MATH201 – Fall Mid-term Solutions

Multiple Choice Questions

1. The solution to the initial value problem

$$\frac{dx}{dt}+\frac{x}{t}=e^{t^2}, \qquad x(1)=\frac{e}{2}$$

Linear 1st-order DE

Let $\mu(t)=e^{\int\frac{dt}{t}}=|t|, \Rightarrow \qquad \frac{d}{dt}\{tx\}=te^{t^2}$

Hence, $tx=\int e^{t^2}dt=\frac{e^{t^2}}{2}+c$. Using IC, $c=0,\Rightarrow \qquad x(t)=\frac{e^{t^2}}{2t} \qquad \Rightarrow \qquad x(2)=\frac{e^4}{4}$

2. The solution y(t) of

$$(e^t y+te^t y)dt+(te^t+2)dy=0, \qquad y(0)=-1$$

See Assignment Solution #1, 10(b)

3. The solution y(θ) of $\theta dy - y d\theta=\sqrt{\theta y}\,d\theta$

See Assignment Solution #1, 12(f)

4. The solution y(x) of $\frac{dy}{dx}+y=\frac{1}{y}, \qquad y(0)=\sqrt{2}$

Bernoulli equation, dividing by $y^{-1}\Rightarrow \qquad y\frac{dy}{dt}+y^{2}=1$

Let $v=y^2,\Rightarrow \qquad \frac{dv}{dx}=2y\frac{dy}{dx},\Rightarrow \qquad \frac{dv}{dx}+2v=2$

Linear 1st-order DE

Let $\mu(x)=e^{\int 2dx}=e^{2x},\Rightarrow \qquad \frac{d}{dx}\{e^{2x}v\}=2e^{2x}, \Rightarrow \qquad e^{2x}v=e^{2x}+c$

$y^2=1+ce^{-2x}$. Using IC, $c=1$

Long Questions

1. (a) Solve the initial value problem $ty'' + 3y' + \frac{y}{t} = 0, \quad y(1) = 1, y'(1) = 0$

Cauchy-Euler equation: $t^2 y'' + 3ty' + y = 0$

Let $t = e^x, \Rightarrow \quad ty' = \frac{dy}{dx}, \quad t^2 y'' = \frac{d^2 y}{dx^2} - \frac{dy}{dx}$

Hence,

$$\frac{d^2 y}{dx^2} - \frac{dy}{dx} + 3\frac{dy}{dx} + 1 = 0 ,$$

Let $y(x) = e^{mx}$, the characteristic equation $m^2 + 2m + 1 = (m+1)^2 = 0$

Hence, $y(x) = c_1 e^{-x} + c_2 x e^{-x} \Rightarrow \quad y(t) = c_1 \frac{1}{t} + c_2 \frac{\ln t}{t}$

Using IC, $c_1 = c_2 = 1$.

(b) Determine the Laplace transform of $f(t) = \begin{cases} 1, 0 < t \le 3 \\ 4 - t, 3 < t \le 4 \\ 0, 4 < t \le \infty \end{cases}$

For any s>0, $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$

$$F(s) = \int_0^3 e^{-st} dt + \int_3^4 e^{-st}(4-t)dt + \int_4^\infty 0 dt$$

Hence,

$$= \frac{e^{-st}}{s}\Big|_0^3 - \frac{e^{-st}}{s}(4-t)\Big|_3^4 - \int_3^4 \frac{e^{-st}}{s} dt$$

$$= \frac{1}{s}(1 - e^{-3s}) + \frac{e^{-3s}}{s} + \frac{1}{s^2}(e^{-4s} - e^{-3s})$$

$$= \frac{1}{s} + \frac{1}{s^2}(e^{-4s} - e^{-3s})$$

2. See Assignment Solution #3, 8

3. Find a particular solution of $y'' - y = 2e^{x} + \frac{4e^{-x}}{1-e^{-2x}}$

The characteristic equation for the homogeneous equation is $m^2 - 1 = (m-1)(m+1) = 0$

Hence, we have $y_1(x) = e^{x}$, $\quad y_2(x) = e^{-x}$

Use the undetermined coefficient method for $2e^{x}$, let $y_p(x) = Axe^{x}$.

Since $y_p'(x) = Axe^{x} + Ae^{x}$, $\quad y_p''(x) = Axe^{x} + 2Ae^{x}$, we found $A = 1$

For $\frac{4e^{-x}}{1-e^{-2x}}$, use the variation of parameters, let $y_p(x) = v_1(x)e^{x} + v_2(x)e^{-x}$, solve

$$e^{x}v_1' + e^{-x}v_2' = 0$$
$$e^{x}v_1' - e^{-x}v_2' = \frac{4e^{-x}}{1-e^{-2x}}$$

By adding, we get

$$2e^{x}v_1' = \frac{4e^{-x}}{1-e^{-2x}}, \quad \Rightarrow \quad v_1' = \frac{2e^{-2x}}{1-e^{-2x}} \quad \Rightarrow \quad v_1 = \ln(1-e^{-2x})$$

From the first equation, we have $v_2' = -e^{2x}v_1'$

Hence, $\quad v_2' = \frac{-2}{1-e^{-2x}} = \frac{-2e^{2x}}{e^{2x}-1} \quad \Rightarrow \quad v_2 = -\ln(e^{2x}-1)$

The particular solution is $y_p(x) = xe^{x} + e^{x}\ln(1-e^{-2x}) - e^{-x}\ln(e^{2x}-1)$