Math 201 (Fall 2009)
Differential Equations

Assignment #2

1. Find the general solution of the given differential equation.
   (a) $y'' + 7y' + 12y = 0$
   (b) $y'' - 4y' + 4y = 0$
   (c) $16y'' - 8y' + 145y = 0$
   (d) $y'' + 5y' = 0$
   (e) $9y'' - 12y' + 4y = 0$
   (f) $y'' - 2y' + 6y = 0$

2. Find the solution of the given initial value value problem and describe its behavior as t increases.
   (a) $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$
   (b) $y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3$
   (c) $y'' - 2y' + 5y = 0, \quad y(\pi/2) = 0, \quad y'(\pi/2) = 2$

3. The Wronskian (or Wronskian determinant) of two functions y_1 and y_2 and three functions y_1, y_2, y_3 are defined respectively by

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad \text{and} \qquad W[y_1, y_2, y_3] = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Determine by Wronskian whether the functions in the following sets are linearly dependent or linearly independent on the indicated intervals. (See Text book: Exercises 4.2 No. 34)
   (a) $y_1(t) = e^{3t}, \quad y_1(t) = e^{-4t}, \quad (0, 1)$
   (b) $y_1(t) = t^{1/2}, \quad y_2(t) = t^{-1}, \quad (0, \infty)$
   (c) $y_1(t) = e^t, \quad y_2 = e^{-t}, \quad y_3(t) = \cosh t, \quad (-\infty, \infty)$
   (d) $y_1(t) = t^2, \quad y_2(t) = x^2 - 1, \quad y_3(t) = x^2 + x + 1, \quad (-\infty, \infty)$

4. Prove the sum of angle formula for the sine function

$$\sin(t + x) = \sin t \cos x + \cos t \sin x$$

by the theory of differential equation.
   (a) Let $f(t) := \sin(t + x)$. Fix x. Show that

$$f''(t) + f(t) = 0, \qquad f(0) = \sin x, \qquad f'(0) = \cos x$$

   (b) Use the auxiliary equation technique to solve the initial value problem

$$y'' + y = 0, \qquad y(0) = \sin x, \qquad y'(0) = \cos x$$

(c) By uniqueness, the solution in part (b) is the same as $f(t)$ from part (a).

Remark. You can apply the same technique to prove the other trigonometric formulas. For example,

$$\sin(t-x) = \sin t \cos x - \cos t \sin x$$

$$\cos(t+x) = \cos t \cos x - \sin t \sin x$$

$$\cos(t-x) = \cos t \cos x + \sin t \sin x$$

all satisfy the same differential equation but different initial conditions.

5. Determine the value of α, if any, for which all solutions tend to zero as $t \to \infty$; also determine the value of α, if any, for which all (nonzero) solutions become unbounded as $t \to \infty$.

(a) $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$

(b) $y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$

6. Use the method of undetermined coefficients to find a particular solution to the given differential equation.

(a) $y'' + 4y' + 3y = 5e^{2t}$

(b) $y'' + 4y' + 3y = 5 \sin 2t$

(c) $y''' - 3y'' + 3y' - y = 4e^{t}$

7. Given that $y_1(t) = \frac{1}{4} \sin 2t$ is a solution to $y'' + 2y' + 4y = \cos 2t$ and that $y_2(t) = \frac{t}{4} - \frac{1}{8}$ is a solution to $y'' + 2y' + 4y = t$. Use the superposition principle to find the particular solution to the following:

(a) $y'' + 2y' + 4y = t + \cos 2t$

(b) $y'' + 2y' + 4y = 2t - 3\cos 2t$

(a) $y'' + 2y' + 4y = 11t - 12\cos 2t$

8. Find the general solution of the equation

$$y'' + 5y' + 6y = 3e^{-2t} + e^{3t}$$

9. The general solution of the 2nd-order differential equation

$$y'' + y = 0$$

is given by

$$y(t) = c_1 \cos t + c_2 \sin t$$

where c_1 and c_2 are arbitrary constants, show that

(a) There is a unique solution satisfies the boundary conditions

$$y(0) = 2 \qquad \text{and} \qquad y(\pi/2) = 0.$$

(b) There is no solution $y(t)$ satisfies

$$y(0) = 2 \qquad \text{and} \qquad y(\pi) = 0.$$

(c) There is infinitely many solutions $y(t)$ satisfies

$$y(0) = 2 \qquad \text{and} \qquad y(\pi) = -2.$$

10. Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin m\pi t$$

where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, 2, \cdots, N$.