

**Math 201** (Fall 2009)  
Differential Equations

**Solution #2**

1. Find the general solution of the given differential equation.

- (a)  $y'' + 7y' + 12y = 0$
- (b)  $y'' - 4y' + 4y = 0$
- (c)  $16y'' - 8y' + 145y = 0$
- (d)  $y'' + 5y' = 0$
- (e)  $9y'' - 12y' + 4y = 0$
- (f)  $y'' - 2y' + 6y = 0$

*Solution:*

- (a) The auxiliary (characteristic) equation in this case is

$$r^2 + 7r + 12 = 0$$

and its roots are

$$r_1 = -3, \quad r_2 = -4$$

Since these values are different, a general solution is

$$y = c_1 e^{-3t} + c_2 e^{-4t}$$

- (b) The auxiliary (characteristic) equation of the differential equation is

$$r^2 + 4r + 4 = 0$$

Its roots are  $r_1 = r_2 = 2$ ; hence a general solution is

$$y = c_1 e^{2t} + c_2 t e^{2t}$$

- (c) The auxiliary (characteristic) equation is  $16r^2 - 8r + 145 = 0$  and its roots are  $r = \frac{1}{4} \pm 3i$ . Thus the general solution of the differential equation is

$$y = c_1 e^{t/4} \cos 3t + c_2 e^{t/4} \sin 3t$$

- (d)  $y = c_1 + c_2 e^{-5t}$
- (e)  $y = c_1 e^{2t/3} + c_2 t e^{2t/3}$
- (f)  $y = c_1 e^t \cos \sqrt{5}t + c_2 e^t \sin \sqrt{5}t$

2. Find the solution of the given initial value value problem and describe its behavior as  $t$  increases.

- (a)  $y'' + 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = -1$

- (b)  $y'' + 3y' = 0$ ,  $y(0) = -2$ ,  $y'(0) = 3$   
(c)  $y'' - 2y' + 5y = 0$ ,  $y(\pi/2) = 0$ ,  $y'(\pi/2) = 2$

*Solution:*

- (a) The general solution is

$$y(t) = c_1 e^{-t} + c_2 e^{-3t}$$

Substituting the given initial data into the equations for  $y$  and  $y'$ , respectively, we have

$$c_1 + c_2 = 2 \quad \text{and} \quad -c_1 - 3c_2 = -1$$

Hence  $c_1 = \frac{5}{2}$ ,  $c_2 = -\frac{1}{2}$ , and required solution is

$$y(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}$$

Since  $e^{-t}, e^{-3t} \rightarrow 0$  as  $t \rightarrow \infty$ , so  $y \rightarrow 0$  as  $t \rightarrow \infty$ .

- (b) The general solution is

$$y(t) = c_1 + c_2 e^{-3t}$$

Applying the initial conditions, we obtain the following two equations for  $c_1$  and  $c_2$ :

$$c_1 + c_2 = -2, \quad -3c_2 = 3$$

The solution of these equations is  $c_1 = -1$ ,  $c_2 = -1$ , so the solution of the initial value problem is

$$y(t) = -1 - e^{-3t}$$

It is obvious that  $y \rightarrow -1$  as  $t \rightarrow \infty$ .

- (c) The general solution is

$$y(t) = c_1 e^t \cos 2t + c_2 e^t \sin 2t$$

Substituting the first initial data into the equation for  $y$  we have

$$y(\pi/2) = c_1 e^{\pi/2} \cos \pi + c_2 e^{\pi/2} \sin \pi = -c_1 e^{\pi/2} = 0, \quad c_1 = 0$$

Hence  $y(t) = c_2 e^t \sin 2t$ . Applying the second initial condition, we have

$$y'(\pi/2) = c_2 e^{\pi/2} \sin \pi + 2c_2 e^{\pi/2} \cos \pi = -2c_2 e^{\pi/2} = 2, \quad c_2 = -e^{-\pi/2}$$

and required solution is

$$y(t) = -e^{t-\pi/2} \sin 2t$$

We observe that the solution  $y$  is a *growing oscillation* and  $y \rightarrow -\infty$  as  $t \rightarrow \infty$ .

3. The Wronskian (or Wronskian determinant) of two functions  $y_1$  and  $y_2$  and three functions  $y_1, y_2, y_3$  are defined respectively by

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad \text{and} \quad W[y_1, y_2, y_3] = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Determine by Wronskian whether the functions in the following sets are linearly dependent or linearly independent on the indicated intervals. (See Text book: Exercises 4.2 No. 34)

- (a)  $y_1(t) = e^{3t}, \quad y_2(t) = e^{-4t}, \quad (0, 1)$
- (b)  $y_1(t) = t^{1/2}, \quad y_2(t) = t^{-1}, \quad (0, \infty)$
- (c)  $y_1(t) = e^t, \quad y_2 = e^{-t}, \quad y_3(t) = \cosh t, \quad (-\infty, \infty)$
- (d)  $y_1(t) = t^2, \quad y_2(t) = x^2 - 1, \quad y_3(t) = x^2 + x + 1, \quad (-\infty, \infty)$

*Solution:*

- (a) We calculate the Wronskian  $W$  of  $y_1 = e^{3t}$  and  $y_2 = e^{-4t}$ :

$$W = \begin{vmatrix} e^{3t} & e^{-4t} \\ 3e^{3t} & -4e^{-4t} \end{vmatrix} = -7e^{-t}.$$

Since  $W \neq 0$  for  $t \in (0, 1)$ , we conclude that  $y_1 = e^{3t}$  and  $y_2 = e^{-4t}$  are linearly independent.

- (b) We calculate the Wronskian  $W$  of  $y_1 = t^{1/2}$  and  $y_2 = t^{-1}$ :

$$W = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -\frac{3}{2}t^{-3/2}.$$

Since  $W \neq 0$  for  $t > 0$ , we conclude that  $y_1 = t^{1/2}, y_2 = t^{-1}$  are linearly independent. Let us remark that  $y_1$  and  $y_2$  are two independent solutions of the D.E.

$$2t^2y'' + 3ty' - y = 0, \quad t > 0.$$

- (c) The Wronskian of  $y_1(t) = e^t, y_2 = e^{-t}, y_3(t) = \cosh t$  is

$$W = \begin{vmatrix} e^t & e^{-t} & \cosh t \\ e^t & -e^{-t} & \sinh t \\ e^t & e^{-t} & \cosh t \end{vmatrix} = 0 \quad (\text{1st row} = \text{3rd row})$$

These functions are linearly dependent. Indeed, by the definition of hyperbolic cosine function

$$y_3(t) = \cosh t = \frac{1}{2}(e^t + e^{-t}) = \frac{1}{2}y_1(t) + \frac{1}{2}y_2(t)$$

and so the given functions are linearly dependent on  $(-\infty, \infty)$ .

(c) These functions are linearly independent since the Wronskian is not zero for all  $t \in (-\infty, \infty)$ :

$$W = \begin{vmatrix} t^2 & t^2 - 1 & t^2 + t + 1 \\ 2t & 2t & 2t + 1 \\ 2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} t^2 & -1 & t + 1 \\ 2t & 0 & 1 \\ 2 & 0 & 0 \end{vmatrix} = -2 \neq 0$$

4. Prove the sum of angle formula for the sine function

$$\sin(t + x) = \sin t \cos x + \cos t \sin x$$

by the theory of differential equation.

(a) Let  $f(t) := \sin(t + x)$ . Fix  $x$ . Show that

$$f''(t) + f(t) = 0, \quad f(0) = \sin x, \quad f'(0) = \cos x$$

(b) Use the auxiliary equation technique to solve the initial value problem

$$y'' + y = 0, \quad y(0) = \sin x, \quad y'(0) = \cos x$$

(c) By uniqueness, the solution in part (b) is the same as  $f(t)$  from part (a).

*Solution:*

(a) It is straightforward by computation.

(b) The auxiliary equation is  $r^2 + 1 = 0$  and the roots are  $i$  and  $-i$ . Thus the general solution is  $y(t) = c_1 \cos t + c_2 \sin t$ . Substituting the given initial data into the equations for  $y$  and  $y'$  we have

$$y(0) = c_1 = \sin x, \quad y'(0) = c_2 = \cos x$$

and the required solution is

$$y(t) = \sin x \cos t + \cos x \sin t$$

(c) By uniqueness we conclude that  $f(t) = y(t)$  i.e.,

$$\sin(t + x) = \sin(x + t) = \sin x \cos t + \cos x \sin t$$

*Remark.* You can apply the same technique to prove the other trigonometric formulas. For example,

$$\sin(t - x) = \sin t \cos x - \cos t \sin x$$

$$\cos(t + x) = \cos t \cos x - \sin t \sin x$$

$$\cos(t - x) = \cos t \cos x + \sin t \sin x$$

all satisfy the same differential equation but different initial conditions.

5. Determine the value of  $\alpha$ , if any, for which all solutions tend to zero as  $t \rightarrow \infty$ ; also determine the value of  $\alpha$ , if any, for which all (nonzero) solutions become unbounded as  $t \rightarrow \infty$ .

(a)  $y'' - (2\alpha - 1)y' + \alpha(\alpha - 1)y = 0$

(b)  $y'' + (3 - \alpha)y' - 2(\alpha - 1)y = 0$

*Solution:*

(a) The auxiliary equation is

$$r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = (r - \alpha)(r - 1 + \alpha) = 0$$

and its roots are  $r_1 = \alpha$  and  $r_2 = \alpha - 1$ . Hence the general solution is

$$y(t) = c_1 e^{\alpha t} + c_2 e^{(\alpha-1)t}$$

It is obvious that  $y \rightarrow 0$  for  $\alpha < 0$ ;  $y$  becomes unbounded for  $\alpha > 1$ .

(b) The auxiliary equation is

$$r^2 + (3 - \alpha)r - 2(\alpha - 1) = (r + 2)(r - 1 + \alpha) = 0$$

and its roots are  $r_1 = -2$  and  $r_2 = \alpha - 1$ . Hence the general solution is

$$y(t) = c_1 e^{-2t} + c_2 e^{(\alpha-1)t}$$

It is obvious that  $y \rightarrow 0$  for  $\alpha < 1$ ; there is no  $\alpha$  for which all nonzero solutions become unbounded.

6. Use the method of undetermined coefficients to find a particular solution to the given differential equation.

(a)  $y'' + 4y' + 3y = 5e^{2t}$

(b)  $y'' + 4y' + 3y = 5 \sin 2t$

(c)  $y''' - 3y'' + 3y' - y = 4e^t$

*Solution:*

(a) Since differentiating an exponential of the form  $e^{kt}$  merely reproduces that function with, at most, a change in its numerical coefficient, it is natural to "guess" that it may be possible to determine  $A$  so that

$$y_p(t) = Ae^{2t}$$

will be a solution. To check this, we substitute  $y_p = Ae^{2t}$  for  $y$  in the given equation, getting

$$4Ae^{2t} + 8Ae^{2t} + 3Ae^{2t} = 5e^{2t} \quad \text{or} \quad 15Ae^{2t} = 5e^{2t}$$

which will be an identity if and only if  $A = \frac{1}{3}$ . Thus, the required particular solution is

$$y_p(t) = \frac{1}{3}e^{2t}$$

Instead of guessing, we can apply the operator  $D - 2$  to the original equation and reduce it to homogeneous D.E.

$$y'' + 4y' + 3y = (D^2 + 4D + 3)y = 5e^{2t} \implies (D - 2)(D + 1)(D + 3)y = 0$$

The general solution of the new equation is

$$y(t) = (c_1e^{-t} + c_2e^{-3t}) + c_3e^{2t} = y_h(t) + y_p(t)$$

Note that  $y_h(t) = c_1e^{-t} + c_2e^{-3t}$  is the homogeneous solution and the particular solution must be  $y_p(t) = c_3e^{2t}$ .

(b) The particular solution is assumed to be

$$y_p(t) = A \sin 2t + B \cos 2t$$

Substituting into the original equation yields (after some computation)

$$(-A - 8B) \sin 2t + (8A - B) \cos 2t = 5 \sin 2t$$

and for this to be an identity requires that

$$-A - 8B = 5 \quad \text{and} \quad 8A - B = 0$$

from which we find immediately that  $A = -\frac{1}{13}$  and  $B = -\frac{8}{13}$ . Hence, finally,

$$y_p(t) = -\frac{\sin 2t + 8 \cos 2t}{13}$$

Similarly, we apply the operator  $D^2 + 2^2$  to the original equation

$$y'' + 4y' + 3y = (D^2 + 4D + 3)y = 5 \sin 2t \implies (D^2 + 2^2)(D + 1)(D + 3)y = 0$$

The general solution of the new equation is

$$y(t) = (c_1e^{-t} + c_2e^{-3t}) + (c_3 \cos 2t + c_4 \sin 2t) = y_h(t) + y_p(t)$$

Thus the particular solution is  $y_p(t) = c_3 \cos 2t + c_4 \sin 2t$ .

(c) The auxiliary (characteristic) equation for the homogeneous equation is

$$r^3 - 3r^2 + 3r - 1 = (r - 1)^3$$

so the general solution of the homogeneous equation is

$$y_h(t) = c_1e^t + c_2te^t + c_3t^2e^t$$

To guess the particular solution we observe from the differential equation

$$\begin{aligned} -y &\approx 4e^t &\implies & y_p \approx e^t \\ 3y' &\approx 4e^t &\implies & y_p \approx te^t \\ -3y'' &\approx 4e^t &\implies & y_p \approx t^2e^t \\ y''' &\approx 4e^t &\implies & y_p \approx t^3e^t \end{aligned}$$

However, since  $e^t$ ,  $te^t$ , and  $t^2e^t$  are all solutions of the homogeneous equation. Thus our final assumption is that  $y_p(t) = At^3e^t$ . Indeed,  $A = \frac{2}{3}$  and the particular solution is

$$y_p(t) = \frac{2}{3}t^3e^t$$

We can transform the original equation into a homogeneous differential equation by applying the operator  $D - 1$  to the original differential equation

$$(D - 1)^4y = 0 \implies y = (c_1e^t + c_2te^t + c_3t^2e^t) + c_4t^3e^t$$

from which it is obvious that the particular solution must be  $y_p(t) = c_4t^3e^t$ .

7. Given that  $y_1(t) = \frac{1}{4} \sin 2t$  is a solution to  $y'' + 2y' + 4y = \cos 2t$  and that  $y_2(t) = \frac{t}{4} - \frac{1}{8}$  is a solution to  $y'' + 2y' + 4y = t$ . Use the superposition principle to find the particular solution to the following:

(a)  $y'' + 2y' + 4y = t + \cos 2t$

(b)  $y'' + 2y' + 4y = 2t - 3 \cos 2t$

(a)  $y'' + 2y' + 4y = 11t - 12 \cos 2t$

*Solution:*

(a)  $y_p = y_2 + y_1 = \left(\frac{t}{4} - \frac{1}{8}\right) + \left(\frac{1}{4} \sin 2t\right)$

(b)  $y_p = 2y_2 - 3y_1 = 2\left(\frac{t}{4} - \frac{1}{8}\right) - 3\left(\frac{1}{4} \sin 2t\right)$

(c)  $y_p = 11y_2 - 12y_1 = 11\left(\frac{t}{4} - \frac{1}{8}\right) - 12\left(\frac{1}{4} \sin 2t\right)$

8. Find the general solution of the equation

$$y'' + 5y' + 6y = 3e^{-2t} + e^{3t}$$

*Solution:* The roots of the auxiliary(characteristic) equation

$$r^2 + 5r + 6 = 0$$

are  $r_1 = -2$  and  $r_2 = -3$ . Hence the homogeneous solution is

$$y_h(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

According to superposition principle, we assume the particular solution  $y_p = y_1 + y_2$  where  $y_1$  and  $y_2$  satisfy

$$y_1'' + 5y_1' + 6y_1 = 3e^{-2t} \quad \text{and} \quad y_2'' + 5y_2' + 6y_2 = e^{3t}$$

Since  $-2$  is a root and we have  $y_1 = Ate^{-2t}$ . For the term  $e^{3t}$  the normal choice is  $y_2 = Be^{3t}$ . Hence we have the particular solution

$$y_p(t) = Ate^{-2t} + Be^{3t}$$

After some computation, we find  $A = 3$  and  $B = \frac{1}{30}$ . Hence

$$y_p(t) = 3te^{-2t} + \frac{1}{30}e^{3t}$$

and the general solution is

$$y(t) = y_h(t) + y_p(t) = c_1 e^{-2t} + c_2 e^{-3t} + 3te^{-2t} + \frac{1}{30}e^{3t}$$

9. The general solution of the 2nd-order differential equation

$$y'' + y = 0$$

is given by

$$y(t) = c_1 \cos t + c_2 \sin t$$

where  $c_1$  and  $c_2$  are arbitrary constants, show that

(a) There is a unique solution satisfies the boundary conditions

$$y(0) = 2 \quad \text{and} \quad y(\pi/2) = 0.$$

(b) There is no solution  $y(t)$  satisfies

$$y(0) = 2 \quad \text{and} \quad y(\pi) = 0.$$

(c) There is infinitely many solutions  $y(t)$  satisfies

$$y(0) = 2 \quad \text{and} \quad y(\pi) = -2.$$

*Solution:*

(a) Substituting the boundary condition into  $y$  yields

$$y(0) = c_1 = 2, \quad y(\pi/2) = c_1 \cos(\pi/2) + c_2 \sin(\pi/2) = c_2 = 0$$

Thus  $y(t) = 2 \cos t$  is the required *unique* solution.

(b) Same as (a)

$$y(0) = c_1 = 2, \quad y(\pi) = c_1 \cos \pi + c_2 \sin \pi = -c_1 = 0$$

This is contradiction. Thus the solution does not exist!

(c) Same as (a)

$$y(0) = c_1 = 2, \quad y(\pi) = c_1 \cos \pi + c_2 \sin \pi = -c_1 = -2$$

There is no restriction of  $c_2$ . Thus the solution is

$$y(t) = 2 \cos t + c_2 \sin t$$

Since  $c_2$  is arbitrary constant. There are infinitely many solutions!

10. Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^N a_m \sin m\pi t$$

where  $\lambda > 0$  and  $\lambda \neq m\pi$  for  $m = 1, 2, \dots, N$ .

*Solution:* By superposition principle, we only need to discuss one term

$$y_m'' + \lambda^2 y_m = \sin m\pi t$$

Since  $\lambda \neq m\pi$ , we can assume  $y_m = A \cos m\pi t + B \sin m\pi t$  then substituting into the equation we have

$$y_m'' + \lambda^2 y_m = A(\lambda^2 - m^2\pi^2) \cos m\pi t + B(\lambda^2 - m^2\pi^2) \sin m\pi t = \sin m\pi t$$

Hence  $A = 0$ ,  $B = \frac{1}{\lambda^2 - m^2\pi^2}$  and  $y_m(t) = \frac{1}{\lambda^2 - m^2\pi^2} \sin m\pi t$ ; therefore the particular solution is

$$y_p = \sum_{m=1}^N a_m y_m = \sum_{m=1}^N \frac{a_m}{\lambda^2 - m^2\pi^2} \sin m\pi t$$

and the general solution is

$$y(t) = y_h(t) + y_p(t) = c_1 \cos \lambda t + c_2 \sin \lambda t + \sum_{m=1}^N \frac{a_m}{\lambda^2 - m^2\pi^2} \sin m\pi t$$