau, \vec{x})] \exp(-S_E(\Phi))$$

$$BC: \ \Phi|_{\Sigma_-} = \Phi(0, y) = \varphi_-(y), \quad \Phi|_{\Sigma_+} = \Phi(0, x) = \varphi_+(x)$$

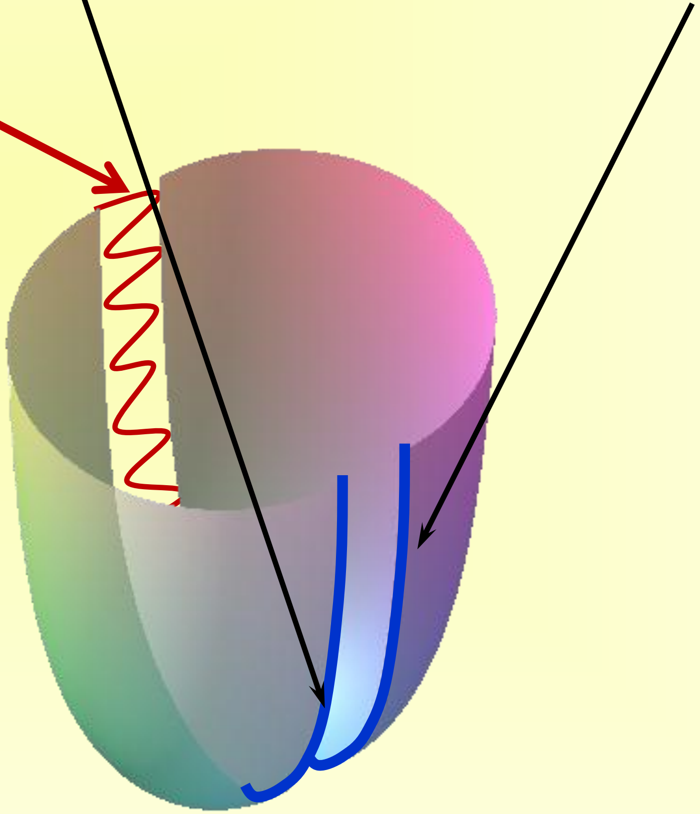
Statement: So defined ground state wavefunction coinsides with Hartle-Hawking vacuum in the eternal black hole.

Density matrix

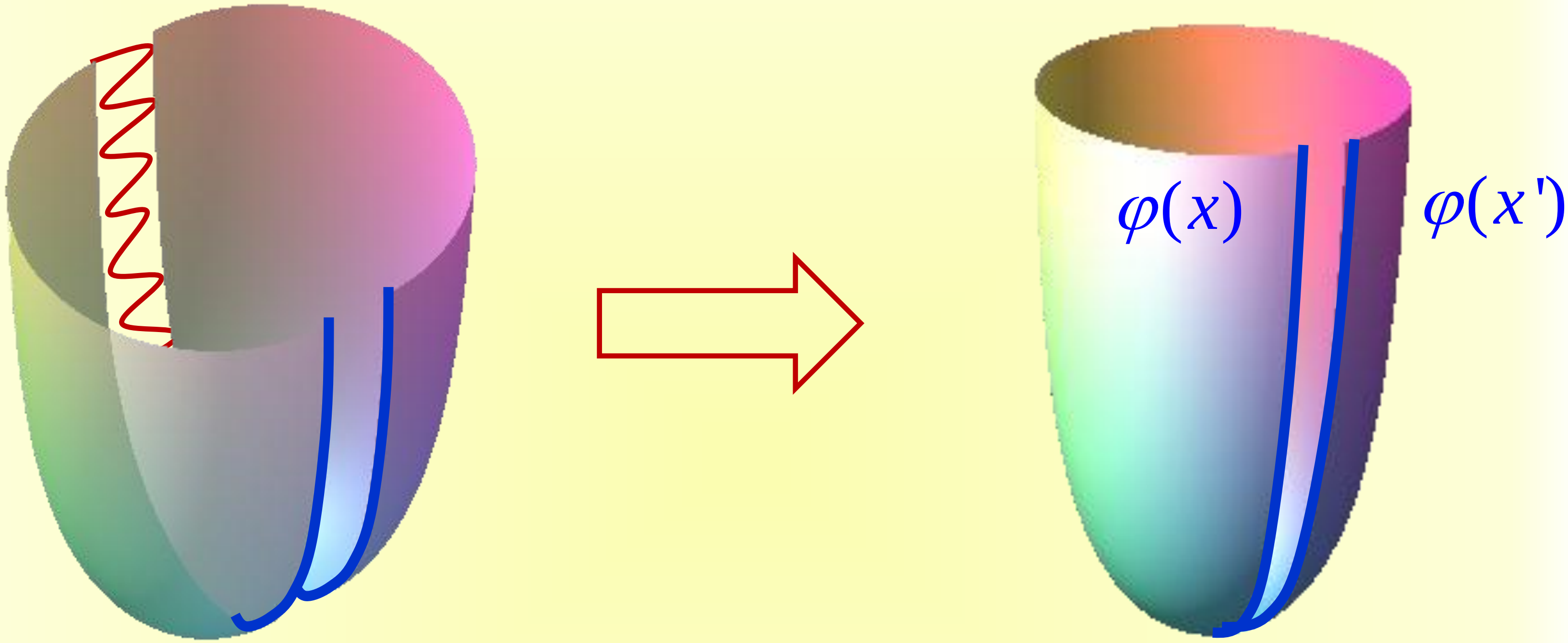
$$\hat{\rho}_+ = Z_+^{-1} Tr_-(|\Psi\rangle\langle\Psi|)$$

Density matrix for quantum fields near BH

$$\rho(\varphi_+(x),\varphi_+(x')) \sim \int [D\varphi_-(y)]\,\Psi(\varphi_+(x),\varphi_-(y))\,\Psi(\varphi_-(y),\varphi_+(x'))$$



Density matrix



$$\rho(\varphi_+(x), \varphi_+(x')) = Z^{-1} \int_{\varphi_+(x)}^{\varphi_+(x')} [D\Phi] \exp(-S_E[\Phi])$$

$$Z = \oint [D\Phi] \exp(-S_E[\Phi])$$

$$\rho_+(\varphi_+(x),\varphi_+(x')) = \langle \varphi_+(x) | \hat{\rho}_+ | \varphi_+(x') \rangle$$

$$\hat{\rho}_+ = Z_+^{-1}\exp(-\beta \hat{H}_+),$$

$\hat{H}_+$ is a Hamiltonian of the field Φ in R_{+},

$\beta = \kappa / 2\pi$ is the inverse Hawking temperature

(period of the Hawking-Gibbons instanton)

Spherical Reduction of Wave equations

For simplicity we consider scalar massless field in the Schwarzschild ST

Action: $S[\varphi] = -\frac{1}{2}\int d^4x\sqrt{|g|}\,g^{\mu\nu}\,\varphi_{,\mu}\varphi_{,\nu}$.

The field equation is $\Box\varphi \equiv \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\,g^{\mu\nu}\varphi_{,\nu}\right)_{,\mu} = -0$.

We write a spherically symmetric metric in the form

$$ds^2 = d\gamma^2 + r^2 d\Omega^2 , \quad d\gamma^2 = \gamma_{AB} dx^A dx^B ,$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2 , \quad \varphi = \psi / r$$

Field equation reads

$$\Box^2 \psi - \frac{\Box^2 r}{r} \psi + \frac{1}{r^2} \Delta \psi = 0$$

In a spherically symmetric spacetime one can decompose the field into spherical modes

$$\varphi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\varphi_{lm}(x)}{r} Y_{lm}(\theta, \phi)$$

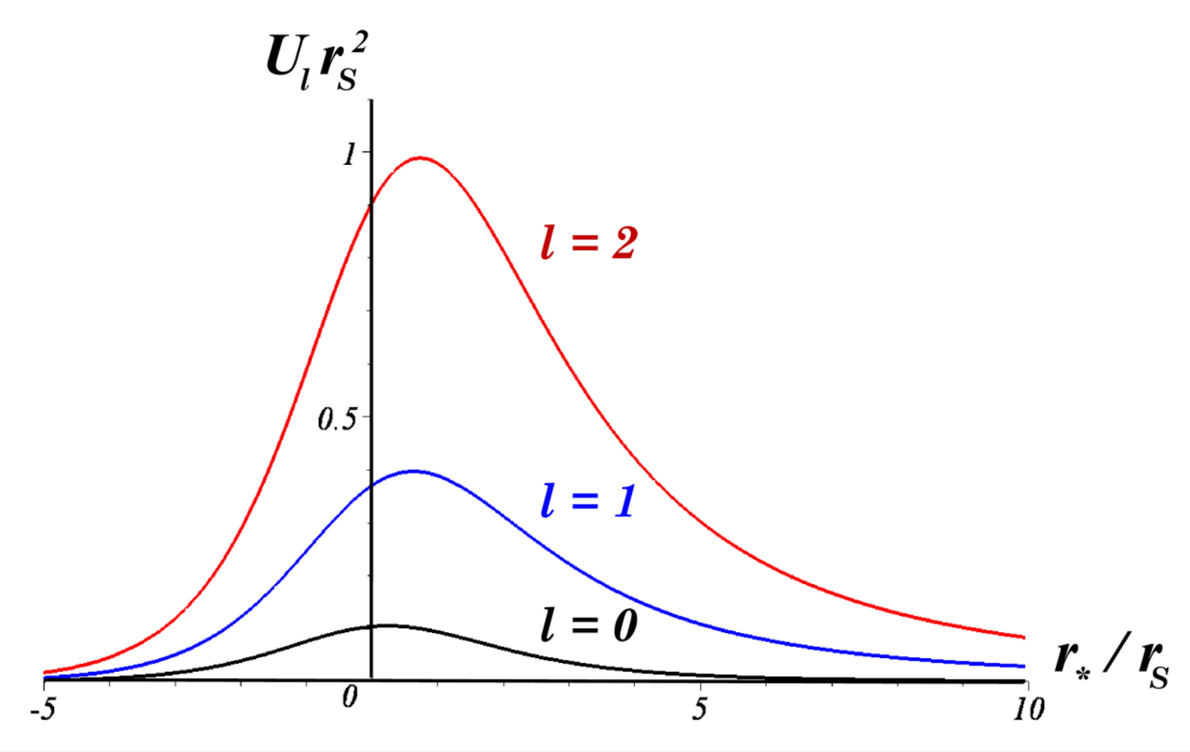
$$\Box^2 \varphi_{lm} - W_l \varphi_{lm} = 0, \quad W = \frac{l(l+1)}{r^2} + \frac{\Box^2 r}{r}$$

$$[-\partial_t^2 + \partial_{r_*}^2 - U_l]\,\varphi_l = 0,$$

$$U_l = \left(1 - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right)$$

$$\varphi_{lm} = \exp(-i\omega t)\, u(r_*),\ dr_* = dr/(1 - r_S/r)$$

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - U_l\right]\varphi_{lm} = 0$$



$$\tilde{u}_{in,\omega}(r_*) = \begin{pmatrix} e^{-i\omega r_*}, & r_* \to -\infty; \\ A_{in,\omega} e^{-i\omega r_*} + A_{out,\omega} e^{i\omega r_*}, & r_* \to +\infty; \end{pmatrix}$$

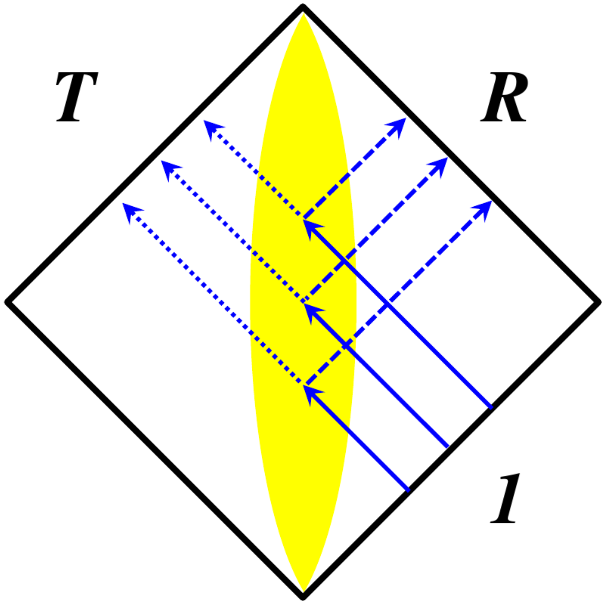
$$\tilde{u}_{up,\omega}(r_*) = \begin{pmatrix} B_{out,\omega} e^{i\omega r_*} + B_{in,\omega} e^{-i\omega r_*}, & r_* \to -\infty; \\ e^{i\omega r_*}, & r_* \to +\infty. \end{pmatrix}$$

$$T_\omega = \frac{1}{A_{in,\omega}}, \quad R_\omega = \frac{A_{out,\omega}}{A_{in,\omega}}, \quad t_\omega = \frac{1}{B_{out,\omega}}, \quad r_\omega = \frac{B_{in,\omega}}{B_{out,\omega}};$$

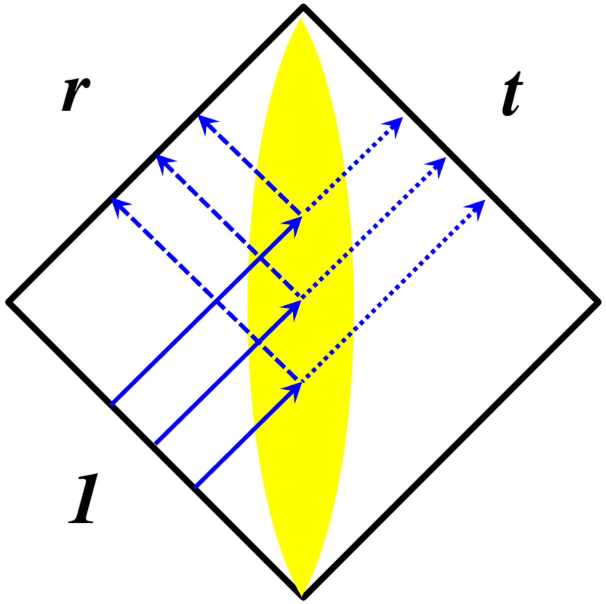
$$|T_\omega|^2 + |R_\omega|^2 = 1, \quad |t_\omega|^2 + |r_\omega|^2 = 1;$$

`Left-movers’

`Right-movers’



in - mode



up - mode

Pole of $A_{in,\omega}$ in a complex plane $\Rightarrow$

resonance (in Quantum Mechanics);

Quasi-normal mode (in Relativity).

$$\varphi = \int_0^\infty d\omega \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ \frac{\exp(-i\omega t)}{r} Y_{lm}(\theta,\phi) \left[a_{in,\omega lm}\, u_{in,\omega lm}(r_*) + a_{up,\omega lm}\, u_{up,\omega lm}(r_*) \right] + H.C. \right\}$$

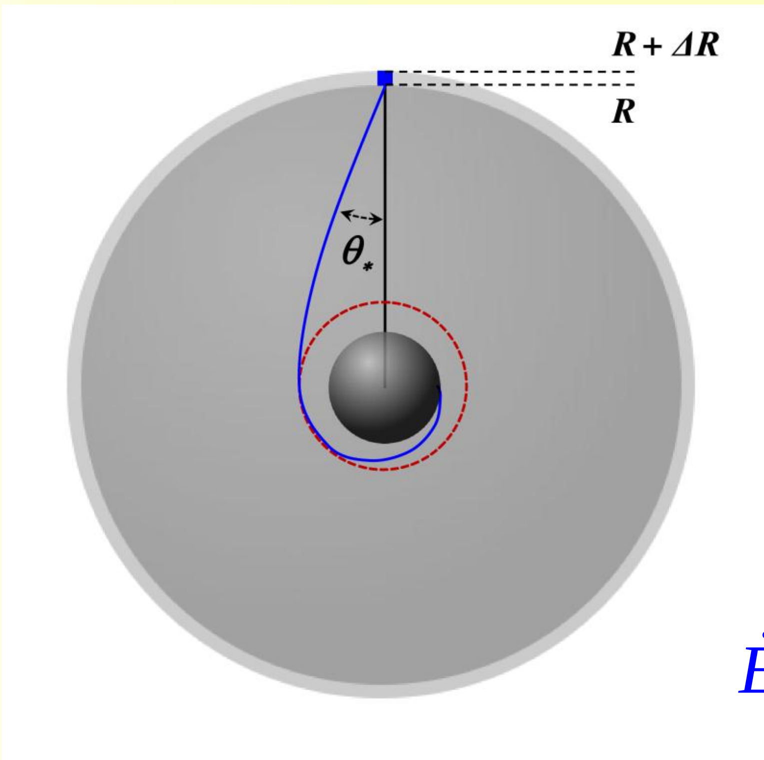
Classical real field: $a_{in,\omega lm}$, $a_{up,\omega lm}$ $-$ complex `c-numbers'

Quantum mechanics, 1 particle state: No H.C.

Quantum field: $a_{in,\omega lm}$, $a_{up,\omega lm}$ - operators of annihilation

Corresponding quanta are called Rindler particles.

Black Hole in a Thermal Bath



$$\dot{E} = \sum_{=0}^{\infty} (2+1)\dot{E} = \frac{1}{2\pi}\int_0^{\infty} \frac{d\omega\,\omega\Gamma_{\omega}}{e^{\beta\omega}-1},$$

$$\Gamma_{\omega} = \sum_{=0}^{\infty} (2+1)\,|\,T_{\omega}\,|^2, \quad \beta = \Theta^{-1}$$

DeWitt approximation:

$$|T_\omega|^2 \approx \theta(3\sqrt{3}M\omega - l),$$

$$l_{min} = 2M\omega\ell_{min}, \quad \ell_{min} = 3\sqrt{3}/2$$

Rate of accretion $\dot{E} \approx \frac{9}{4}\pi^3 r_S^2 \Theta^4$

Black hole thermodynamics

A black hole radiates as a hot black body. Consider a black hole in a thermostat. If its temperature is the same as Hawking temperature of the black hole, the complete system, that is the black hole and surrounding it radiation, is in equilibrium. Such a system can be studied by using standard thermodynamic laws, provided the black hole subsystem has such characteristics as the energy, entropy and so on. The basic ideas of the black hole thermodynamics were formulated by Bekenstein and Hawking in 1975 According to this analogy a non-rotating black hole of mass *M* has the entropy

$$S_{\mathrm{H}} = \frac{A_H}{4\, l_{\mathrm{Pl}}^2}$$

Wheeler seems to have been the first to notice that the very existence of a black hole in the classical theory of gravitation contradicts to the law of non-decreasing entropy. Indeed, imagine that a black hole swallows a hot body possessing a certain amount of entropy. Then the observer outside of it finds that the total entropy in the part of the world accessible to his observation has decreased. This disappearance of entropy could be avoided in a purely formal way if we simply would assign the entropy of the ingested body to the inner region of the black hole.

In fact, this “solution" is patently unsatisfactory because any attempt by an “outside" observer to determine the amount of entropy “absorbed" by the black hole is doomed to failure. Quite soon after the absorption, the black hole becomes stationary and completely “forgets", as a result of “balding", such “details" as the structure of the ingested body and its entropy.

If we are not inclined to forgo the law of non-decreasing entropy because a black hole has formed somewhere in the Universe, we have to conclude that any black hole by itself possesses a certain amount of entropy and that a hot body falling into it not only transfers its mass, angular momentum and electric charge to the black hole, but its entropy S as well. As a result, the entropy of the black hole increases by at least S. Bekenstein (1972, 1973a) noticed that the properties of one of the black hole characteristics -- its area A resemble those of entropy.

κ

$$\Theta = \Theta_H = \frac{\hbar\kappa}{2\pi k_B c}, \quad S^H = \frac{A}{4 l_{\mathrm{Pl}}^2}, \quad E = Mc^2,$$

are temperature, entropy, and internal energy of the black hole.

Zeroth law : The surface gravity κ of a stationary black hole is constant everywhere on the surface of the event horizon.

First law : When the system incorporating a black hole switches from one stationary state to another, its mass changes by

$$dM = \Theta\, dS^H + \Omega\, dJ + \Phi\, dQ + \delta q,$$

where dJ and dQ are changes of angular momentum and charge.

Second law : In any classical process, the area of a black hole A and hence its entropy S_H do not decrease.

Third law : It is impossible to reduce the temperature of any system to the absolute zero in a finite number of operations.

Quantum effects violate the condition for the applicability of Hawking's area theorem. Thus, quantum evaporation reduces the area of black holes, and the **second law** is violated. On the other hand, black hole radiation is thermal in nature, and the black hole evaporation is accompanied by a rise in entropy in the surrounding space.

Generalized entropy: $\tilde{S} = S^H + S^m$

Generalized second law. *In any physical process involving black holes, the generalized entropy*

$$\Delta\tilde{S} = \Delta S^H + \Delta S^m \geq 0.$$