Math 201 (Fall 2009)
Differential Equations

Solution #3

1. Find the particular solution of the following differential equation by variation of parameter

(a) $y'' + y = \csc t$

(b) $t^2y'' + ty' - y = t\ln t, \qquad t > 0$

Solution:

(a) The corresponding homogeneous equation $y''+y = 0$ has $y_h(t) = c_1 \sin t + c_2 \cos t$ as its general solution, so $y_1 = \sin t$, $y_1' = \cos t$, $y_2 = \cos t$, and $y_2' = -\sin t$. The Wronskian of y_1 and y_2 is

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' = -\sin^2 t - \cos^2 t = -1$$

so by variation of parameter we have

$$v_1 = \int \frac{-\cos t \csc t}{-1} dt = \int \frac{\cos t}{\sin t} dt = \int \frac{d \sin t}{\sin t} = \log(\sin t)$$

$$v_2 = \int \frac{\sin t \csc t}{-1} dt = -t$$

According,

$$y_p = \sin t \log(\sin t) - t\cos t$$

is the desired particular solution.

(b) The corresponding homogeneous equation $t^2y'' + ty' - y = 0$ is the Cauchy-Euler equation and $\{y_1(t) = t, y_2(t) = 1/t\}$ is a pair of independent solution. The Wronskian is

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = t(-1/t^2) - (1/t) = -2/t, \qquad t > 0$$

To apply the method of variation parameter, we have to write the differential equation in the standard form

$$y'' + \frac{1}{t}y' - \frac{1}{t^2}y = \frac{\ln t}{t}, \qquad t > 0\,.$$

then we have

$$v_1(t) = -\int \frac{y_2(t)\frac{\ln t}{t}}{W[y_1, y_2]} dt = \frac{1}{2}\int \frac{\ln t}{t} dt = \frac{1}{2}\int \ln t \, d\ln t = \frac{1}{4}(\ln t)^2$$

$$v_2(t) = \int \frac{y_1(t)\frac{\ln t}{t}}{W[y_1, y_2]} dt = -\frac{1}{2}\int t\ln t dt = -\frac{1}{4}\int \ln t dt^2 = -\frac{1}{4}t^2 \ln t - \frac{1}{8}t^2$$

and

$$v_1(t)y_1(t) + v_2(t)y_2(t) = \frac{1}{4}t(\ln t)^2 - \frac{1}{4}t\ln t - \frac{1}{8}t\,.$$

Since $-\frac{1}{8}t$ is a part of the homogeneous solution, so the required particular solution is

$$y_p(t) = \frac{1}{4}t(\ln t)^2 - \frac{1}{4}t\ln t$$

2. Let g be a continue function and y be defined by

$$y(t) = \frac{1}{2}\int_0^t \sin 2(t-\tau)\, g(\tau)d\tau \qquad (*)$$

(a) Apply the Leibniz formula (the generalization of the fundamental theorem of calculus) to show that the function y defined by $(*)$ satisfies the initial value problem

$$y'' + 4y = g(t), \qquad y(0) = 0, \qquad y'(0) = 0$$

(b) Use the variation of parameters to solve the initial value problem and show that the solution is given by $(*)$.

Solution :

(a) By Leibniz formula we have

$$y'(t) = \frac{\sin 2(t-t)}{2}g(t) + \int_0^t \frac{d}{dt}\frac{\sin 2(t-\tau)}{2}\,g(\tau)d\tau = \int_0^t \cos 2(t-\tau)g(\tau)d\tau$$

$$\begin{aligned} y''(t) &= \cos 2(t-t)g(t) + \int_0^t \frac{d}{dt}\cos 2(t-\tau)g(\tau)d\tau \\ &= g(t) - 2\int_0^t \sin 2(t-\tau)g(\tau)d\tau = g(t) - 4y(t) \end{aligned}$$

Therefore y satisfies the differential equation $y''+4y = g$. It is obvious from the expression of $y(t)$ and $y'(t)$ that $y(0) = y'(0) = 0$.

(b) Form the homogeneous solution

$$y_h(t) = c_1\sin 2t + c_2\cos 2t$$

we have $y_1(t) = \sin 2t, y_2(t) = \cos 2t$, $y_1'(t) = 2\cos 2t, y_2'(t) = -2\sin 2t$ and the Wronskian $W = y_1y_2' - y_2y_1' = -2$, then

$$v_1(t) = -\int_0^t \frac{y_2(\tau)g(\tau)}{W[y_1,y_2]}d\tau = \frac{1}{2}\int_0^t \cos 2\tau\, g(\tau)d\tau$$

$$v_2(t) = \int_0^t \frac{y_1(t)g(t)}{W[y_1,y_2]}dt = -\frac{1}{2}\int_0^t \sin 2\tau g(\tau)d\tau$$

and the required particular solution is

$$\begin{aligned} y_p(t) &= v_1(t)y_1(t) + v_2(t)y_2(t) \\ &= \frac{1}{2}\int_0^t (\sin 2t \cos 2\tau - \cos 2t \sin 2\tau) g(\tau) d\tau \\ &= \frac{1}{2}\int_0^t \sin 2(t-\tau) g(\tau) d\tau \end{aligned}$$

Thus the general solution of the differential equation is

$$y(t) = c_1 \sin 2t + c_2 \cos 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau) g(\tau) d\tau$$

By Leibniz formula we have

$$y'(t) = 2c_1 \cos 2t - 2c_2 \sin 2t + \int_0^t \cos 2(t-\tau) g(\tau) d\tau$$

To determine the parameters c_1, c_2, we set $t = 0$ then

$$y(0) = c_2 = 0, \qquad y'(0) = 2c_1 = 0$$

Therefore the function y defined by $(*)$ is the solution of the initial value problem.

3. Find all values of α for which all solutions of

$$t^2 y'' + \alpha t y' + \frac{5}{2} y = 0$$

approach zero as $t \to 0$.

Solution: Substituting $y = t^r$, we find that

$$r(r-1) + \alpha r + \frac{5}{2} = 0 \qquad \text{or} \qquad r^2 + (\alpha - 1)r + \frac{5}{2} = 0\,.$$

Thus

$$r_1 = \frac{1 - \alpha + \sqrt{(\alpha-1)^2 - 10}}{2}, \qquad r_2 = \frac{1 - \alpha - \sqrt{(\alpha-1)^2 - 10}}{2}$$

and the general solution is

$$y(t) = c_1 t^{r_1} + c_2 t^{r_2}$$

In order for solutions to approach zero as $t \to 0$ it is necessary that the real parts of r_1 and r_2 be positive. Suppose that $\alpha > 1$, then $\sqrt{(\alpha-1)^2 - 10}$ is either imaginary or real and less than $\alpha - 1$; hence the real parts of r_1 and r_2 will be negative. Suppose $\alpha = 1$, then $r_1, r_2 = \pm i\sqrt{10}$ and the solutions are oscillatory. Suppose $\alpha < 1$, then $\sqrt{(\alpha-1)^2 - 10}$ is either imaginary or real and less than $|\alpha - 1| = 1 - \alpha$; hence the real parts of r_1 and r_2 will be positive. Thus, if $\alpha < 1$ the solutions of the D.E. will approach zero as $t \to 0$.

4. Determine the general solution of the given differential equation that is valid in any interval not including the singular point.

(a) $(t+1)^2y'' + 3(t+1)y' + 0.75y = 0$
(b) $t^2y'' + 3ty' + 5y = 0$
(c) $t^2y'' - 5ty' + 9y = 0$

Solution:

(a) Assume $y = (t+1)^r$ for $t+1 > 0$. Substitution of y into the D. E. yields

$$r(r-1) + 3r + \frac{3}{4} = 0 \quad \text{or} \quad r^2 + 2r + \frac{3}{4} = 0$$

which yields $r = -\frac{3}{2}, -\frac{1}{2}$. The general solution of the D.E. is then

$$y = c_1|x+1|^{-1/2} + c_2|x+1|^{-3/2}, \qquad x \neq -1$$

(b) If $y = t^r$ then

$$r(r-1) + 3r + 5 = 0 \qquad \text{or} \qquad r^2 + 2r + 5 = 0$$

and $r = -1 \pm 2i$. Thus the general solution of the D. E. is

$$y(t) = c_1t^{-1}\cos(2\ln|t|) + c_2t^{-1}\sin(2\ln|t|), \qquad t \neq 0$$

(c) Again let $y = t^r$ to obtain

$$r(r-1) - 5r + 9 = 0 \qquad \text{or} \qquad (r-3)^2 = 0$$

and the general solution is

$$y(t) = c_1t^3 + c_2t^3\ln|t|, \qquad t \neq 0$$

5. Find a particular solution of

$$y'' - y' - 6y = e^{-t}$$

first by undetermined coefficients and then by variation of parameters.

Solution: The proper particular solution should be $y_p(t) = Ae^{-t}$, then substituting into the D.E. we have

$$y_p'' - y_p' - 6y_p = Ae^{-t} + Ae^{-t} - 6Ae^{-t} = -4Ae^{-t} = e^{-t} \quad \Longrightarrow \quad A = -\frac{1}{4}$$

We can also guess the particular solution by eliminating the right hand side;

$$(D-3)(D+2)(D+1)y = \quad \Longrightarrow \quad y = (c_1e^{3t} + c_2e^{-2t}) + c_3e^{-t} = y_h + y_p$$

Thus the particular solution is $y_p = c_3e^{-t}$. Next, from the homogeneous solution

$$y_h = c_1y_1 + c_2y_2 = c_1e^{3t} + c_2e^{-2t}$$

The Wronskian is

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{3t}(-2e^{-2t}) - e^{-2t}(3e^{3t}) = -5e^t$$

and

$$v_1(t) = -\int \frac{y_2(t)e^{-t}}{W[y_1, y_2]} dt = -\frac{1}{20}e^{-4t}$$
$$v_2(t) = \int \frac{y_1(t)e^{-t}}{W[y_1, y_2]} dt = -\frac{1}{5}e^{t}$$

Therefore the particular solution is

$$y_p(t) = v_1(t)y_1(t) + v_1(t)y_2(t) = -\frac{1}{20}e^{-4t}e^{3t} - \frac{1}{5}e^t e^{-2t} = -\frac{1}{4}e^{-t}$$

6. Solving the following Cauchy-Euler equations by using the substitution $t = e^x$, $Y(x) = y(t) = y(e^x)$ to change them to constant coefficient equation.

(a) $t^2y'' + ty' - 9y = 0$

(b) $t^2y'' + 3ty' + y = t + \frac{1}{t}$

(c) $t^3y''' + 4t^2y'' - 5ty' - 15y = t^4$

Solution:

(a) The chain rule implies the following relations:

$$t\frac{dy}{dt} = \frac{dY}{dx}, \qquad t^2\frac{d^2y}{dt^2} = \frac{d^2Y}{dx^2} - \frac{dY}{dx}$$

and the equation becomes

$$\frac{d^2Y}{dx^2} - \frac{dY}{dx} + \frac{dY}{dx} - 9Y = \frac{d^2Y}{dx^2} - 9Y = 0$$

Thus the general solution of the original equation is

$$Y(x) = c_1e^{3x} + c_2e^{-3x} \quad \Longrightarrow \quad y(t) = c_1t^3 + c_2t^{-3}$$

(b) Same computation as (a), the equation becomes

$$Y'' + 2Y' + Y = e^x + e^{-x}$$

The homogeneous solution is

$$Y_h(x) = c_1e^{-x} + c_2xe^{-x}$$

By superposition principle and undetermined coefficients method we compute the particular solution $Y_p = Y_1 + Y_2$ separately;

$$Y_1'' + 2Y_1' + Y_1 = e^x \quad \Longrightarrow \quad Y_1 = \frac{1}{4}e^x$$

$$Y_2'' + 2Y_2' + Y_2 = e^{-x} \quad \Longrightarrow \quad Y_2 = \frac{1}{2}x^2 e^{-x}$$

Thus the general solution of the new equation is

$$Y = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{4}e^{x} + \frac{1}{2}x^2 e^{-x}$$

Finally, replacing x by $\ln|t|$, we obtain a general solution of the original equation

$$y(t) = c_1 t^{-1} + c_2 t^{-1}\ln|t| + \frac{1}{4}t + \frac{1}{2}t^{-1}(\ln|t|)^2$$

(c) Following the same computation as (a) by chain rule we have the 3rd derivative

$$t^3 \frac{d^3y}{dt^3} = D(D-1)(D-2)Y, \qquad D = \frac{d}{dx}$$

We can transform the original equation into

$$D(D-1)(D-2)Y = 4D(D-1)Y - 5DY - 15Y = e^{4x}$$

From this, by expanding the various operational products and then collecting terms, we find

$$(D^3 + D^2 - 7D - 15)Y = e^{4x}$$

The auxiliary (characteristic) equation of this equation is

$$r^3 + r^2 - 7r - 15 = (r-3)(r^2 + 4r + 5) = 0$$

Form its roots, $r_1 = 3, r_2, r_3 = -2 \pm i$, we obtain the homogeneous solution

$$Y_h(x) = c_1 e^{3x} + e^{-2x}(c_2 \cos x + c_2 \sin x)$$

For a particular solution we try $Y_p(x) = Ae^{4x}$:

$$64Ae^{4x} + 16Ae^{4x} - 7(4Ae^{4x}) - 15(Ae^{4x}) = e^{4x} \quad \Longrightarrow \quad A = \frac{1}{37}$$

Therefore $Y_p(x) = \frac{1}{37}e^{4x}$, and a general solution is

$$Y(x) = Y_h(x) + Y_p(x) = c_1 e^{3x} + e^{-2x}(c_2 \cos x + c_3 \sin x) + \frac{1}{37}e^{4x}$$

Finally, replacing x by $\ln|t|$, we have as a general solution of the given differential equation

$$\begin{aligned} y(t) &= c_1 e^{3\ln|t|} + e^{-2\ln|t|}(c_2 \cos \ln|t| + c_3 \sin \ln|t|) + \frac{1}{37}e^{4\ln|t|} \\ &= c_1 t^3 + \frac{1}{t^2}\Big(c_2 \cos(\ln|t|) + c_3 \sin(\ln|t|)\Big) + \frac{t^4}{37}, \qquad t \neq 0 \end{aligned}$$

7. Let y_1 be a given nontrivial solution of the associated solution. Find a second linearly independent solution using reduction of order.

(a) $t^2y'' - 3ty' + 4y = 0, \qquad y_1(t) = t^2$
(b) $ty'' - (2t+1)y' + (t+1)y = 0, \qquad y_1(t) = e^t$
(c) $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0, \qquad y_1(x) = x^{-1/2}\sin x, \qquad$ (Bessel equation)

Solution:

(a) To find all solution from this one, we put

$$y(t) = v(t)t^2, \qquad y'(t) = v't^2 + 2vt, \qquad y'' = v''t^2 + 4v't + 2v$$

and substitute into the original equation to obtain

$$v''t^4 + 4v't^3 + 2vt^2 - 3v't^3 - 6vt^2 + 4vt^2 = 0$$

or

$$v''t^4 + v't^3 = 0$$

Put $w = v'$

$$w't^4 + wt^3 = 0$$

Assuming for now that $t \neq 0$, we get

$$w' + \frac{w}{t} = 0$$

The solution of this separable equation are $w(t) = \frac{C}{t}$, C an arbitrary constant, so we get

$$v(t) = \int w(t)dt = C\ln|t| + D$$

Therefore, $y(t) = v(t)t^2 = \big(C\ln|t| + D\big)t^2$ is the general solution and $y_2(t) = t^2\ln|t|$ is the second linearly independent solution. For convenience, we also apply Theorem 8 (Section 4.7) to this problem. We rewrite the equation;

$$y'' - \frac{3}{t}y' + \frac{4}{t^2}y = 0$$

We identify $p(t)$ as $-\frac{3}{t}$ then

$$-\int p(t)dt = \int \frac{3}{t}dt = 3\ln|t|$$

and the second independent solution is

$$y_2(t) = y_1(t)\int \frac{e^{3\ln|t|}}{y_1^2}dt = t^2\int \frac{t^3}{t^4}dt = t^2\int \frac{1}{t}dt = t^2\ln|t|$$

(b) We begin by writing the given equation in the standard form

$$y'' - \frac{2t+1}{t}y' + \frac{t+1}{t}y = 0$$

Since $p(t) = -\frac{2t+1}{t}$, a second linearly independent solution is given by $y_2 = vy_1$, where

$$v(t) = \int \frac{e^{-\int p(t)dt}}{e^{2t}} dt = \int \frac{e^{2t+\ln|t|}}{e^{2t}} dt = \int t dt = \frac{t^2}{2}$$

This yields $y_2(t) = v(t)y_1(t) = \frac{t^2}{2}e^t$, so the general solution is

$$y = c_1 e^t + c_2 t^2 e^t$$

(c) Same as (b), we rewrite the original equation as the standard form

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = 0$$

then

$$\begin{aligned} v(x) &= \int \frac{e^{-\int p(x)dx}}{y_1^2} dx = \int \frac{1/x}{(x^{-1/2}\sin x)^2} dx \\ &= \int \frac{1}{\sin^2 x} dx = \int \csc^2 x dx = -\cot x = -\frac{\cos x}{\sin x} \end{aligned}$$

Thus the second linearly independent solution is

$$y_2(x) = v(x)y_1(x) = -x^{-1/2}\cos x$$

and the general solution is

$$y(x) = c_1 x^{-1/2}\cos x + c_2 x^{-1/2}\sin x$$

8. A weight of 49g is suspended from a spring of modulus $\frac{5}{2}$ g/cm (corresponding to stiffness).The coefficient of friction in the system is estimated to be $\frac{1}{10}$ g/(cm/sec). At $t = 0$, the weight is pulled down 6 cm from its equilibrium position and released from that point with an upward velocity of 20 cm/sec. Find the subsequent displacement of the weight as a function of time. When does the weight pass through its equilibrium position? Take $g = 980\text{cm/sec}^2$

Solution: The differential equation to be solved is

$$\frac{49}{980}\frac{d^2y}{dt^2} + \frac{1}{10}\frac{dy}{dt} + \frac{5}{2}y = 0 \qquad \text{or} \qquad \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 50y = 0$$

The auxiliary (characteristic) equation os this equation is $r^2 + 2r + 50 = 0$, and its roots are $r_1, r_2 = -1 \pm 7i$. Hence

$$y = e^{-t}(c_1 \cos 7t + c_2 \sin 7t)$$

and, differentiating

$$v = \frac{dy}{dt} = -e^{-t}(c_1 \cos 7t + c_2 \sin 7t) + e^{-t}(-7c_1 \sin 7t + 7c_2 \cos 7t)$$

Substituting the data $y = -6$, $t = 0$ into the equation for y, we find

$$c_1 = -6$$

Substituting the data $v = 20$, $t = 0$ into the velocity equation, we find

$$20 = -c_1 + 7c_2, \qquad \text{or} \qquad c_2 = 2$$

The displacement of the weight is thus a damped oscillation described by the equation

$$y = e^{-t}(-6\cos 7t + 2\sin 7t), \qquad t \geq 0$$

The weight passes through its equilibrium position when $y = 0$, that is, when

$$-6\cos 7t + 2\sin 7t = 0, \qquad \tan 7t = 3$$

Thus

$$t = \frac{1}{7}\text{Tan}^{-3} + \frac{n\pi}{7} \approx 0.178 + \frac{n\pi}{7} \quad \text{sec}$$

9. The position of a certain spring-mass system satisfies the initial value problem

$$\frac{3}{2}y'' + ky = 0, \qquad y(0) = 2, \qquad y'(0) = v\,.$$

If the period and amplitude of the resulting motion are observed to be π and 3, respectively, determine the values of k and v.
Solution: The general solution is

$$y(t) = c_1 \cos\sqrt{\frac{2k}{3}}\,t + c_2 \sin\sqrt{\frac{2k}{3}}\,t$$

Substituting the initial conditions into the equation yields

$$y(0) = c_1 = 2, \qquad y'(0) = \sqrt{\frac{2k}{3}}\,c_2 = v, \qquad c_2 = \sqrt{\frac{3}{2k}}\,v$$

Thus the solution of the initial value problem is

$$y(t) = 2\cos\sqrt{\frac{2k}{3}}\,t + \sqrt{\frac{3}{2k}}\,v \sin\sqrt{\frac{2k}{3}}\,t$$

The period is

$$\frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{2k}{3}}} = \pi \quad \Longrightarrow \quad k = 6$$

The amplitude is

$$A = \sqrt{2^2 + \frac{3}{2k}v^2} = \sqrt{4 + \frac{v^2}{4}} = 3 \quad \Longrightarrow \quad v = \pm 2\sqrt{5}$$

10. (Abel formula) If y_1 and y_2 are solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous functions, then the Wronskian $W[y_1, y_2]$ is given by Abel formula

$$W[y_1, y_2] = C \exp\Big(- \int p(t)dt \Big)$$

(a) Show that the Wronskian W satisfies the first order differential equation

$$W' + pW = 0$$

(b) Solve the separable equation in (a).

(c) Apply Abel formula to the following differential equation

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0, \qquad y_1 = t^{1/2}, \qquad y_2 = t^{-1}$$

Solution:

(a) We start by noting that Y_1 and y_2 satisfy

$$\begin{aligned} y_1'' + p(t)y_1 + q(t)y_1 &= 0 \\ y_2'' + p(t)y_2 + q(t)y_2 &= 0 \end{aligned}$$

If we multiply the first equation by $-y_2$, multiply the second by y_1, and add the resulting equations, we obtain

$$(y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0$$

Nest, we let $W = W[y_1, y_2]$ and observe

$$W' = y_1y_2'' - y_1''y_2$$

Then we can write the previous equation in the form

$$W' + p(t)W = 0$$

(b) The first differential equation obtained in (a) is separable.

$$\frac{dW}{W} = -p(t)dt \quad \Longrightarrow \quad W = C \exp\Big(- \int p(t)dt \Big)$$

(c) Direct computation shows that $W = -\frac{3}{2}t^{-3/2}$. On the other hand, to apply the Abel formula, we must write the differential equation in the standard form

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0$$

so $p(t) = 3/2t$. Hence

$$W[y_1, y_2] = C \exp\Big(- \int \frac{3}{2t}dt \Big) = Ce^{-\frac{3}{2}\ln t} = Ct^{-3/2}$$

This formula gives you the Wronskian of any pair of solution of the given equation. For particular solutions given in this example we must choose $C = -3/2$.