

**Solution for MATH 201 Problem Set 10 (CH. 10.2-10.5)**

1. Find the values of  $\lambda$  for which the given problem has a nontrivial solution. Also determine the corresponding nontrivial solution.

$$y'' + \lambda y = 0, \quad 0 < x < \frac{\pi}{2}$$

$$y'(0) = 0, \quad y'\left(\frac{\pi}{2}\right) = 0$$

**Solution:**  $\lambda_n = 4n^2$ ,  $y_n = c_n \cos(2nx)$ ,  $n = 0, 1, 2, \dots$

2.

$$y'' + \lambda y = 0, \quad 0 < x < \pi$$

$$y(0) - y'(0) = 0, \quad y(\pi) = 0$$

**Solution:**

The char. equation:  $m^2 + \lambda = 0$ ,

$$\lambda < 0, \quad \Rightarrow \quad m = \pm\sqrt{-\lambda}$$

$$y(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x},$$

$$y'(x) = c_1 \sqrt{-\lambda} e^{\sqrt{-\lambda}x} - c_2 \sqrt{-\lambda} e^{-\sqrt{-\lambda}x}$$

$$y(0) - y'(0) = c_1 + c_2 - c_1 \sqrt{-\lambda} + c_2 \sqrt{-\lambda} = 0$$

$$\Rightarrow (1 - \sqrt{-\lambda})c_1 + (1 + \sqrt{-\lambda})c_2 = 0$$

$$y(\pi) = c_1 e^{\sqrt{-\lambda}\pi} + c_2 e^{-\sqrt{-\lambda}\pi} = 0 \quad \Rightarrow \quad c_2 = -c_1 e^{-2\sqrt{-\lambda}\pi}$$

From the two equations, we found that  $c_1 = c_2 = 0$ .

For  $\lambda = 0$ ,

$$y(x) = c_1 + c_2 x, \quad y'(x) = c_2$$

$$y(0) - y'(0) = c_1 - c_2 = 0$$

$$y(\pi) = c_1 + c_2 \pi e^{-\sqrt{\lambda}\pi} = 0$$

Hence,  $c_1 = c_2 = 0$ . Consider

$$\lambda > 0,$$

$$\Rightarrow y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x),$$

$$y'(x) = -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}x) + c_2 \sqrt{\lambda} \cos(\sqrt{\lambda}x)$$

$$y(0) - y'(0) = c_1 - c_2 \sqrt{\lambda} = 0, \quad \Rightarrow \quad c_1 = c_2 \sqrt{\lambda}$$

$$y(\pi) = c_1 \cos(\sqrt{\lambda}\pi) + c_2 \sin(\sqrt{\lambda}\pi) = 0$$

$$\text{Hence, } c_2[\sqrt{\lambda} \cos(\sqrt{\lambda}\pi) + \sin(\sqrt{\lambda}\pi)] = 0$$

$$\text{If } c_2 \neq 0, \quad \Rightarrow \quad \sqrt{\lambda} \cos(\sqrt{\lambda}\pi) + \sin(\sqrt{\lambda}\pi) = 0$$

$$\text{Hence, } \lambda_n \text{ satisfy } \tan(\sqrt{\lambda_n}\pi) + \sqrt{\lambda_n} = 0.$$

3. Solve the heat flow problem

$$\frac{\partial u}{\partial t} = 3 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0$$

$$u(0, t) = u(\pi, t) = 0, \quad t > 0,$$

$$u(x, 0) = \sin 4x + 3 \sin 6x - \sin 10x, \quad 0 < x < \pi$$

**Solution:**  $u(x, t) = e^{-48t} \sin 4x + 3e^{-108t} \sin 6x - e^{-300t} \sin 10x$

4. Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 9 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0$$

$$u(0, t) = u(\pi, t) = 0, \quad t > 0,$$

$$u(x, 0) = 6 \sin 2x + 2 \sin 6x, \quad 0 < x < \pi$$

$$\frac{\partial u}{\partial t}(x, 0) = 11 \sin 9x - 14 \sin 15x, \quad 0 < x < \pi$$

**Solution:**

The solution is

$$u(x, t) = \sum_{n=1}^{\infty} [a_n \cos(3nt) + b_n \sin(3nt)] \sin(nx)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} [-3na_n \sin(3nt) + 3nb_n \cos(3nt)] \sin(nx)$$

From ICs,

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin(nx) = 6 \sin(2x) + 2 \sin(6x)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} 3nb_n \sin(nx) = 11 \sin(9x) - 14 \sin(15x)$$

Hence,

$$a_2 = 6, \quad a_6 = 2, \quad \text{other } a_n = 0$$

$$n = 9, \quad 3 \cdot 9 \cdot b_9 = 11, \quad n = 15, \quad 3 \cdot 15 \cdot b_{15} = -14, \quad \text{other } b_n = 0$$

$$u(x, t) = 6 \cos 6t \sin 2x + 2 \cos 18t \sin 6x + \frac{11}{27} \sin 27t \sin 9x - \frac{14}{45} \sin 45t \sin 15x$$

5. Compute the Fourier series for the given function f on the specific interval.

$$(a) \quad f(x) = x, -\pi < x < \pi$$

$$(b) \quad f(x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

$$(c) \quad f(x) = e^x, -\pi < x < \pi$$

**Solutions:**

(a)  $f(x) = x$  is odd, for any  $n$ ,  $\cos(nx)$  is even, but  $x\cos(nx)$  is odd, and  $x\sin(nx)$  is even.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx) dx = 0, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx$$

$$b_n = \frac{2}{\pi} \left( -\frac{1}{n} \right) \left[ x \cos(nx) \Big|_0^{\pi} - \int_0^{\pi} \cos(nx) dx \right] = -\frac{2}{n\pi} \left[ \pi \cos(n\pi) - \frac{1}{n} \sin(nx) \Big|_0^{\pi} \right] = \frac{2(-1)^{n+1}}{n}$$

$$\text{Hence, } f(x) \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

$$(b) \quad f(x) \sim \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{\sin(2nx)}{n}$$

$$(c) \quad f(x) \sim \frac{\sinh \pi}{\pi} \left[ 1 + 2 \sum_{n=1}^{\infty} \left( \frac{(-1)^n}{1+n^2} \cos nx + \frac{(-1)^{n+1} n}{1+n^2} \sin nx \right) \right]$$

6. Compute the Fourier sine series for the given function.

$$(a) \quad f(x) = x^2, \quad 0 < x < \pi$$

$$(b) \quad f(x) = \pi - x, \quad 0 < x < \pi$$

$$(c) \quad f(x) = e^x, \quad 0 < x < 1$$

**Solutions:**

$$(a) \quad f(x) \sim \sum_{n=1}^{\infty} \left( \frac{2\pi(-1)^{n+1}}{n} + \frac{4[(-1)^n - 1]}{\pi n^3} \right) \sin nx$$

$$(b) \quad f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

$$(c) \quad f(x) \sim \sum_{n=1}^{\infty} \frac{2\pi n [1 - e(-1)^n]}{1 + \pi^2 n^2} \sin n\pi x$$

7. Compute the Fourier cosine series for the given function.

(a)  $f(x) = 1 + x, \quad 0 < x < \pi$

(b)  $f(x) = e^x, \quad 0 < x < 1$

(c)  $f(x) = \sin x, \quad 0 < x < \pi$

**Solutions:**

(a)  $f(x) \sim 1 + \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)x$

(b)  $f(x) \sim e - 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n e - 1}{1 + \pi^2 n^2} \cos n\pi x$

(c)  $f(x) \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos 2nx$

8. Find a formal solution to the given initial-boundary value problem.

(a)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0$$

$$u(0, t) = u(\pi, t) = 0, \quad t > 0,$$

$$u(x, 0) = x^2, \quad 0 < x < \pi$$

**Solution:**

$$u(x, t) = \sum_{n=1}^{\infty} \left( \frac{2\pi(-1)^{n+1}}{n} + \frac{4[(-1)^n - 1]}{\pi n^3} \right) e^{-n^2 t} \sin nx$$

(b)

$$\frac{\partial u}{\partial t} = 7 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0$$

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0, \quad t > 0,$$

$$u(x, 0) = 1 - \sin x, \quad 0 < x < \pi$$

**Solution:**

$$u(x, t) = 1 - \frac{2}{\pi} + 2e^{-7t} \cos x + \sum_{n=1}^{\infty} \frac{4}{\pi(4n^2 - 1)} e^{-28n^2 t} \cos 2nx$$

(c)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0$$

$$u(0, t) = 0, \quad u(\pi, t) = 3\pi, \quad t > 0,$$

$$u(x, 0) = 0, \quad 0 < x < \pi$$

**Solution:**

$$u(x,t) = 3x + 6 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2 t} \sin nx$$

(d)

$$\frac{\partial u}{\partial t} = 3 \frac{\partial^2 u}{\partial x^2} + x, \quad 0 < x < \pi, \quad t > 0$$

$$u(0,t) = u(\pi,t) = 0, \quad t > 0,$$

$$u(x,0) = \sin x, \quad 0 < x < \pi$$

**Solution:**

$$u(x,t) = \left(\frac{\pi^2}{18}\right)x - \left(\frac{1}{18}\right)x^3 + \left(\frac{1}{3}\right)e^{-3t} \sin x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{3n^3} e^{-3n^2 t} \sin nx$$

(e)

$$\begin{aligned}\frac{\partial u}{\partial t} &= 4 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0 \\ \frac{\partial u}{\partial t}(0, t) &= 0, \quad u(\pi, t) = 0, \quad t > 0, \\ u(x, 0) &= f(x), \quad 0 < x < \pi\end{aligned}$$

Solution:

$$\text{Let } u(x, t) = X(x)T(t), \quad \Rightarrow \quad T' X = 4X'' T \quad \Rightarrow \quad \frac{T'}{4T} = \frac{X''}{X} = K$$

Now, solve the eigenvalue problem:

$$\begin{aligned}X''(x) &= KX(x), \quad 0 < x < \pi, \\ \text{BCs: } X'(0) &= 0, \quad X(\pi) = 0\end{aligned}$$

The Char. Eq. is  $m^2 - K = 0$ .

$$\text{For } K = 0, \quad \Rightarrow \quad X'' = 0, \quad \Rightarrow \quad X(x) = c_1 + c_2 x.$$

$$\text{For } K > 0, \quad \Rightarrow \quad m^2 - K = 0, \quad \Rightarrow \quad m = \pm\sqrt{K}$$

$$\Rightarrow \quad X(x) = c_1 e^{\sqrt{K}x} + c_2 e^{-\sqrt{K}x}$$

Using the procedure as in Question #2, you could verify that  $c_1 = c_2 = 0$ .

Now, consider

$$K < 0, \quad -K > 0, \quad \Rightarrow \quad m = \pm i\sqrt{-K}$$

$$\Rightarrow \quad X(x) = c_1 \sin(\sqrt{-K}x) + c_2 \cos(\sqrt{-K}x)$$

$$\Rightarrow \quad X'(x) = c_1 \sqrt{-K} \cos(\sqrt{-K}x) - c_2 \sqrt{-K} \sin(\sqrt{-K}x)$$

$$BC: \quad X'(0) = 0 \quad \Rightarrow \quad c_1 = 0$$

$$BC: \quad X(\pi) = 0 \quad \Rightarrow \quad c_2 \cos(\sqrt{-K}\pi) = 0 \quad \Rightarrow \quad \sqrt{-K} = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad \Rightarrow \quad K_n = -\frac{(2n+1)^2}{4}, \quad n = 0, 1, 2, \dots$$

$$\text{Hence,} \quad X_n(x) = c_n \cos\left(\frac{2n+1}{2}x\right)$$

$$\text{Then, solve the Eq. } T'(t) - 4KT(t) = 0, \quad \Rightarrow \quad T'(t) + (2n+1)^2 T(t) = 0$$

This is a linear first order DE, and the solution is

$$T_n(t) = b_n e^{-(2n+1)^2 t}, \quad n = 0, 1, 2, \dots$$

Since,  $u(x, t) = X(x)T(t)$ , we have

$$u(x, t) = \sum_{n=0}^{\infty} b_n e^{-(2n+1)^2 t} c_n \cos\left(\frac{2n+1}{2}x\right) = \sum_{n=0}^{\infty} a_n e^{-(2n+1)^2 t} \cos\left(\left(n + \frac{1}{2}\right)x\right)$$

**The coefficients**  $a_n$  are determined from the IC:

$$u(x, 0) = f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\left(n + \frac{1}{2}\right)x\right)$$