Regularized quantile regression under heterogeneous sparsity with application to quantitative genetic traits

CrossMark

Qianchuan He [a],[*],[1], Linglong Kong [b],[1], Yanhua Wang [c], Sijian Wang [d], Timothy A. Chan [e], Eric Holland [f]

[a] Public Health Sciences Division, Fred Hutchinson Cancer Research Center, Seattle, WA 98109, USA

[b] Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, Canada T6G 2G1

[c] School of Mathematics, Beijing Institute of Technology, Beijing, 100081, China

[d] Department of Biostatistics & Medical Informatics and Department of Statistics, The University of Wisconsin-Madison, Madison, WI 53706, USA

[e] Human Oncology and Pathogenesis Program, Memorial Sloan-Kettering Cancer Center, New York, NY 10065, USA

[f] Human Biology Division, Fred Hutchinson Cancer Research Center, Seattle, WA 98109, USA

A R T I C L E   I N F O

A B S T R A C T

Genetic studies often involve quantitative traits. Identifying genetic features that influence quantitative traits can help to uncover the etiology of diseases. Quantile regression method considers the conditional quantiles of the response variable, and is able to characterize the underlying regression structure in a more comprehensive manner. On the other hand, genetic studies often involve high-dimensional genomic features, and the underlying regression structure may be heterogeneous in terms of both effect sizes and sparsity. To account for the potential genetic heterogeneity, including the heterogeneous sparsity, a regularized quantile regression method is introduced. The theoretical property of the proposed method is investigated, and its performance is examined through a series of simulation studies. A real dataset is analyzed to demonstrate the application of the proposed method.

1. Introduction

In many genetic studies, quantitative traits are collected for studying the associations between the traits and certain genomic features. For example, body mass index, lipids and blood pressure have been investigated with respect to single nucleotide polymorphisms (SNPs) (Avery et al., 2011). With the rapid progress of high-throughput genome technology, new types of quantitative traits have emerged and attracted considerable research interest, such as gene expression, DNA methylation, and protein quantification (Landmark-Høyvik et al., 2013). The analysis of these quantitative traits yields new insight into biological processes and sheds light on the genetic basis of diseases.

Typically, quantitative genetic traits are analyzed by least-square based methods, which seek to estimate the $E(Y|Z)$, where Y is the trait and Z is the set of covariates of interest. Quantile regression (Koenker and Bassett, 1978) instead considers the conditional quantile function of Y given Z, $Q_\tau(Y|Z)$, at a given $\tau \in (0, 1)$. When τ is fixed at 0.5, quantile regression is simply the median regression, which is well known to be more robust than the least-square estimation. In examining

* Corresponding author.
  E-mail address: qhe@fhcrc.org (Q. He).

[1] Equally contributed.

quantiles at different τ, quantile regression provides a more complete picture of the underlying regression structure between Y and Z.

Like least-square methods, traditional quantile regression methods only consider a handful of covariates. With the emergence of high-dimensional data, penalized quantile regression methods have been developed in recent years, and can be broadly classified into two classes. The first class seeks to harness the information shared among different quantiles to jointly estimate the regression coefficients. Jiang et al. (2014) proposed two novel methods, Fused Adaptive Lasso (FAL) and Fused Adaptive Sup-norm estimator (FAS), for variable selection in interquantile regression. FAL combines the LASSO penalty (Tibshirani, 1996) with the fused lasso penalty (Tibshirani et al., 2005), while FAS imposes the grouped sup-norm penalty and the fused lasso penalty. The fused lasso penalty delivers the effect of smoothing the regression slopes of adjacent quantiles, hence FAL and FAS can be used to identify common quantile slopes if such a smoothing property is desired. Zou and Yuan (2008) adopted an F_∞ penalty, which either eliminates or retains all the regression coefficients for a covariate at multiple quantiles. The method by Jiang et al. (2013) seeks to shrink the differences among adjacent quantiles by resorting to the fused lasso penalty; however, this method does not perform variable selection at the covariate level, i.e., it does not remove any covariates from the model.

The second class of methods focuses on a single quantile at a time. Koenker (2004) imposed a LASSO penalty on the random effects in the mixed-effect quantile regression model; Li and Zhu (2008) adopted the LASSO penalty; Wu and Liu (2009) explored the SCAD penalty (Fan and Li, 2001) and the adaptive LASSO penalty (Zou, 2006), and proved the selection consistency and normality of the proposed estimators (for a fixed dimension of covariates). Wang et al. (2012) investigated several penalties for quantile regression under the scenario of $p > n$, i.e., the dimension is larger than the sample size, and proved the selection consistency of their proposed methods through a novel use of the subgradient theory. A recent approach by Peng et al. (2014) shares characteristics with both classes; its loss function targets a single τ, while its penalty borrows information across different quantiles; their proposed penalty was shown to achieve more accurate estimation than the one that uses information from only a single quantile.

If the regression coefficients associated with a covariate are treated as a group, then some groups may be entirely zero and some other groups may be partially zero. Thus, sparsity can occur both at the group level and within the group level, and we call this type of sparsity as the heterogeneous sparsity. In this paper, we propose an approach that conducts joint variable selection and estimation for multiple quantiles under the situation that p can diverge with n. Our proposed method is able to achieve sparsity both at the group level and within the group level. We note that FAL can potentially yield sparsity at the two levels, but this approach has not been evaluated under the scenario where the dimension p is high. To the best of our knowledge, this is the first paper that explicitly investigates the heterogeneous sparsity for quantile regression. We show that our method tends to be more effective than the compared methods in handling heterogeneous sparsity when the dimension is high. We also provide theoretical justification for the proposed method. The paper is organized as follows. In Section 2, we describe the proposed method and the implementation details. In Section 3, we prove the theoretical properties of the proposed method. In Section 4, we show the results of simulation studies regarding several related methods, and in Section 5, we present an example of real data analysis for the proposed method.

2. Method

2.1. Data and model

Let Z represent the vector consisting of p covariates, such as SNPs or genes. Let γ_τ be the p-dimension coefficient vector at the τth quantile. Let Y be the random variable that denotes the phenotype we are interested in, such as quantitative traits in genetic studies. For a given $\tau \in (0, 1)$, the linear quantile regression model is known as

$$Q_\tau(Y|Z) = \gamma_0 + Z^{\mathrm{T}}\gamma_\tau,$$

where γ_0 is the intercept, and $Q_\tau(Y|Z)$ is the τ-th conditional quantile of Y given Z, that is, $P_{Y|Z}(Y \leq Q_\tau(Y|Z)) = \tau$.

The dimension p can be potentially very high in genomic studies, but typically it is assumed that only a limited number of genomic features contribute to the phenotype. For this reason, one needs to find a sparse estimation of γ_τ to identify those important genomic features. On the other hand, we also wish to consider multiple quantile levels simultaneously so that information shared among different quantile levels can be utilized. To this end, we propose the following model for the joint estimation of the regression coefficients for multiple quantiles. Given M quantile levels, $0 < \tau_1 < \cdots < \tau_M < 1$, our linear quantile regression model is defined as, for τ_m ($m = 1, \ldots, M$),

$$Q_{\tau_m}(Y|Z) = \gamma_{m0} + Z^{\mathrm{T}}\gamma_{\tau_m}. \tag{1}$$

where γ_{m0} is the intercept, and γ_{τ_m} is the p-dimension coefficient vector. For ease of notation, we write $\gamma_m = \gamma_{\tau_m}$. For the above model, we further define $\gamma \equiv (\gamma_1^{\mathrm{T}}, \ldots, \gamma_M^{\mathrm{T}})^{\mathrm{T}}$ with $\gamma_m = (\gamma_{m1}, \ldots, \gamma_{mp})^{\mathrm{T}}$, and intercept parameter $\gamma_0 \equiv (\gamma_{10}, \ldots, \gamma_{M0})^{\mathrm{T}}$.

We now focus on the sample version of the model (1). Let $\{(Y_i, Z_i^T)^T\}_{i=1}^n$ be an i.i.d. random sample of size n from population $(Y, Z^T)^T$, where $Z_i = (Z_{i1}, Z_{i2}, \ldots, Z_{ip})^T$. The sample quantile loss function is defined as

$$Q_n(\gamma_0, \gamma) = \sum_{m=1}^{M} \sum_{i=1}^{n} \rho_m(Y_i - Z_i^T \gamma_m - \gamma_{m0})$$

where $\rho_m(u) = u(\tau_m - I(u < 0))$ is the quantile check loss function with $I(\cdot)$ being the indicator function. To introduce sparsity to the model, we add to the loss function a penalty function

$$P_n(\gamma) = n\lambda_n \sum_{j=1}^{p} \left(\sum_{m=1}^{M} \omega_{mj} |\gamma_{mj}| \right)^{\frac{1}{2}},$$

where λ_n is the tuning parameter, and $\omega_n = (\omega_{mj} : 1 \leq m \leq M, 1 \leq j \leq p)$ is the weight vector whose component $\omega_{mj} > 0$ is the weight of parameter γ_{mj}. Note that the penalty is a nonconvex function. It essentially divides the regression coefficients into p groups, and each group consists of M parameters associated with the jth covariate. The motivation is that, while each quantile may have its own set of regression parameters, we wish to borrow strengths from each quantile to select covariates that are important across all quantiles as well as those covariates that are important to only some of the quantiles. This type of nonconvex penalty has been considered in the Cox regression model and other settings (see Wang et al. (2009) for an example), but to the best of our knowledge has not been studied in the quantile regression model. We can choose $\omega_{mj} = \tilde{\gamma}_{mj}^{-1}$, where $\tilde{\gamma}_{mj}$ is some consistent estimate for γ_{mj}; for example, we may use the estimates from the unpenalized quantile regression conducted at each individual quantile level. When $p < n$ but is fixed, the consistency of the unpenalized estimates has been proved by Koenker and Bassett (1978). When $p < n$ but is diverging with n, the estimates from unpenalized quantile regression are consistent by adapting to Lemma A.1 of Wang et al. (2012). Thus, our objective function is defined as

$$\sum_{m=1}^{M} \sum_{i=1}^{n} \rho_m(Y_i - Z_i^T \gamma_m - \gamma_{m0}) + n\lambda_n \sum_{j=1}^{p} \left(\sum_{m=1}^{M} \omega_{mj} |\gamma_{mj}| \right)^{\frac{1}{2}}.$$

For the sake of convenience, we define $\theta = (\theta_1^T, \ldots, \theta_M^T)^T$ with $\theta_m = (\gamma_{m0}, \gamma_m^T)^T$, $m = 1, \ldots, M$, and the corresponding parameter space by $\Theta_n \subset \mathbb{R}^{M(p+1)}$. Further define $U_i = (1, Z_i^T)^T$, $i = 1, \ldots, n$. Then, $Q_n(\gamma_0, \gamma)$ can be written as $Q_n(\theta)$. Emphasizing that γ is a subset of θ, we can write the objective function as

$$L_n(\theta) \equiv Q_n(\theta) + P_n(\gamma) = \sum_{m=1}^{M} \sum_{i=1}^{n} \rho_m(Y_i - U_i^T \theta_m) + n\lambda_n \sum_{j=1}^{p} \left(\sum_{m=1}^{M} \omega_{mj} |\gamma_{mj}| \right)^{\frac{1}{2}}. \tag{2}$$

Let $\hat{\theta}$ be a local minimizer of $L_n(\theta)$ in (2) for $\theta \in \Theta_n$. Because the heterogeneity of sparsity is explicitly taken into account in this model, we name our proposed method as Heterogeneous Quantile Regression (Het-QR). Our model can be modified to accommodate different weights for the losses at different quantiles. That is, $Q_n(\theta)$ may take the form $\sum_{m=1}^{M} \pi_m \sum_{i=1}^{n} \rho_m(Y_i - U_i^T \theta_m)$, where π_m is the weight for the mth quantile. Some examples on the choice of weight π_m can be found in Koenker (2004) and Zhao and Xiao (2014).

2.2. Implementation

We design the following algorithm to implement the proposed method. First, we show in the Appendix that the objective function can be transformed into

$$\text{argmin}_{\theta, \xi} \sum_{m=1}^{M} \sum_{i=1}^{n} \rho_m(Y_i - U_i^T \theta_m) + \lambda_1 \sum_{j=1}^{p} \xi_j + \sum_{j=1}^{p} \xi_j^{-1} \left(\sum_{m=1}^{M} \omega_{mj} |\gamma_{mj}| \right),$$

where $\xi = (\xi_1, \ldots, \xi_p)$ are newly introduced nonnegative parameters. Then, the new objective function can be solved by the following iterative algorithm:

Step 1: We first fix θ to solve ξ_j, $j = 1, \ldots, p$. To this end, ξ_j has a closed-form solution. That is, $\hat{\xi}_j = \left(\sum_{m=1}^{M} \omega_{mj} |\gamma_{mj}| \right)^{1/2} \lambda_1^{-1/2}, j = 1, \ldots, p$.

Step 2: We fix $\xi_j, j = 1, \ldots, p$, to solve θ. That is, we aim to solve

$$\text{argmin}_{\theta} \sum_{m=1}^{M} \sum_{i=1}^{n} \rho_m(Y_i - U_i^T \theta_m) + \sum_{j=1}^{p} \sum_{m=1}^{M} \xi_j^{-1} \omega_{mj} |\gamma_{mj}|.$$

We can formulate this objective function as a linear program and derive its dual form (see Appendix for details), then the optimization can be conducted by recoursing to existing linear programming packages; we utilize the Quantreg R package (Koenker, 2015) in our implementation.

Step 3: Iterate step 1 and step 2 until convergence. Due to the nonconvexity of the penalty function, the estimate is a local minimizer.

3. Theoretical properties

Now we investigate the asymptotic properties of the proposed method. FAL and FAS considered p to be fixed. We study the situation where p can diverge with n. Let the true value of θ be θ^*, where the corresponding true values of $\gamma_{m0}, \gamma_{mj}, \gamma$ are $\gamma_{m0}^*, \gamma_{mj}^*, \gamma^*$, respectively. Let the number of nonzero elements in γ^* be s. To emphasize that s and p can go to infinity, we use s_n and p_n when necessary. For Theorems 1 and 2 (to be shown), p_n is at the order lower than $O(n^{1/2})$; for Theorem 3, p_n is at the order lower than $O(n^{1/6})$.

We define some index sets to be used in our theorems. Let $\mathcal{N} = \{(m, j) : 1 \leq m \leq M, 1 \leq j \leq p_n\}$. For the true parameters, define the oracle index set $I = \{(m, j) \in \mathcal{N} : \gamma_{mj}^* \neq 0\}$ and its complementary set $II = \{(m, j) \in \mathcal{N} : \gamma_{mj}^* = 0\}$. Assume that I has cardinality $|I| = s_n$.

We define some notations used for our theorems.

Define $d_{nI} = \max_{(m,j) \in I} \omega_{mj}$ and $d_{nII} = \min_{(m,j) \in II} \omega_{mj} \left(\max_{(m,j) \in \mathcal{N}} \omega_{mj} \right)^{-\frac{1}{2}}$. Define θ_I^* and $\hat{\theta}_I$ as the subvectors of the vectors θ^* and $\hat{\theta}$ corresponding to the oracle index set I, respectively. For every fixed $1 \leq m \leq M$, define the index set $I_m = \{1 \leq j \leq p_n : \gamma_{mj}^* \neq 0\}$. Let

$$\Sigma_n = (\Sigma_{lm})_{M \times M} \quad \text{with} \quad \Sigma_{lm} = (\min(\tau_m, \tau_l) - \tau_m \tau_l) E(U_{ill} U_{imI}^T), \tag{3}$$

$$\mathbf{B}_{nI} = \text{Diag}(B_1, B_2, \ldots, B_M) \quad \text{with} \quad B_m = \sum_{i=1}^{n} f(U_i^T \theta_m^* | Z_i) U_{imI} U_{imI}^T, \tag{4}$$

where $U_{imI} = (1, Z_{imI}^T)^T$ with Z_{imI} being the subvector of Z_i corresponding to index set I_m.

Let $F(y|z)$ and $f(y|z)$ be the conditional distribution function and the conditional density function of Y given $Z = z$, respectively. For any proper square matrix A, let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalue of A, respectively.

Before stating the main theorems, we need the following regularity conditions labeled by $\mathcal{L}$:

(L1) The conditional density $f(y|z)$ has first order derivative $f'(y|z)$ with respect to y; And $f(y|z)$ and $f'(y|z)$ are uniformly bounded away from 0 and ∞ on the support set of Y and the support set of Z;

(L2) For random sample $Z_i = (Z_{i1}, Z_{i2}, \ldots, Z_{ip})^T$, $1 \leq i \leq n$, there exists a positive constant C_1 such that $\max_{1 \leq i \leq n, 1 \leq j \leq p} |Z_{ij}| \leq C_1$;

(L3) For $U_i = (1, Z_i^T)^T$, $i = 1, \ldots, n$, let $\mathbf{S}_n = \sum_{i=1}^{n} U_i U_i^T$. There exist positive constants $C_2 < C_3$ such that $C_2 \leq \lambda_{\min}(n^{-1} \mathbf{S}_n) \leq \lambda_{\max}(n^{-1} \mathbf{S}_n) \leq C_3$;

(L4) The dimension s_n satisfies that $s_n = a_0 n^{\alpha_0}$, and the dimension p_n satisfies that $p_n = a_1 n^{\alpha_1}$, where $0 < \alpha_0 < \alpha_1 < \frac{1}{2}$, and a_0 and a_1 are two positive constants;

(L5) The matrix $\mathbf{B}_n$ given in (6) (see Appendix) satisfies that $C_4 \leq \lambda_{\min}(n^{-1} \mathbf{B}_n) \leq \lambda_{\max}(n^{-1} \mathbf{B}_n) \leq C_5$, where C_4 and C_5 are positive constants.

(L6) The matrix Σ_n satisfies that $\lambda_{\min}(\Sigma_n) \geq C_6$, where C_6 is a positive constant.

Conditions (L1)–(L3) and (L5)–(L6) are seen in typical theoretical investigation of quantile regression. Condition (L4) specifies the magnitude of s_n and p_n with respect to the sample size. Under the aforementioned regularity conditions, we present the following three theorems. The proof is relegated to the Appendix. Define θ_{II}^* and $\hat{\theta}_{II}$ as the subvectors of the vectors θ^* and $\hat{\theta}$ corresponding to the index set II, respectively. Clearly, $\theta_{II}^* = 0$. Due to the nonconvexity of the penalty function, all the following theorems and their proof (in the Appendix) are regarding to a local minimizer of the objective function.

Theorem 1. *Under conditions* (L1)–(L5), *if* $\lambda_n d_{nI}^{\frac{1}{2}} = o(s_n^{-\frac{3}{4}} p_n^{\frac{3}{4}} n^{-\frac{3}{4}})$, *then the estimator* $\hat{\theta}$ *of* θ^* *exists, is a local minimizer, and satisfies the estimation consistency that* $\|\hat{\theta} - \theta^*\|_2 = O_p(n^{-\frac{1}{2}} p_n^{\frac{1}{2}})$.

Theorem 1 shows that the proposed method is consistent in parameter estimation. The convergence rate $O_p(n^{-\frac{1}{2}} p_n^{\frac{1}{2}})$ is typical for the settings where p diverges with n.

Theorem 2. *Under conditions* (L1)–(L5), *if* $\lambda_n d_{nI}^{\frac{1}{2}} = o(n^{-\frac{3}{4}} s_n^{-\frac{3}{4}} p_n^{\frac{3}{4}})$ *and* $n^{-\frac{1}{2}} p_n^{\frac{1}{2}} = o(\lambda_n d_{nII})$, *then* $P(\hat{\theta}_{II} = 0) \to 1$.

Theorem 2 indicates that our method can distinguish the truly zero coefficients from the nonzero coefficients with probability tending to 1. It can be seen that the penalty weight d_{nII} plays a critical role in the property of selection consistency.

Theorem 3. *Under conditions* (L1)–(L3) *and* (L5)–(L6) *, if* $\lambda_n d_{nl}^{\frac{1}{2}} = O(n^{-1} p_n^{\frac{1}{2}})$*, and* $n^{-\frac{1}{2}} p_n^{\frac{1}{2}} = o(\lambda_n d_{nll})$*, and the powers of* s_n *and* p_n *in condition* (L4) *satisfy* $0 < \alpha_0 < \alpha_1 < \frac{1}{6}$*, then for any unit vector* $b \in \mathbb{R}^{M+s_n}$ *we have*

$$(n b^{\mathrm{T}} \Sigma_n b)^{-\frac{1}{2}} b^{\mathrm{T}} \mathbf{B}_{nl} (\hat{\theta}_l - \theta_l^*) \rightarrow N(0, 1).$$

Theorem 3 suggests that the estimated nonzero coefficients have the asymptotic normality. Heuristically, for given n, λ_n and ω_{mj}, the considered penalty in (2) has its slope tending to infinity when γ_{mj} goes to 0, thus the penalty tends to dominate small γ_{mj}. On the other hand, when λ_n is sufficiently small, the penalty has little impact on the estimation of relatively large γ_{mj}. These properties, in combination with proper choice of the tuning parameter, play major roles in the oracle property of the proposed estimator. The oracle property for coefficients within a group is mainly due to the penalty weights, which put large penalty on small coefficients (and small penalty on large coefficients).

4. Simulation studies

We conduct simulation studies to evaluate the proposed method along with the following methods: the QR method, which applies quantile regression to each individual quantile level without any variable selection; the QR-LASSO method, which adopts the L_1-penalized quantile regression for each quantile level; the QR-aLASSO method, which imposes the adaptive LASSO penalty on each quantile level (Wu and Liu, 2009); the FAL and the FAS method (Jiang et al., 2014). Both FAL and FAS contain a fused-LASSO type of penalty, which encourages the equality of the regression coefficients among different quantiles. FAL allows within-group sparsity, while FAS generates sparsity only at the group level. For Het-QR, we set the penalty weight ω_{mj} to be the inverse of the estimate from the unpenalized quantile regression (unless specified otherwise).

We first consider a model where important covariates have nonzero regression coefficients across all (or almost all) quantiles. We simulate 6 independent covariates, each of which follows the uniform(0,1) distribution. Then we simulate the trait as

$$Y = 1.0 + \beta_1 Z_1 + \beta_2 Z_2 + \beta_6 Z_6 + \kappa Z_6 \epsilon,$$

where $\beta_1 = 1$, $\beta_2 = 1$, $\beta_6 = 2$, $\kappa = 2$ and $\epsilon \sim N(0, 1)$. Under this set up, Z_1 and Z_2 have constant regression coefficients across all quantiles, while Z_6's regression coefficient is determined by $2 + 2 \times \Phi^{-1}(\tau)$, which varies across different quantiles. That is, the τth quantile of Y given Z_1, Z_2 and Z_6 is

$$Q_\tau (Y|Z_1, Z_2, Z_6) = 1.0 + Z_1 + Z_2 + (2 + 2 \times \Phi^{-1}(\tau)) Z_6.$$

This model is in line with the model considered by Jiang et al. (2014). All the other 3 covariates, Z_3, Z_4 and Z_5, have no contribution to Y. The sample size n is set to 500. To select the tuning parameter, we follow the lines of Mazumder et al. (2011) and Wang et al. (2012) to generate another dataset with sample size of $10n$, and then pick the tuning parameter at which the check loss function is minimized. The total number of simulations for each experiment is 100.

We consider various criteria to evaluate the performance of the compared methods, such as the model size and the parameter estimation error (PEE). The model size refers to the number of estimated non-zero coefficients among the M quantile levels. The PEE is calculated by $\sum_{m=1}^{M} \sum_{j=1}^{p} |\hat{\gamma}_{mj} - \gamma_{mj}^*| / M$. To evaluate the prediction error, we simulate an independent dataset, (Y_{pred}, Z_{pred}), with sample size of $100n$, and then calculate the F-measure (FM) (Gasso et al., 2009), the quantile prediction error (QPE) and the prediction error (PE). The FM is equal to $2 \times S_a / M_a$, where S_a is the number of truly nonzero slopes being captured, and M_a is the sum of the estimated model size and the true model size. The QPE is defined as the sample version of the $\sum_{m=1}^{M} (Q_{\tau_m}(Y_{pred}|Z_{pred}) - Z_{pred}^{\mathrm{T}} \hat{\gamma}_{\tau_m} - \hat{\gamma}_{m0})^2 / M$, averaged across all the subjects. The PE is defined as $Q_n(\hat{\theta})/n$, i.e., the check loss averaged across all considered quantiles for all the test samples, evaluated on (Y_{pred}, Z_{pred}).

For the purpose of illustration, we consider three quantiles, $\tau = 0.25, 0.5, 0.75$. The results are shown in the upper panel of Table 1. It can be seen that when p is 6, FAL has the lowest parameter estimation error and FAS has the lowest PE, though the difference between these two methods and the other compared methods is generally quite small. Next, we increase p to 100 to evaluate the methods under a higher dimension. As shown in the lower panel of Table 1, when p is equal to 100, both FAL and FAS have deteriorated performance; for instance, their model sizes tend to be twice (or more) as the true model size and their PEE and PE are higher than Het-QR. This experiment shows that the performance of FAL and FAS is suboptimal when the dimensionality grows large; one potential explanation is that the penalties of FAL and FAS may overemphasize the interquantile shrinkage, which make them less efficient when many noise covariates are present. Further research is merited. As to computation, we did not observe non-convergence for Het-QR in our experiments.

Next, we systematically evaluate the situation where within-group sparsity exists. To introduce correlations into covariates, we simulate 20 blocks of covariates, each block containing 5 correlated covariates. For each block, we first simulate a multivariate normal distribution with mean being the unit vector and covariance matrix following either the compound symmetry or the auto-regressive correlation structure with correlation coefficient $\rho = 0.5$; next, we take the absolute value of the simulated random normal variables as the covariates Z. The total number of covariates is 100. We specify the conditional quantile regression coefficient function $\gamma(\tau)$ as follows. For $\tau \in (0, 0.3]$, the first 8 regression

Table 1
Comparison of Het-QR and other methods in the absence of within-group sparsity (standard error of the sample mean shown in the parenthesis).

Method	Model-size	FM (%)	$PEE \times 100$	$QPE \times 10^3$	$PE \times 10^3$
	$p = 6$				
QR	18	–	53.3(1.5)	10.2(0.6)	1041.2(0.6)
QR-LASSO	15.0(0.2)	75(0.7)	36.1(1.2)	6.7(0.5)	1038.4(0.6)
QR-aLASSO	10.9(0.2)	91(0.7)	25.6(0.9)	5.8(0.4)	1037.2(0.5)
FAL	11.1(0.2)	91(0.9)	25.3(1.0)	6.2(0.5)	1037.2(0.5)
FAS	12.1(0.2)	86(0.9)	26.4(1.1)	5.9(0.4)	1037.1(0.5)
Het-QR	9.6(0.1)	97(0.6)	26.0(0.9)	6.5(0.5)	1037.5(0.5)
	$p = 100$				
QR	300	–	1556.4(12.7)	325.0(5.1)	1242.1(2.7)
QR-LASSO	47.3(1.2)	33(0.7)	120.5(3.5)	23.4(1.2)	1052.4(0.9)
QR-aLASSO	16.8(0.4)	72(1.1)	46.9(1.9)	10.5(0.8)	1042.1(0.7)
FAL	17.7(0.6)	70(1.4)	41.9(1.8)	10.2(0.8)	1041.0(0.6)
FAS	23.7(0.7)	58(1.2)	58.0(2.4)	13.9(0.9)	1043.4(0.7)
Het-QR	9.3(0.1)	99(0.4)	29.5(1.2)	8.4(0.6)	1039.9(0.6)

slopes for Z are $(0.5, 0, 0, 0, 0, 0.6, 0, 0)$; for $\tau \in (0.3, 0.7]$, the first 8 regression slopes are $(0.5, 0, 0, 0, 0, 0.6, 0, 0.7)$; for $\tau \in (0.7, 1.0)$, the corresponding slopes are $(0.6, 0, 0, 0, 0, 0.7, 0, 0.7)$. All other regression slopes are 0. Thus, the first and the sixth covariates are active among all quantiles, while the eighth covariate is active only for the last two quantile levels. To generate Y, we first simulate a random number $\tau \in$ Uniform $(0, 1)$, and then determine the $\gamma(\tau)$ based on τ; subsequently, we obtain

$$Y = 1.0 + Z^T \gamma(\tau) + F^{-1}(\tau),$$

where F^{-1} is the inverse cumulative function of some distribution F. That is, the τth quantile of Y given Z is

$$Q_\tau(Y|Z) = \begin{cases} 1.0 + F^{-1}(\tau) + 0.5Z_1 + 0.6Z_6 & \text{if } 0 < \tau \leq 0.3 \\ 1.0 + F^{-1}(\tau) + 0.5Z_1 + 0.6Z_6 + 0.7Z_8 & \text{if } 0.3 < \tau \leq 0.7 \\ 1.0 + F^{-1}(\tau) + 0.6Z_1 + 0.7Z_6 + 0.7Z_8 & \text{if } 0.7 < \tau < 1. \end{cases}$$

We explore different distributions for F: the standard normal distribution, the T-distribution with degrees of freedom equal to 3 (T_3), and the exponential distribution with shape parameter equal to 1.

We first consider the normal distribution for F. The results are shown in Table 2. Because no variable selection is conducted, QR has much larger PEE, QPE, and PE than the other methods; for example, the PEE and QPE of QR are more than 10 times higher than the compared methods. QR-LASSO, QR-aLASSO, FAL, and FAS have more tamed model sizes, but still contain a number of noise features. Het-QR yields a model that is closer to the true model, in which the three considered quantiles contain 2, 2, and 3 nonzero slopes, respectively. Het-QR also appears to have the highest FM, and lowest errors for parameter estimation and prediction. Next, we consider the distribution to be the T_3 (Table 3) and the exponential distribution (Table 4), and the results show a similar pattern. These experiments indicate that Het-QR can handle higher dimension as well as the heterogeneous sparsity better than the other methods.

We finally consider the situation where $p > n$. While theoretical development is still needed for this setting, our experiment is to evaluate the practical performance of the proposed approach. We let $n = 500$ and $p = 600$. For $\tau \in (0, 0.3]$, the first 8 regression slopes for Z are $(0.6, 0, 0, 0, 0, 0.6, 0, 0)$; for $\tau \in (0.3, 0.7]$, the first 8 regression slopes are $(0.6, 0, 0.8, 0, 0, 0.7, 0, 0.8)$; for $\tau \in (0.7, 1.0)$, the corresponding slopes are $(0.8, 0, 0.8, 0, 0, 0.8, 0, 1.0)$. In this scenario, Z_3 and Z_8 have zero coefficients for the first quantile, but nonzero coefficients for the other two quantiles. That is, the τth quantile of Y given Z is

$$Q_\tau(Y|Z) = \begin{cases} 1.0 + F^{-1}(\tau) + 0.6Z_1 + 0.6Z_6 & \text{if } 0 < \tau \leq 0.3 \\ 1.0 + F^{-1}(\tau) + 0.6Z_1 + 0.8Z_3 + 0.7Z_6 + 0.8Z_8 & \text{if } 0.3 < \tau \leq 0.7 \\ 1.0 + F^{-1}(\tau) + 0.8Z_1 + 0.8Z_3 + 0.8Z_6 + 1.0Z_8 & \text{if } 0.7 < \tau < 1. \end{cases}$$

We omit QR, FAL and FAS because they are not designed to handle the setting of '$p > n$'. QR-LASSO can be directly applied to data that have dimension higher than sample size. For QR-aLASSO, we derive the penalty weights using the estimates from QR-LASSO. For Het-QR, we first run Het-QR with the penalty weights equal to 1 to obtain the initial estimators for γ_m^*, $m = 1, \ldots, M$, and then use the inverse of the initial estimators as the penalty weights; finally, we run Het-QR to obtain the $\hat{\theta}$. The results are shown in Table 5. Het-QR tends to yield a smaller model than the compared methods and have better performance in estimating the regression coefficients as well as in prediction.

5. Real data analysis

We collect 206 brain tumor patients each with 91 gene expression levels. All patients were de-identified. All patients were diagnosed to have glioma, one of the deadliest cancers among all cancer types. Indeed, many patients died within 1

Table 2
Comparison of Het-QR and other methods for $p = 100$ under the normal distribution (standard error of the sample mean shown in the parenthesis).

Method	Model-size	FM (%)	$PEE \times 100$	$QPE \times 10^3$	$PE \times 10^3$
	Correlation structure: auto-regressive				
QR	300	–	1189.3(7.2)	791.6(9.1)	1606.5(3.1)
QR-LASSO	41.1(1.3)	34(0.9)	108.7(3.1)	78.5(3.1)	1353.1(1.2)
QR-aLASSO	17.8(0.5)	63(1.2)	60.4(2.5)	49.2(3.2)	1341.5(1.2)
FAL	20.3(0.8)	60(1.6)	59.5(2.6)	49.9(3.5)	1340.4(1.2)
FAS	24.8(1.1)	53(1.5)	85.1(2.6)	86.4(3.1)	1351.2(1.1)
Het-QR	8.6(0.1)	96(0.7)	34.7(1.6)	32.6(2.8)	1334.9(1.1)
	Correlation structure: compound symmetry				
QR	300	–	1218.1(8.4)	792.7(9.5)	1606.1(3.4)
QR-LASSO	40.1(1.1)	35.0(0.9)	105.9(3.1)	74.8(3.0)	1351.9(1.3)
QR-aLASSO	18.9(0.6)	61(1.3)	61.2(2.7)	48.0(3.1)	1341.6(1.3)
FAL	19.9(0.8)	61(1.4)	61.0(2.9)	54.4(4.0)	1341.2(1.3)
FAS	24.1(1.1)	55(1.6)	83.8(2.8)	89.3(3.2)	1351.3(1.2)
Het-QR	8.9(0.2)	94(0.8)	34.6(1.9)	31.4(2.9)	1334.7(1.2)

Table 3
Comparison of the Het-QR and other methods for $p = 100$ under the T_3 distribution (standard error of the sample mean shown in the parenthesis).

Method	Model-size	FM (%)	$PEE \times 100$	$QPE \times 10^3$	$PE \times 10^3$
	Correlation structure: auto-regressive				
QR	300	–	1452.1(10.2)	1149.1(15.2)	2114.3(4.5)
QR-LASSO	39.4(1.3)	35(0.9)	123.6(3.3)	103.7(3.7)	1796.8(1.4)
QR-aLASSO	19.7(0.6)	58(1.3)	81.0(3.0)	77.3(4.2)	1787.7(1.6)
FAL	20.6(0.7)	58(1.3)	76.4(3.1)	74.8(4.6)	1786.2(1.6)
FAS	24.2(0.9)	53(1.3)	99.6(3.4)	107.8(4.2)	1795.6(1.6)
Het-QR	8.9(0.2)	92(1.0)	47.3(2.3)	53.5(4.0)	1779.4(1.5)
	Correlation structure: compound symmetry				
QR	300	–	1487.4(11.2)	1156.3(15.6)	2115.2(4.7)
QR-LASSO	38.7(1.0)	35(1.2)	120.5(3.2)	99.4(3.4)	1795.5(1.6)
QR-aLASSO	20.3(0.7)	56(1.3)	81.6(3.3)	76.0(4.1)	1787.6(1.8)
FAL	22.5(0.9)	56(1.5)	82.5(3.9)	78.1(5.1)	1786.6(1.8)
FAS	25.6(1.0)	51(1.3)	99.1(3.5)	107.0(4.1)	1794.7(1.7)
Het-QR	9.1(0.3)	91(1.1)	48.6(2.9)	55.4(4.7)	1780.1(1.8)

Table 4
Comparison of Het-QR and other methods for $p = 100$ under the exponential distribution (standard error of the sample mean shown in the parenthesis).

Method	Model-size	FM (%)	$PEE \times 100$	$QPE \times 10^3$	$PE \times 10^3$
	Correlation structure: auto-regressive				
QR	300	–	1024.2(7.1)	618.4(8.8)	1446.4(3.1)
QR-LASSO	39.3(1.1)	36(0.9)	88.7(2.5)	60.6(2.2)	1220.9(1.0)
QR-aLASSO	17.1(0.5)	66(1.2)	47.9(1.9)	37.0(2.3)	1210.2(0.9)
FAL	18.9(0.7)	63(1.5)	43.0(1.9)	33.3(3.0)	1209.5(1.0)
FAS	26.9(1.2)	51(1.7)	74.5(2.3)	79.1(3.5)	1223.8(1.1)
Het-QR	8.7(0.1)	96(0.6)	28.8(1.3)	24.2(1.8)	1205.4(0.8)
	Correlation structure: compound symmetry				
QR	300	–	1041.6(7.5)	610.9(8.3)	1444.4(2.9)
QR-LASSO	39.6(1.2)	36(0.9)	86.1(2.6)	58.0(2.4)	1220.2(1.1)
QR-aLASSO	17.0(0.5)	66(1.3)	46.6(2.0)	35.7(2.3)	1210.1(1.1)
FAL	18.7(0.7)	63(1.4)	41.8(2.0)	31.0(3.0)	1208.8(1.1)
FAS	27.6(1.4)	51(1.8)	74.3(2.5)	79.6(3.3)	1223.5(1.1)
Het-QR	8.7(0.1)	96(0.7)	27.2(1.3)	22.3(1.9)	1205.1(0.9)

year after the diagnosis. Glioma is associated with a number of genes. We focus on the PDGFRA gene, which encodes the alpha-type platelet-derived growth factor receptor and has been shown to be an important gene for brain tumors (Holland, 2000; Puputti et al., 2006). We use this dataset to investigate how the expression of PDGFRA is influenced by other genes.

For demonstration, we set τ to 0.25, 0.5, and 0.75. For QR-LASSO, QR-aLASSO, and Het-QR, we use cross-validation to ascertain the tuning parameter. That is, (1) we divide the data into 3 folds; (2) we use 2 folds to build the model and 1 fold to calculate the prediction error, and this is done three times; (3) we choose the λ_n that minimizes the prediction error as the best tuning parameter (we were not able to obtain an independent dataset with sample size $10n$ to determine

Table 5
Comparison of Het-QR and other methods for $p > n$ (standard error of the sample mean shown in the parenthesis).

Method	Model-size	FM (%)	$PEE \times 100$	$QPE \times 10^3$	$PE \times 10^3$
Correlation structure: auto-regressive					
QR-LASSO	67.6(1.8)	25(0.6)	199.0(4.2)	215.5(6.0)	1801.4(1.7)
QR-aLASSO	15.2(0.3)	73(1.1)	81.3(3.0)	113.3(6.2)	1768.8(1.6)
Het-QR	10.6(0.1)	96(0.6)	44.6(2.5)	61.0(7.0)	1753.9(1.6)
Correlation structure: compound symmetry					
QR-LASSO	62.9(1.8)	27(0.7)	182.4(4.2)	199.8(5.9)	1796.4(1.8)
QR-aLASSO	15.2(0.3)	74(1.0)	76.7(2.9)	103.5(5.9)	1766.6(1.6)
Het-QR	10.6(0.1)	96(0.7)	46.4(3.0)	63.1(7.8)	1754.2(1.7)

the tuning parameter; it would be meaningful to compare the two procedures when such a dataset becomes available in future). For FAL and FAS, we follow Jiang et al. (2014) to use BIC and AIC for determining the tuning parameter, and the corresponding methods are named as FAL-BIC, FAL-AIC, FAS-BIC, FAS-AIC. Hence, in total 7 approaches are compared. We first examine the model sizes. For all the three quantiles combined, the number of nonzero covariates of the seven models are 89 (QR-LASSO), 47 (QR-aLASSO), 25 (Het-QR), 49 (FAL-BIC), 182 (FAL-AIC), 93 (FAS-BIC), 167 (FAS-AIC). For illustration, we list some of the estimated regression coefficients in Table 6. It can be seen that the coefficients for a given gene often differ among different quantiles. For a better view of the regression coefficients among different quantiles, we plot the estimated coefficients for the first 30 covariates (Fig. 1). Table 6 and Fig. 1 show that most models (except FAS-BIC and FAS-AIC) demonstrate heterogeneous sparsity, i.e., some covariates have nonzero effects in only one or two of the three quantiles. FAS-BIC and FAS-AIC do not show this type of sparsity due to the sup-norm penalty they adopt, as this penalty either selects or removes a covariate for all the quantiles. FAL-AIC and FAS-AIC models contain more nonzero estimates than FAL-BIC and FAS-BIC, consistent with the fact that BIC favors smaller models than AIC. Compared to other methods, Het-QR yields a smaller model which may be easier to interpret and prioritize candidate genes for further functional study.

The covariates selected by Het-QR are shown in Table 7. Consistent with the model assumption, the estimated regression coefficients show heterogeneity among quantiles. For example, the CDKN2C gene has zero coefficient at $\tau = 0.25$, and nonzero coefficients at $\tau = 0.5$ and 0.75. In contrast, some other genes, such as BMP2 and SLC4A4, have nonzero coefficients across all the considered quantiles. This suggests that the expression of PDGFRA is influenced by other genes in a delicate manner that may not be fully characterized by least square methods or quantile regression methods that fail to account for the genetic heterogeneity. CDKN2C encodes a cyclin-dependent kinase, and BMP2 and SLC4A4 encode a bone morphogenetic protein and a sodium bicarbonate cotransporter, respectively. This indicates that PDGFRA's expression is associated with genes with a wide spectrum of cellular functions. The gene EGFR has non-positive regression coefficients, suggesting that there may be some negative control between PDGFRA and EGFR. Future biological studies may provide new insight into the gene regulation of PDGFRA.

One main purpose of variable selection is to apply the selected variables from one dataset to other datasets to guide statistical analysis. Along this line, we further collect the brain tumor data from the cancer genome atlas (TCGA) project, which contains 567 subjects. We apply the models selected by different methods from the training data to the TCGA data to assess the prediction accuracy of the different models. We randomly split the TCGA data into two halves, and use one half to estimate the regression coefficients and the other half to calculate the prediction error; the prediction error is then averaged across the two halves. We repeat the random-splitting 400 times, and calculate the average of the prediction errors. Het-QR appears to have a slightly lower prediction error than the other compared ones, but the difference among the seven methods is generally small; in detail, the observed prediction errors are 1.349 (QR-LASSO), 1.351 (QR-aLASSO), 1.345 (Het-QR), 1.362 (FAL-BIC), 1.513 (FAL-AIC), 1.355 (FAS-BIC), and 1.430 (FAS-AIC).

6. Discussion

In this article, we have proposed a variable selection method that is able to conduct joint variable selection and estimation for multiple quantiles simultaneously. The joint selection/estimation allows one to harness the strength shared among multiple quantiles and to achieve a model that is closer to the truth. In particular, our approach is able to handle the heterogeneous sparsity, under which a covariate contributes to some (but not all) of the quantiles. By considering the heterogeneous sparsity, one can better dissect the regression structure of the trait over the covariates, which in turn leads to more accurate characterization of the underlying biological mechanism.

We have conducted a series of simulation studies to evaluate the performance of our proposed approach and other approaches. Our simulation studies show that the proposed method has superior performance to its peer methods. In real data analysis, our method tends to yield a sparser model than the compared methods. The benefit of achieving a sparse model is of great importance to biological studies, because it helps biological investigators to narrow down important candidate covariates (such as genes or proteins), so that research efforts can be leveraged more efficiently. Our analysis indicates that the regression coefficients at different quantiles can be quite heterogeneous. We suggest that the interpretation of the results be guided by biological knowledge and scientific insight, and that the variability be examined by experimental
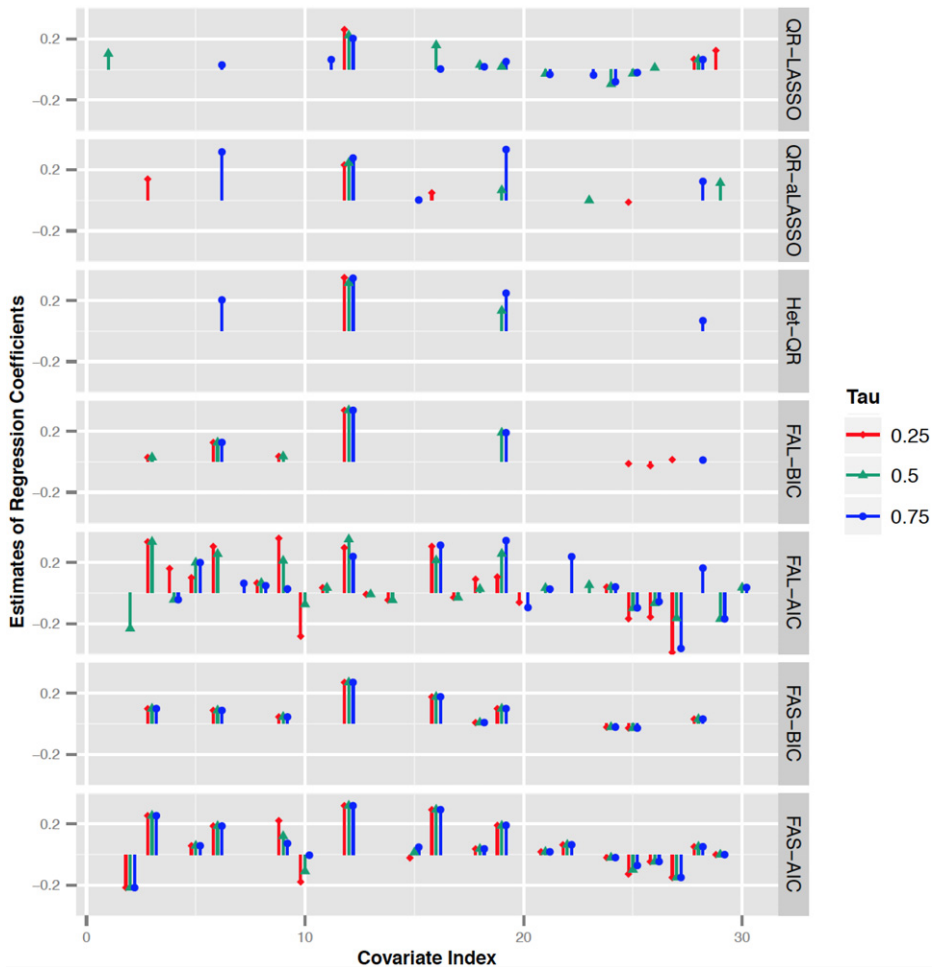
Fig. 1. Graphic view of the first 30 regression coefficients estimated by different methods. Estimates are thresholded at 0.4 and −0.4, and only nonzero estimates are shown.

Table 6
A snapshot of the estimated regression coefficients (only 5 covariates are shown).

Gene	τ	QR-LASSO	QR-aLASSO	Het-QR	FAL-BIC	FAL-AIC	FAS-BIC	FAS-AIC
POLR2A	0.25					0.10		0.06
	0.5					0.20		0.06
	0.75					0.20		0.06
SDHA	0.25				0.13	0.30	0.09	0.19
	0.5				0.13	0.26	0.09	0.19
	0.75	0.03	0.32	0.2	0.13	0.58	0.09	0.19
CDKN2A	0.25					0.09	0.01	0.04
	0.5	0.03				0.03	0.01	0.04
	0.75	0.02					0.01	0.04
CDKN2C	0.25					0.10	0.10	0.19
	0.5	0.02	0.07	0.13	0.19	0.26	0.10	0.19
	0.75	0.05	0.33	0.25	0.19	0.34	0.10	0.19
DLL3	0.25	0.07					0.03	0.05
	0.5	0.06					0.03	0.05
	0.75	0.06	0.12	0.07	0.01	0.16	0.03	0.05

Note: Zero estimates are left blank.

studies. FAL and FAS were mainly designed to generate interquantile shrinkage for quantile regression; when a smooth $\gamma(\tau)$ (with respect to τ) is desired, these two methods are highly suitable and are indeed the only available methods to achieve such a goal.

Table 7
The model selected by the Het-QR method.

Gene	Estimated regression coefficients		
	$\tau = 0.25$	$\tau = 0.5$	$\tau = 0.75$
SDHA			0.2
BMP2	0.35	0.31	0.34
CDKN2C		0.13	0.25
DLL3			0.07
EGFR		−0.08	−0.29
GRIA2	0.28	0.22	0.18
LTF			0.07
OLIG2	0.14	0.30	0.38
PLAT	0.20	0.21	0.25
SLC4A4	−0.21	−0.25	−0.24
TAGLN			−0.20
TMEM100	0.20	0.17	

We have also provided theoretical proof for the proposed method under the situation that p can grow to infinity. Our exploratory experiments suggest that Het-QR can be potentially applied to '$p > n$', although theoretical work is still needed to guide future experiments in this direction. Wang et al. (2012) proposed a novel approach for studying asymptotics under the '$p > n$' situation, and they focused on the penalties that can be written as the difference of two convex functions. The group penalty considered herein does not seem to fall into their framework. Further theoretical development is merited. While we have imposed equal weights for multiple quantiles in this paper, our method can be easily extended to accommodate different weights for different quantiles. Properly chosen weights may lead to improved efficiency of the estimated parameters (Zhao and Xiao, 2014).

Acknowledgments

Appendix

Transformation of the objective function

Our proof is in vein with the proof of Proposition 1 in Huang et al. (2009). Consider the transformed objective function

$$\min_{\theta,\xi} \sum_{m=1}^{M} \sum_{i=1}^{n} \rho_m(Y_i - U_i^{\mathrm{T}}\theta_m) + \lambda_1 \sum_{j=1}^{p} \xi_j + \sum_{j=1}^{p} \xi_j^{-1} \left(\sum_{m=1}^{M} \omega_{mj}|\gamma_{mj}| \right)$$

$$= \min_{\theta} \left\{ \sum_{m=1}^{M} \sum_{i=1}^{n} \rho_m(Y_i - U_i^{\mathrm{T}}\theta_m) + \min_{\xi} \left\{ \lambda_1 \sum_{j=1}^{p} \xi_j + \sum_{j=1}^{p} \xi_j^{-1} \left(\sum_{m=1}^{M} \omega_{mj}|\gamma_{mj}| \right) \right\} \right\} \quad (5)$$

By Cauchy–Schwarz inequality, we have

$$\lambda_1 \sum_{j=1}^{p} \xi_j + \sum_{j=1}^{p} \xi_j^{-1} \left(\sum_{m=1}^{M} \omega_{mj}|\gamma_{mj}| \right) \geq 2\sqrt{\lambda_1} \sum_{j=1}^{p} \left(\sum_{m=1}^{M} \omega_{mj}|\gamma_{mj}| \right)^{\frac{1}{2}},$$

then it follows that (5) is equivalent to

$$\min_{\theta} \sum_{m=1}^{M} \sum_{i=1}^{n} \rho_m(Y_i - U_i^{\mathrm{T}}\theta_m) + 2\sqrt{\lambda_1} \sum_{j=1}^{p} \left(\sum_{m=1}^{M} \omega_{mj}|\gamma_{mj}| \right)^{\frac{1}{2}}. \quad (6)$$

Now, let $2\sqrt{\lambda_1} = n\lambda_n$, then (6) is identical to the original objective function (2). $\quad \square$

Derivation of the primal and dual problem

Let $\lambda_{mj} = \xi_j^{-1}\omega_{mj}$, then in step 2 of Section 2.2, we aim to solve

$$\min_{\gamma,\gamma_0}\left\{\sum_{m=1}^{M}\sum_{i=1}^{n}\rho_m(Y_i - Z_i^{\mathsf{T}}\gamma_m - \gamma_{m0}) + \sum_{j=1}^{p}\sum_{m=1}^{M}\lambda_{mj}|\gamma_{mj}|\right\}.$$

Let e_n denote the unit vector of length n, and $\boldsymbol{\lambda}_m$ the vector of $\lambda_{mj}(j = 1, \ldots, p)$. With slight abuse of notation, let Y be the $n \times 1$ vector consisting of Y_i, and Z the $n \times p$ matrix consisting of Z_i. The above objective function is equivalent to

$$\min_{u_m,v_m,\gamma_m,s_m,t_m}\sum_{m=1}^{M}\tau_m e_n^T u_m + (1-\tau_m)e_n^T v_m + \boldsymbol{\lambda}_m^T s_m + \boldsymbol{\lambda}_m^T t_m,$$

subject to $u_m - v_m = Y - Z\gamma_m - \gamma_{m0}e_n$, $s_m - t_m = \gamma_m$, $u_m \geq 0$, $v_m \geq 0$, $s_m \geq 0$, and $t_m \geq 0$.

Let $\mathbf{0}_r$ be the zero vector of length r, $\boldsymbol{\lambda}^* = (\boldsymbol{\lambda}_1^T, \ldots, \boldsymbol{\lambda}_M^T)^T$, $s^* = (s_1^T, \ldots, s_M^T)^T$, $t^* = (t_1^T, \ldots, t_M^T)$, $u^* = (u_1^T, \ldots, u_M^T)^T$, $v^* = (v_1^T, \ldots, v_M^T)^T$. Let $Y_{(M)}$ denote the vector in which Y is stacked by M times. Let

$$c = (\mathbf{0}_{Mp}^T, \mathbf{0}_M^T, \lambda^{*T}, \lambda^{*T}, \tau_1 e_n^T, \ldots, \tau_M e_n^T, (1-\tau_1)e_n^T, \ldots, (1-\tau_M)e_n^T)^T,$$

$x = (\gamma^T, \gamma_0^T, s^{*T}, t^{*T}, u^{*T}, v^{*T})^T$, and $b = (Y_{(M)}^T, \mathbf{0}_{Mp}^T)^T$. Then, the linear program primal of the above objective function can be written as

$$\min_x c^T x,$$

subject to $Ax = b$ and $(s^{*T}, t^{*T}, u^{*T}, v^{*T})^T \geq 0$, where A is defined as follows. A is a matrix consisting of two rows of blocks. The first row of A consists of 6 blocks, $A_{11} = I_M \otimes Z$, $A_{12} = I_M \otimes e_n$, $A_{13} = [\mathbf{0}]_{nM \times Mp}$, $A_{14} = [\mathbf{0}]_{nM \times Mp}$, $A_{15} = I_{nM}$, and $A_{16} = -I_{nM}$. The second row of A consists of the following 6 blocks, $A_{21} = I_{Mp}$, $A_{22} = [0]_{Mp \times M}$, $A_{23} = -I_{Mp}$, $A_{24} = I_{Mp}$, $A_{25} = [\mathbf{0}]_{nM \times nM}$, and $A_{26} = [\mathbf{0}]_{nM \times nM}$.

Then using standard linear program arguments, we obtain the dual as

$$\max_{\tilde{d}} b^T \tilde{d},$$

subject to $\tilde{Z}^T \tilde{d} = S_1 + S_2$ and $\tilde{d} \in [0, 1]^{nM+Mp}$, where

$$S_1 = \left((1-\tau_1)e_n^T Z, \ldots, (1-\tau_M)e_n^T Z, n(1-\tau_1), \ldots, n(1-\tau_M)\right)^T,$$

$S_2 = 1/2 \times (R, [\mathbf{0}]_{Mp \times M})^T e_{Mp}$, $R = \mathrm{diag}((2\boldsymbol{\lambda}^T, \ldots, 2\boldsymbol{\lambda}_M^T)^T)$, and $\tilde{Z}$ is defined as follows. $\tilde{Z}$ consists of two rows of blocks. The first row of $\tilde{Z}$ includes two blocks, $\tilde{Z}_{11} = I_M \otimes Z$ and $\tilde{Z}_{12} = I_M \otimes e_n$. The second row includes two blocks, $\tilde{Z}_{21} = R$ and $\tilde{Z}_{22} = [\mathbf{0}]_{Mp \times M}$.

Computation time

For $p = 100$ under the auto-regressive structure, i.e., Table 2, we calculate the summary statistics of the CPU time (seconds). The average time (and the standard error of the sample mean) for QR, QR-LASSO, QR-aLASSO, FAL, FAS, Het-QR is 1.6(0.003), 10.2(0.017), 10.0(0.030), 867.0(8.059), 455.3(2.641), 90.3(0.391), respectively. For $p = 600$ under the auto-regressive structure, the CPU time for QR-LASSO is 372.6(1.453); the time for QR-aLASSO and Het-QR is 362.0(1.729) and 3717.5(36.401), respectively (excluding the time for calculating penalty weights).

Proof of the theorems

We now give the proof of the theorems. We note that throughout the proof, the upper letter C in different formulas stands for different constants.

Recall that the definitions of $\mathcal{N}$, I, II, s_n and I_m have been given in the main text. Define the index set $J = \{1 \leq j \leq p_n : \text{there exists } 1 \leq m \leq M \text{ such that } \gamma_{mj}^* \neq 0\}$ with the cardinality $|J| = k_n$. For every fixed $j \in J$, define the index set $M_j = \{1 \leq m \leq M : \gamma_{mj}^* \neq 0\}$. Clearly, the oracle index set $I = \{(m, j) : m \in M_j, j \in J\}$. For every fixed $1 \leq m \leq M$, define $II_m = \{1 \leq j \leq p_n : \gamma_{mj}^* = 0\}$.

We need to define some notations for our proof. Let the vectors γ_{ml}, γ_{ml}^* and $\hat{\gamma}_{ml}$ be the subvectors of γ_m, γ_m^* and $\hat{\gamma}_m$ corresponding to the index set I_m, respectively. Define the subvectors of γ, γ^* and $\hat{\gamma}$ corresponding to the oracle index set I as $\gamma_I = (\gamma_{1I}^T, \ldots, \gamma_{MI}^T)^T$, $\gamma_I^* = (\gamma_{1I}^{*T}, \ldots, \gamma_{MI}^{*T})^T$ and $\hat{\gamma}_I = (\hat{\gamma}_{1I}^T, \ldots, \hat{\gamma}_{MI}^T)^T$. Let $\theta_{ml} = (\gamma_{m0}, \gamma_{ml}^T)^T$, $\theta_{ml}^* = (\gamma_{m0}^*, \gamma_{ml}^{*T})^T$ and $\hat{\theta}_{ml} = (\hat{\gamma}_{m0}, \hat{\gamma}_{ml}^T)^T$. Define the vector θ_I as the subvector of the parameter vector θ corresponding to the oracle index set I. Recall that the vectors θ_I^* and $\hat{\theta}_I$ are the subvectors of the vectors θ^* and $\hat{\theta}$ corresponding to the oracle index set I. Clearly, $\theta_I = (\theta_{1I}^{\mathsf{T}}, \ldots, \theta_{MI}^{\mathsf{T}})^{\mathsf{T}}$, $\theta_I^* = (\theta_{1I}^{*\mathsf{T}}, \ldots, \theta_{MI}^{*\mathsf{T}})^{\mathsf{T}}$ and $\hat{\theta}_I = (\hat{\theta}_{1I}^{\mathsf{T}}, \ldots, \hat{\theta}_{MI}^{\mathsf{T}})^{\mathsf{T}}$.

Similarly, let the vectors γ_{mII}, γ_{mII}^* and $\hat{\gamma}_{mII}$ be the subvectors of γ_m, γ_m^* and $\hat{\gamma}_m$ corresponding to the index set II_m, respectively. Define the subvectors of γ, γ^* and $\hat{\gamma}$ corresponding to the index set II as $\gamma_{II} = (\gamma_{1II}^\mathrm{T}, \ldots, \gamma_{MII}^\mathrm{T})^\mathrm{T}$, $\gamma_{II}^* = (\gamma_{1II}^{*\mathrm{T}}, \ldots, \gamma_{MII}^{*\mathrm{T}})^\mathrm{T}$ and $\hat{\gamma}_{II} = (\hat{\gamma}_{1II}^\mathrm{T}, \ldots, \hat{\gamma}_{MII}^\mathrm{T})^\mathrm{T}$. Define the vector θ_{II} as the subvector of the parameter vector θ corresponding to the index set II. Recall that the vectors θ_{II}^* and $\hat{\theta}_{II}$ are the subvectors of the vectors θ^* and $\hat{\theta}$ corresponding to the index set II, respectively. Clearly, $\theta_{II} = \gamma_{II}, \theta_{II}^* = \gamma_{II}^* = 0$ and $\hat{\theta}_{II} = \hat{\gamma}_{II}$.

For convenience, write $\gamma = (\gamma_I^\mathrm{T}, \gamma_{II}^\mathrm{T})^\mathrm{T}$, $\gamma^* = (\gamma_I^{*\mathrm{T}}, \gamma_{II}^{*\mathrm{T}})^\mathrm{T}$ and $\hat{\gamma} = (\hat{\gamma}_I^\mathrm{T}, \hat{\gamma}_{II}^\mathrm{T})^\mathrm{T}$; and $\theta = (\theta_I^\mathrm{T}, \theta_{II}^\mathrm{T})^\mathrm{T}$, $\theta^* = (\theta_I^{*\mathrm{T}}, \theta_{II}^{*\mathrm{T}})^\mathrm{T}$ and $\hat{\theta} = (\hat{\theta}_I^\mathrm{T}, \hat{\theta}_{II}^\mathrm{T})^\mathrm{T}$.

We first give a lemma related to the loss function $Q_n(\theta)$. The lemma plays an important role in the proof of our theorems.

Lemma .1. *Under conditions* (L1)–(L4) *, we have*

$$Q_n(\theta) = Q_n(\theta^*) + \mathbf{A}_n^\mathrm{T}(\theta - \theta^*) + \frac{1}{2}(\theta - \theta^*)^\mathrm{T}\mathbf{B}_n(\theta - \theta^*) + R_n(\theta),$$

where $\sup_{1 \le m \le M} \sup_{\|\theta_m - \theta_m^*\|_2 \le \eta_n} |R_n(\theta)| = O_p(n^{\frac{1}{2}}p^{\frac{3}{4}}\eta_n^{\frac{3}{2}}) + O_p(np^{\frac{1}{2}}\eta_n^3)$, *and*

$$\mathbf{A}_n = (A_1^\mathrm{T}, \ldots, A_M^\mathrm{T})^\mathrm{T} \quad \text{with } A_m = \sum_{i=1}^n \left(I(Y_i < U_i^\mathrm{T}\theta_m^*) - \tau_m\right)U_i,$$

$$\mathbf{B}_n = \mathrm{Diag}(B_{11}, B_{22}, \ldots, B_{MM}) \quad \text{with } B_{mm} = \sum_{i=1}^n f(U_i^\mathrm{T}\theta_m^* | Z_i)\, U_i U_i^\mathrm{T}. \tag{7}$$

Proof of Lemma .1. Let $\psi_m(u)$ be a sub-derivative of the quantile function $\rho_m(u)$, then $\psi_m(u) = \tau_m - I(u < 0) + l_m I(u = 0)$ with $l_m \in [-1, 0]$. Let $T_n = Q_n(\theta) - Q_n(\theta^*)$. Then, there exists an $l_{mi} \in [-1, 0]$ for every $1 \le m \le M$ and $1 \le i \le n$ such that

$$
\begin{aligned}
T_n &= -\sum_{m=1}^M \sum_{i=1}^n \psi_m(Y_i - U_i^\mathrm{T}\bar{\theta}_m)\, U_i^\mathrm{T}(\theta_m - \theta_m^*) \\
&= -\sum_{m=1}^M \sum_{i=1}^n \left(\tau_m - I(Y_i < U_i^\mathrm{T}\bar{\theta}_m) + l_{mi}\, I(Y_i = U_i^\mathrm{T}\bar{\theta}_m)\right) U_i^\mathrm{T}(\theta_m - \theta_m^*) \\
&= \sum_{m=1}^M \sum_{i=1}^n \left(I(Y_i < U_i^\mathrm{T}\theta_m^*) - \tau_m\right) U_i^\mathrm{T}(\theta_m - \theta_m^*) \\
&\quad - \sum_{m=1}^M \sum_{i=1}^n \left(I(Y_i < U_i^\mathrm{T}\theta_m^*) - I(Y_i < U_i^\mathrm{T}\bar{\theta}_m)\right)U_i^\mathrm{T}(\theta_m - \theta_m^*) - \sum_{m=1}^M \sum_{i=1}^n l_{mi}\, I(Y_i = U_i^\mathrm{T}\bar{\theta}_m)U_i^\mathrm{T}(\theta_m - \theta_m^*) \\
&\equiv T_{n1} - T_{n2} - T_{n3}
\end{aligned}
\tag{8}
$$

where $\bar{\theta}_m$ is on the linear segment between θ_m and θ_m^*, and may be written as $\bar{\theta}_m = \theta_m^* + \eta_m(\theta_m - \theta_m^*)$ with $\eta_m \in (0, 1)$. For T_{n3}, note that Y_i has a continuous conditional distribution given Z_i, hence almost surely $I(Y_i = U_i^\mathrm{T}\bar{\theta}_m) = 0$ for all $i = 1, \ldots, n$ and $m = 1, \ldots, M$, thus $T_{n3} = 0$ almost surely. Subsequently, we can write

$$T_n = \left(T_{n1} - \mathbf{E}(T_{n1})\right) - \left(T_{n2} - \mathbf{E}(T_{n2})\right) + \mathbf{E}(T_n),$$

where $\mathbf{E}(T_n) = \mathbf{E}(T_{n1}) + \mathbf{E}(T_{n2})$, and $\mathbf{E}$ denotes the conditional expectation given Z. Note that $\mathbf{E}(T_{n1}) = 0$ because $\mathbf{E}(I(Y_i < U_i^\mathrm{T}\theta_m^*) - \tau_m | Z_i) = F(U_i^\mathrm{T}\theta_m^* | Z_i) - \tau_m = 0$ from (1). Rename $\left(T_{n2} - \mathbf{E}(T_{n2})\right)$ as R_{n2} and $\mathbf{E}(T_n)$ as T_{n4}, then we have

$$T_n = T_{n1} - R_{n2} + T_{n4}. \tag{9}$$

For R_{n2} (recall T_{n2} in (8)), let $\zeta_{mi}(t) = I(Y_i - U_i^\mathrm{T}\theta_m^* < 0) - I(Y_i - U_i^\mathrm{T}\theta_m^* < U_i^\mathrm{T}t)$, then

$$
\begin{aligned}
R_{n2} &= \sum_{m=1}^M \left(\sum_{i=1}^n \{\zeta_{mi}(\eta_m(\theta_m - \theta_m^*)) - \mathbf{E}(\zeta_{mi}(\eta_m(\theta_m - \theta_m^*)))\} U_i\right)^\mathrm{T}(\theta_m - \theta_m^*) \\
&= \sum_{m=1}^M \phi_n(m)^\mathrm{T}(\theta_m - \theta_m^*)
\end{aligned}
\tag{10}
$$

where $\phi_n(m) = \sum_{i=1}^n \{\zeta_{mi}(\eta_m(\theta_m - \theta_m^*)) - \mathbf{E}(\zeta_{mi}(\eta_m(\theta_m - \theta_m^*)))\} U_i$.

Note that $|\zeta_{mi}(t)| = |I(Y_i - U_i^T \theta_m^* < 0) - I(Y_i - U_i^T \theta_m^* < U_i^T t)| \leq I(|Y_i - U_i^T \theta_m^*| \leq |U_i^T t|)$, and that $f(t|Z_i)$ is bounded under condition (L1) and $\|U_i\|_2 \leq Cp^{\frac{1}{2}}$ under condition (L2). Hence, making use of independence, under conditions (L1) and (L2), for all $1 \leq m \leq M$ and $\|\theta_m - \theta_m^*\|_2 \leq \eta_n$, we can see

$$
\begin{aligned}
\mathbf{E}\|\phi_n(m)\|_2^2 &= \mathbf{E} \left\| \sum_{i=1}^n \{\zeta_{mi}(\eta_m(\theta_m - \theta_m^*)) - \mathbf{E}(\zeta_{mi}(\eta_m(\theta_m - \theta_m^*)))\} U_i \right\|_2^2 \\
&= \sum_{i=1}^n \mathbf{E}\{\zeta_{mi}(\eta_m(\theta_m - \theta_m^*)) - \mathbf{E}(\zeta_{mi}(\eta_m(\theta_m - \theta_m^*)))\}^2 \|U_i\|_2^2 \\
&\leq Cnp\mathbf{E}\zeta_{mi}(\eta_m(\theta_m - \theta_m^*))^2 \leq Cnp\, P\left(|Y_i - U_i^T \theta_m^*| \leq |U_i^T(\theta_m - \theta_m^*)|\right) \\
&\leq CnpP\left(|Y_i - U_i^T \theta_m^*| \leq Cp^{\frac{1}{2}}\|\theta_m - \theta_m^*\|_2\right).
\end{aligned}
$$

Note that

$$
P\left(|Y_i - U_i^T \theta_m^*| \leq Cp^{\frac{1}{2}}\|\theta_m - \theta_m^*\|_2\right) = F(U_i^T \theta_m^* + Cp^{\frac{1}{2}}\|\theta_m - \theta_m^*\|_2) - F(U_i^T \theta_m^* - Cp^{\frac{1}{2}}\|\theta_m - \theta_m^*\|_2)
$$

$$
\leq |f(\xi_{mi}|Z_i)|p^{\frac{1}{2}}\|\theta_m - \theta_m^*\|_2 \leq Cp^{\frac{1}{2}}\|\theta_m - \theta_m^*\|_2,
$$

where ξ_{mi} is between $U_i^T \theta_m^* + Cp^{\frac{1}{2}}\|\theta_m - \theta_m^*\|_2$ and $U_i^T \theta_m^* - Cp^{\frac{1}{2}}\|\theta_m - \theta_m^*\|_2$. Thus, $\mathbf{E}\|\phi_n(m)\|_2^2 \leq Cnp^{\frac{3}{2}}\eta_n$. By Chebyshev's inequality, we get $\sup_{1 \leq m \leq M} \sup_{\|\theta_m - \theta_m^*\|_2 \leq \eta_n} \|\phi_n(m)\|_2 = O_p(n^{\frac{1}{2}}p^{\frac{3}{4}}\eta_n^{\frac{1}{2}})$. Together with (10), by Cauchy–Schwarz's inequality, we get

$$
\sup_{1 \leq m \leq M} \sup_{\|\theta_m - \theta_m^*\|_2 \leq \eta_n} |R_{n2}| = O_p(n^{\frac{1}{2}}p^{\frac{3}{4}}\eta_n^{\frac{1}{2}})O(\eta_n) = O_p(n^{\frac{1}{2}}p^{\frac{3}{4}}\eta_n^{\frac{3}{2}}). \tag{11}
$$

For T_{n4}, the third term in (9), write $\mathbf{E}(T_n) = e_n(\theta) - e_n(\theta^*)$, where

$$
e_n(\theta) \equiv \sum_{m=1}^M \sum_{i=1}^n \mathbf{E}\, \rho_m\left(Y_i - U_i^T \theta_m\right).
$$

$\mathbf{E}\,\rho_m\left(Y_i - U_i^T \theta_m\right)$ is second order differentiable with respect to θ_m under condition (L1), with gradient $G_{mi}(\theta) = -\mathbf{E}\left\{(\tau_m - I(Y_i < U_i^T \theta_m))U_i\right\} = \left(F(U_i^T \theta_m|Z_i) - \tau_m\right)U_i$, and Hessian matrix $H_{mi}(\theta) = f((U_i^T \theta_m)|Z_i)U_iU_i^T$.

Let $G(\theta)$ and $H(\theta)$ be gradient and Hessian matrix of $e_n(\theta)$, then $G(\theta) = \sum_{m=1}^M \sum_{i=1}^n G_{mi}(\theta)$ and $H(\theta) = \sum_{m=1}^M \sum_{i=1}^n H_{mi}(\theta) = \sum_{m=1}^M \sum_{i=1}^n f((U_i^T \theta_m)|Z_i) U_iU_i^T$. It is easy to see that $G(\theta^*) = 0$ by $F(U_i^T \theta_m^*|Z_i) = \tau_m$ in (1). By Taylor expansion of $T_{n4} = \mathbf{E}(T_n) = e_n(\theta) - e_n(\theta^*)$ at θ^*, we have

$$
\begin{aligned}
T_{n4} &= \frac{1}{2}(\theta - \theta^*)^T H\left(\theta^* + \xi(\theta - \theta^*)\right)(\theta - \theta^*) \\
&= \frac{1}{2} \sum_{m=1}^M \sum_{i=1}^n f(\zeta_{mi}|Z_i)\, (\theta_m - \theta_m^*)^T U_iU_i^T(\theta_m - \theta_m^*), \\
&= \frac{1}{2} \sum_{m=1}^M \sum_{i=1}^n \left\{f(\zeta_{mi}|Z_i) - f(U_i^T \theta_m^*|Z_i)\right\}(\theta_m - \theta_m^*)^T U_iU_i^T(\theta_m - \theta_m^*) \\
&\quad + \frac{1}{2} \sum_{m=1}^M \sum_{i=1}^n f(U_i^T \theta_m^*|Z_i)(\theta_m - \theta_m^*)^T U_iU_i^T(\theta_m - \theta_m^*) \\
&\equiv R_{n4} + T_{n41}. \tag{12}
\end{aligned}
$$

where $\xi \in (0, 1)$ and ζ_{mi} is between $U_i^T \theta_m$ and $U_i^T \theta_m^*$. Trivially,

$$
|R_{n4}| \leq C \sum_{m=1}^M \sup_{1 \leq i \leq n} \left|f(\zeta_{mi}|Z_i) - f(U_i^T \theta_m^*|Z_i)\right|(\theta_m - \theta_m^*)^T \sum_{i=1}^n U_iU_i^T(\theta_m - \theta_m^*). \tag{13}
$$

Note that $f'(t|Z_i)$ is bounded under condition (L1), $\|U_i\|_2 \leq Cp^{\frac{1}{2}}$ under condition (L2), and $\lambda_{\max}\left(n^{-1}\sum_{i=1}^n U_iU_i^T\right) \leq C$ under condition (L3). Hence, for all $1 \leq i \leq n$, $1 \leq m \leq M$, and $\|\theta_m - \theta_m^*\|_2 \leq \eta_n$, we have $|f(\zeta_{mi}|Z_i) - f(U_i^T \theta_m^*|Z_i)| \leq C|U_i^T(\theta_m - \theta_m^*)| \leq C\|U_i\|_2\|\theta_m - \theta_m^*\|_2 \leq Cp^{\frac{1}{2}}\eta_n$, and $(\theta_m - \theta_m^*)^T \sum_{i=1}^n U_iU_i^T(\theta_m - \theta_m^*) \leq n\lambda_{\max}\left(n^{-1}\sum_{i=1}^n U_iU_i^T\right)\|\theta_m - \theta_m^*\|_2^2 \leq Cn\eta_n^2$. Hence, from (13), we get $\sup_{1 \leq m \leq M} \sup_{\|\theta_m - \theta_m^*\|_2 \leq \eta_n} |R_{n4}| \leq Cnp^{\frac{1}{2}}\eta_n^3$. Together with (9), (11) and (12), we obtain

$T_n = T_{n1} + T_{n41} + R_n(\theta)$, where $\sup_{1 \leq m \leq M} \sup_{\|\theta_m - \theta_m^*\|_2 \leq \eta_n} |R_n(\theta)| = O_p(n^{\frac{1}{2}} p^{\frac{3}{4}} \eta_n^{\frac{3}{2}}) + O_p(np^{\frac{1}{2}} \eta_n^3)$, and

$$T_{n1} = \sum_{m=1}^{M} \sum_{i=1}^{n} \left(I(Y_i < U_i^{\mathsf{T}} \theta_m^*) - \tau_m \right) U_i^{\mathsf{T}} (\theta_m - \theta_m^*) = \mathbf{A}_n^{\mathsf{T}} (\theta - \theta^*),$$

$$T_{n41} = \frac{1}{2} \sum_{m=1}^{M} \sum_{i=1}^{n} f(U_i^{\mathsf{T}} \theta_m^* | Z_i)(\theta_m - \theta_m^*)^{\mathsf{T}} U_i U_i^{\mathsf{T}} (\theta_m - \theta_m^*) = \frac{1}{2} (\theta - \theta^*)^{\mathsf{T}} \mathbf{B}_n (\theta - \theta^*).$$

This completes the proof of the lemma. □

Proof of Theorem 1. Recall the definition of $L_n(\theta) = Q_n(\theta) + P_n(\gamma)$ in (2). Let $\theta - \theta^* = v_n u$, where $v_n > 0$, $u \in \mathbb{R}^{M(p+1)}$ and $\|u\|_2 = 1$. It is easy to see that $\|\theta - \theta^*\|_2 = v_n$. Based on the continuity of L_n, if we can prove that in probability

$$\inf_{\|u\|_2 = 1} L_n(\theta^* + v_n u) > L_n(\theta^*), \tag{14}$$

then the minimal value point of $L_n(\theta^* + v_n u)$ on $\{u : \|u\|_2 \leq 1\}$ exists and lies in the unit ball $\{u : \|u\|_2 \leq 1\}$ in probability. We will prove that (14) holds.

For $Q_n(\theta)$, because $\|\theta_m - \theta_m^*\|_2 \leq \|\theta - \theta^*\|_2$ for all $m = 1, 2, \ldots, M$, by Lemma .1 with $\eta_n = v_n$ under conditions (L1) to (L4), we get

$$q_n(\theta) \equiv Q_n(\theta) - Q_n(\theta^*) = \mathbf{A}_n^{\mathsf{T}} (\theta - \theta^*) + \frac{1}{2} (\theta - \theta^*)^{\mathsf{T}} \mathbf{B}_n (\theta - \theta^*) + R_n(\theta), \tag{15}$$

where $\sup_{\|\theta - \theta^*\|_2 \leq v_n} |R_n(\theta)| = O_p(n^{\frac{1}{2}} p^{\frac{3}{4}} v_n^{\frac{3}{2}}) + O_p(np^{\frac{1}{2}} v_n^3)$.

For $P_n(\gamma)$, let $p_n(\gamma) = P_n(\gamma) - P_n(\gamma^*)$. Define $p_{1n}(\gamma) = p_n(\gamma_I, 0)$, that is,

$$p_{1n}(\gamma) = n\lambda_n \sum_{j \in J} \left(\sum_{m \in M_j} \omega_{mj} |\gamma_{mj}| \right)^{\frac{1}{2}} - n\lambda_n \sum_{j \in J} \left(\sum_{m \in M_j} \omega_{mj} |\gamma_{mj}^*| \right)^{\frac{1}{2}}. \tag{16}$$

Clearly, $p_{1n}(\gamma) \leq p_n(\gamma)$ and $p_{1n}(\gamma^*) = p_n(\gamma^*) = 0$.

Define $l_n(\theta) = q_n(\theta) + p_n(\gamma)$ and $l_{1n}(\theta) = q_n(\theta) + p_{1n}(\gamma)$, both of which are continuous. Clearly, $l_{1n}(\theta) \leq l_n(\theta)$ and $l_{1n}(\theta^*) = l_n(\theta^*) = 0$. Note that (14) is equivalent to that in probability

$$\inf_{\|u\|_2 = 1} l_{1n}(\theta^* + v_n u) > 0. \tag{17}$$

Note that

$$|p_{1n}(\gamma)| = n\lambda_n \left| \sum_{j \in J} \left(\sum_{m \in M_j} \omega_{mj} |\gamma_{mj}| \right)^{\frac{1}{2}} - \sum_{j \in J} \left(\sum_{m \in M_j} \omega_{mj} |\gamma_{mj}^*| \right)^{\frac{1}{2}} \right|$$

$$\leq n\lambda_n \sum_{j \in J} \left| \left(\sum_{m \in M_j} \omega_{mj} |\gamma_{mj}| \right)^{\frac{1}{2}} - \left(\sum_{m \in M_j} \omega_{mj} |\gamma_{mj}^*| \right)^{\frac{1}{2}} \right| \leq n\lambda_n \sum_{j \in J} \left(\sum_{m \in M_j} \omega_{mj} |\gamma_{mj} - \gamma_{mj}^*| \right)^{\frac{1}{2}},$$

where the last inequality follows from the fact that $|\sqrt{|x|} - \sqrt{|y|}| \leq \sqrt{|x - y|}$. Note that $\gamma_{mj} - \gamma_{mj}^* = v_n u_{mj}$ where u_{mj} is a component of u. Hence,

$$|p_{1n}(\gamma)| \leq n\lambda_n d_{nl}^{\frac{1}{2}} v_n^{\frac{1}{2}} \sum_{j \in J} \left(\sum_{m \in M_j} |u_{mj}| \right)^{\frac{1}{2}} \leq n\lambda_n d_{nl}^{\frac{1}{2}} v_n^{\frac{1}{2}} k_n^{\frac{1}{2}} \left(\sum_{(m,j) \in I} |u_{mj}| \right)^{\frac{1}{2}}$$

$$\leq n\lambda_n d_{nl}^{\frac{1}{2}} v_n^{\frac{1}{2}} k_n^{\frac{1}{2}} s_n^{\frac{1}{4}} \|u\|_2^{\frac{1}{2}} = n\lambda_n d_{nl}^{\frac{1}{2}} v_n^{\frac{1}{2}} s_n^{\frac{3}{4}} \|u\|_2^{\frac{1}{2}}, \tag{18}$$

where $k_n = |J| \leq s_n$. By (15) and the above, we have

$$l_{1n}(\theta^* + v_n u) = q_n(\theta^* + v_n u) + p_{1n}(\theta^* + v_n u) = v_n \mathbf{A}_n^{\mathsf{T}} u + \frac{1}{2} v_n^2 u^{\mathsf{T}} \mathbf{B}_n u + R_L(u), \tag{19}$$

where $\sup_{\|u\|_2 = 1} |R_L(u)| = O_p(n^{\frac{1}{2}} p^{\frac{3}{4}} v_n^{\frac{3}{2}}) + O_p(np^{\frac{1}{2}} v_n^3) + O(n\lambda_n d_{nl}^{\frac{1}{2}} v_n^{\frac{1}{2}} s_n^{\frac{3}{4}})$.

For the quadratic term in (19), from condition (L5),

$$\frac{1}{2} v_n^2 u^{\mathsf{T}} \mathbf{B}_n u \geq \frac{C_4}{2} n v_n^2 \|u\|_2^2. \tag{20}$$

For the linear term $\mathbf{A}_n^{\mathrm{T}}u$ in (19), by the independence of (Z_i, Y_i) and (Z_j, Y_j) for all $i \neq j$ and the fact that $E(I(Y_i < U_i^{\mathrm{T}}\theta_m^*) - \tau_m|Z_i) = 0$, we get

$$E(A_n^{\mathrm{T}}A_n) = E\sum_{m=1}^{M}\sum_{i=1}^{n}\left(I(Y_i < U_i^{\mathrm{T}}\theta_m^*) - \tau_m\right)U_i^{\mathrm{T}}\sum_{j=1}^{n}\left(I(Y_j < U_j^{\mathrm{T}}\theta_m^*) - \tau_m\right)U_j$$

$$= \sum_{m=1}^{M}\sum_{i=1}^{n}E\left(\left(I(Y_i < U_i^{\mathrm{T}}\theta_m^*) - \tau_m\right)^2\|U_i\|_2^2\right) \leq Cnp.$$

Then, it follows that

$$\|\mathbf{A}_n\|_2 = O_p((np)^{\frac{1}{2}}), \tag{21}$$

which implies that $\sup_{\|u\|_2 \leq 1}|\mathbf{A}_n^{\mathrm{T}}u| = O_p((np)^{\frac{1}{2}})$. Together with (19) and (20), in probability,

$$\inf_{\|u\|_2=1} l_{1n}(\theta^* + v_n u) \geq \frac{C_4}{2}nv_n^2 - C(np)^{\frac{1}{2}}v_n - Cn^{\frac{1}{2}}p^{\frac{3}{4}}v_n^{\frac{3}{2}} - Cnp^{\frac{1}{2}}v_n^3 - Cn\lambda_n d_{nI}^{\frac{1}{2}}v_n^{\frac{1}{2}}s_n^{\frac{3}{4}}$$

$$\geq \frac{C_4}{2}nv_n\left\{v_n - Cn^{-\frac{1}{2}}p^{\frac{1}{2}} - Cn^{-\frac{1}{2}}p^{\frac{3}{4}}v_n^{\frac{1}{2}} - Cp^{\frac{1}{2}}v_n^2 - C\lambda_n d_{nI}^{\frac{1}{2}}s_n^{\frac{3}{4}}v_n^{-\frac{1}{2}}\right\}. \tag{22}$$

Now take $v_n = C_0(n^{-\frac{1}{2}}p^{\frac{1}{2}})$ where C_0 is a sufficiently large constant. Under condition (L4), i.e., $\lambda_n d_{nI}^{\frac{1}{2}} = o(s_n^{-\frac{3}{4}}p_n^{\frac{3}{4}}n^{-\frac{3}{4}})$, for the last three terms in (22), we can check that $n^{-\frac{1}{2}}p^{\frac{3}{4}}v_n^{\frac{1}{2}} \leq Cn^{-\frac{1}{4}+\frac{1}{2}\alpha_1}v_n = o(v_n)$, $p^{\frac{1}{2}}v_n^2 \leq Cn^{-\frac{1}{2}+\alpha_1}v_n = o(v_n)$, and $\lambda_n d_{nI}^{\frac{1}{2}}s_n^{\frac{3}{4}}v_n^{-\frac{1}{2}} \leq C\lambda_n d_{nI}^{\frac{1}{2}}s_n^{\frac{3}{4}}p_n^{-\frac{3}{4}}n^{\frac{3}{4}}v_n = o(v_n)$. Hence,

$$\inf_{\|u\|_2=1} l_{1n}(\theta^* + v_n u) \geq Cnv_n^2 \to \infty \quad \text{in probability.} \tag{23}$$

Therefore, in probability there exists a local minimizer $\hat{\theta}$ of $L_n(\theta)$ such that $\|\hat{\theta} - \theta^*\|_2 < v_n$. This completes the proof of the theorem. $\square$

Proof of Theorem 2. For the quantile function $Q_n(\theta)$, because $\|\theta_m - \theta_m^*\|_2 \leq \|\theta - \theta^*\|_2$ for all $1 \leq m \leq M$, by Lemma .1 with $\eta_n = v_n$, under conditions (L1)–(L4), we have

$$Q_n(\theta) - Q_n(\theta^*) = \mathbf{A}_n^{\mathrm{T}}(\theta - \theta^*) + \frac{1}{2}(\theta - \theta^*)^{\mathrm{T}}\mathbf{B}_n(\theta - \theta^*) + R_n(\theta), \tag{24}$$

where $\sup_{\|\theta-\theta^*\|_2 \leq v_n}|R_n(\theta)| = O_p(n^{\frac{1}{2}}p^{\frac{3}{4}}v_n^{\frac{3}{2}}) + O_p(np^{\frac{1}{2}}v_n^3)$.

Let $\theta - \theta^* = v_n u$ where $v_n > 0$ and $u \in \mathbb{R}^{M(p+1)}$. Then $\|\theta - \theta^*\|_2 \leq v_n$ if and only if $\|u\|_2 \leq 1$. Let $u = (u_I^{\mathrm{T}}, u_{II}^{\mathrm{T}})^{\mathrm{T}}$, where u_I and u_{II} are subvectors of u corresponding to the index sets I and II, respectively. Clearly, $\|u\|_2^2 = \|u_I\|_2^2 + \|u_{II}\|_2^2$. Note that $\theta_I = \theta_I^* + v_n u_I$ and $\theta_{II} = \theta_{II}^* + v_n u_{II} = v_n u_{II}$.

Define the ball $\tilde{\Theta}_n = \{\theta = \theta^* + v_n u \in \Theta_n : \|u\|_2 \leq 1\}$ with $v_n = C_0 n^{-\frac{1}{2}}p^{\frac{1}{2}}$. For any $\theta = (\theta_I^{\mathrm{T}}, \theta_{II}^{\mathrm{T}})^{\mathrm{T}} \in \tilde{\Theta}_n$, we can see $\|(\theta_I^{\mathrm{T}}, 0^{\mathrm{T}})^{\mathrm{T}} - \theta^*\|_2 = \|\theta_I - \theta_I^*\|_2 \leq v_n$, and $\|\theta_{II}\|_2 = v_n\|u_{II}\|_2$, where $\|u_{II}\|_2 \leq 1$.

Consider that $Q_n(\theta_I, \theta_{II}) - Q_n(\theta_I, 0) = Q_n(\theta_I, \theta_{II}) - Q_n(\theta^*) - \left(Q_n(\theta_I, 0) - Q_n(\theta^*)\right)$

$$= \frac{1}{2}(0^{\mathrm{T}}, \theta_{II}^{\mathrm{T}})\mathbf{B}_n(0^{\mathrm{T}}, \theta_{II}^{\mathrm{T}})^{\mathrm{T}} + \mathbf{A}_n^{\mathrm{T}}(0^{\mathrm{T}}, \theta_{II}^{\mathrm{T}})^{\mathrm{T}} - (\theta_I^{\mathrm{T}} - \theta_I^{*\mathrm{T}}, 0^{\mathrm{T}})\mathbf{B}_n(0^{\mathrm{T}}, \theta_{II}^{\mathrm{T}})^{\mathrm{T}} + \left(R_n(\theta) - R_n(\theta_I, 0)\right)$$

$$\equiv I_{n1} + I_{n2} + I_{n3} + r_{1n}(\theta).$$

From (21), we can see $\sup_{\theta \in \tilde{\Theta}_n}|I_{n2}| \leq \|\mathbf{A}_n\|_2\|\theta_{II}\|_2 = O_p((np)^{\frac{1}{2}})v_n\|u_{II}\|_2 = O_p(p\|u_{II}\|_2)$. Under condition (L5), $\|\mathbf{B}_n(0^{\mathrm{T}}, \theta_{II}^{\mathrm{T}})^{\mathrm{T}}\|_2^2 = (0^{\mathrm{T}}, \theta_{II}^{\mathrm{T}})\mathbf{B}_n^2(0^{\mathrm{T}}, \theta_{II}^{\mathrm{T}})^{\mathrm{T}} \leq n^2\lambda_{\max}^2(n^{-1}B_n)\|\theta_{II}\|_2^2 \leq Cn^2v_n^2\|u_{II}\|_2^2$, we have $\sup_{\theta \in \tilde{\Theta}_n}|I_{n3}| \leq \|\theta_I - \theta_I^*\|_2\|\mathbf{B}_n(0^{\mathrm{T}}, \theta_{II}^{\mathrm{T}})^{\mathrm{T}}\|_2 \leq Cnv_n^2 \leq Cp$. From (24), we have $\sup_{\theta \in \tilde{\Theta}_n}|r_{1n}(\theta)| = O_p(n^{-\frac{1}{4}}p^{\frac{3}{2}})$. Hence,

$$Q_n(\theta_I, \theta_{II}) - Q_n(\theta_I, 0) = I_{n1} + r_{2n}(\theta), \tag{25}$$

where $\sup_{\theta \in \tilde{\Theta}_n}|r_{2n}(\theta)| = O_p(p\|u_{II}\|_2) + O_p(n^{-\frac{1}{4}}p^{\frac{3}{2}})$. Recall $L_n(\theta_I, \theta_{II}) - L_n(\theta_I, 0) = Q_n(\theta_I, \theta_{II}) - Q_n(\theta_I, 0) + P_n(\gamma_I, \gamma_{II}) - P_n(\gamma_I, 0)$. From (25), we get

$$L_n(\theta_I, \theta_{II}) - L_n(\theta_I, 0) \geq P_n(\gamma_I, \gamma_{II}) - P_n(\gamma_I, 0) + r_{2n}(\theta). \tag{26}$$

Note that $(n\lambda_n)^{-1}\big(P(\gamma_I, \gamma_{II}) - P_n(\gamma_I, 0)\big)$

$$
= \sum_{j=1}^{p} \Big(\sum_{m=1}^{M} \omega_{mj}|\gamma_{mj}|\Big)^{\frac{1}{2}} - \sum_{j\in J} \Big(\sum_{m\in M_j} \omega_{mj}|\gamma_{mj}|\Big)^{\frac{1}{2}}
$$

$$
= \sum_{j\in J^c} \Big(\sum_{m=1}^{M} \omega_{mj}|\gamma_{mj}|\Big)^{\frac{1}{2}} + \sum_{j\in J} \Big(\sum_{m=1}^{M} \omega_{mj}|\gamma_{mj}|\Big)^{\frac{1}{2}} - \sum_{j\in J} \Big(\sum_{m\in M_j} \omega_{mj}|\gamma_{mj}|\Big)^{\frac{1}{2}}
$$

$$
\geq \sum_{j\in J^c} \frac{\sum_{m=1}^{M} \omega_{mj}|\gamma_{mj}|}{2\Big(\sum_{m=1}^{M} \omega_{mj}|\gamma_{mj}|\Big)^{\frac{1}{2}}} + \sum_{j\in J} \frac{\sum_{m\in M_j^c} \omega_{mj}|\gamma_{mj}|}{2\Big(\sum_{m=1}^{M} \omega_{mj}|\gamma_{mj}|\Big)^{\frac{1}{2}}}. \tag{27}
$$

For all $\theta \in \tilde{\Theta}_n$, we have $|\gamma_{mj}| \leq |\gamma_{mj}^*| + 1 \leq C$, which implies that $\sum_{m=1}^{M} \omega_{mj}|\gamma_{mj}| \leq C\|\omega_n\|_\infty$ for all $j = 1, 2, \ldots, p$, where $\|\omega_n\|_\infty = \max_{1\leq m\leq M,\, 1\leq j\leq p} |\omega_{mj}|$. Recall that $d_{nII} = \min_{(m,j)\in II}\{\omega_{mj}\}\|\omega_n\|_\infty^{-\frac{1}{2}}$. From (27), it follows that for all $\theta \in \tilde{\Theta}_n$,

$$
P_n(\gamma_I, \gamma_{II}) - P_n(\gamma_I, 0) \geq Cn\lambda_n \|\omega_n\|_\infty^{-\frac{1}{2}} \Big(\sum_{j\in J^c} \sum_{m=1}^{M} \omega_{mj}|\gamma_{mj}| + \sum_{j\in J} \sum_{m\in M_j^c} \omega_{mj}|\gamma_{mj}|\Big)
$$

$$
= Cn\lambda_n \|\omega_n\|_\infty^{-\frac{1}{2}} \sum_{(m,j)\in II} \omega_{mj}|\gamma_{mj}| \geq Cn\lambda_n d_{nII} \sum_{(m,j)\in II} |\gamma_{mj}|
$$

$$
\geq Cn\lambda_n d_{nII}\|\gamma_{II}\|_2 = C\lambda_n d_{nII}(np)^{\frac{1}{2}}\|u_{II}\|_2. \tag{28}
$$

Define $\Omega_n = \{\theta = \theta^* + \nu_n u \in \tilde{\Theta}_n : \|u_{II}\|_2 > 0\}$ and $\Omega_n^c = \{\theta = \theta^* + \nu_n u \in \tilde{\Theta}_n : u_{II} = 0\}$. Clearly, $\tilde{\Theta}_n = \Omega_n \cup \Omega_n^c$. From (26) and (28), we obtain in probability

$$
\inf_{\theta\in\Omega_n} \big(L_n(\theta_I, \theta_{II}) - L_n(\theta_I, 0)\big) \geq \inf_{\theta\in\Omega_n} \big(P_n(\gamma_I, \gamma_{II}) - P_n(\gamma_I, 0)\big) - \sup_{\theta\in\tilde{\Theta}_n} |r_{2n}(\theta)|
$$

$$
\geq \tilde{C}_1\lambda_n d_{nII}(np)^{\frac{1}{2}}\|u_{II}\|_2 - \tilde{C}_2 p\|u_{II}\|_2 - \tilde{C}_2 n^{-\frac{1}{4}}p^{\frac{3}{2}}
$$

$$
\geq p\Big(\|u_{II}\|_2\big(\tilde{C}_1\lambda_n d_{nII}n^{\frac{1}{2}}p^{-\frac{1}{2}} - \tilde{C}_2\big) - \tilde{C}_2 n^{-\frac{1}{4}}p^{\frac{1}{2}}\Big)
$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are positive constants. Under the given conditions, $\lambda_n d_{nII}n^{\frac{1}{2}}p^{-\frac{1}{2}} \to \infty$ and $n^{-\frac{1}{4}}p^{\frac{1}{2}} \to 0$ as $n \to \infty$. Hence, $\inf_{\theta\in\Omega_n}\big(L_n(\theta_I, \theta_{II}) - L_n(\theta_I, 0)\big) > 0$ in probability. Thus, $\inf_{\theta\in\Omega_n} L_n(\theta) \geq \inf_{\theta\in\Omega_n}\big(L_n(\theta_I, \theta_{II}) - L_n(\theta_I, 0)\big) + \inf_{\theta\in\Omega_n} L_n(\theta_I, 0) > \inf_{\theta\in\Omega_n} L_n(\theta_I, 0) = \inf_{\theta\in\tilde{\Theta}_n} L_n(\theta_I, 0) \geq \inf_{\theta\in\tilde{\Theta}_n} L_n(\theta)$, which implies that $\inf_{\theta\in\tilde{\Theta}_n} L_n(\theta) = \inf_{\theta\in\Omega_n^c} L_n(\theta)$. Therefore, the minimal value point of $L_n(\theta)$ on $\tilde{\Theta}_n$ only lies in its subset Ω_n^c.

From Theorem 1, we know that in probability $\hat{\theta} \in \tilde{\Theta}_n$ and that $\hat{\theta}$ is a local minimizer of $L_n(\theta)$. Hence, $\hat{\theta} \in \Omega_n^c$ in probability, which implies that $\hat{\gamma}_{II} = 0$. $\square$

Proof of Theorem 3. Let $\theta_I - \theta_I^* = \nu_n u$ where $\nu_n > 0$ and $u \in \mathbb{R}^{M+s_n}$. Because $\|\theta_{mI} - \theta_{mI}^*\|_2 \leq \|\theta_I - \theta_I^*\|_2 = \nu_n\|u\|_2$ for all $1 \leq m \leq M$, and due to conditions (L1)–(L3) and that $0 < \alpha_0 < \alpha_1 < \frac{1}{6}$, Lemma .1 implies that

$$
Q_n(\theta_I, 0) - Q_n(\theta_I^*, 0) = Q_n(\theta_I^* + \nu_n u, 0) - Q_n(\theta_I^*, 0) = \nu_n \mathbf{A}_{nI}^{\mathrm{T}} u + \frac{1}{2}\nu_n^2 u^{\mathrm{T}} \mathbf{B}_{nI} u + r_n(u), \tag{29}
$$

where $\sup_{\|u\|_2\leq 1} |r_n(u)| = O_p(n^{\frac{1}{2}}s_n^{\frac{3}{4}}\nu_n^{\frac{3}{2}}) + O_p(ns_n^{\frac{1}{2}}\nu_n^3)$, $\mathbf{B}_{nI}$ is given in (4),

$$
\mathbf{A}_{nI} = (A_1^{\mathrm{T}}, \ldots, A_M^{\mathrm{T}})^{\mathrm{T}} \quad \text{with} \quad A_m = \sum_{i=1}^{n} \big(I(Y_i < U_i^{\mathrm{T}}\theta_m^*) - \tau_m\big) U_{imI}, \tag{30}
$$

and U_{imI} is given in Section 3. Note that A_{nI} and B_{nI} are the sub-vector(matrix) of $\mathbf{A}_n$ and $\mathbf{B}_n$ in (6) corresponding to the index set I, respectively.

For $P_n(\gamma)$, define $p_n(\gamma) = P_n(\gamma) - P_n(\gamma^*)$. Let u_I be the subvector of u corresponding to the subvector γ_I in θ_I. Then, we have

$$
p_n(\gamma_I, 0) = p_n(\gamma_I^* + \nu_n u_I, 0) = n\lambda_n \sum_{j\in J} \Big(\sum_{m\in M_j} \omega_{mj}|\gamma_{mj}|\Big)^{\frac{1}{2}} - n\lambda_n \sum_{j\in J} \Big(\sum_{m\in M_j} \omega_{mj}|\gamma_{mj}^*|\Big)^{\frac{1}{2}},
$$

which is $p_{1n}(\gamma)$ given in (16). By (18) in the proof of Theorem 1, we have $|p_n(\gamma_l, 0)| = |p_n(\gamma_l^* + v_n u_l, 0)| \leq n\lambda_n d_{nl}^{\frac{1}{2}} v_n^{\frac{1}{2}} s_n^{\frac{3}{4}} \|u\|_2^{\frac{1}{2}}$. This, combined with (29), implies that

$$L_n(\theta_l^* + v_n u, 0) - L_n(\theta_l^*, 0) = v_n \mathbf{A}_{nl}^{\mathrm{T}} u + \frac{1}{2} v_n^2 u^{\mathrm{T}} \mathbf{B}_{nl} u + r_l(u), \tag{31}$$

where $\sup_{\|u\|_2 \leq 1} |r_l(u)| = O_p(n^{\frac{1}{2}} s_n^{\frac{3}{4}} v_n^{\frac{3}{2}}) + O_p(n s_n^{\frac{1}{2}} v_n^3) + O(n\lambda_n d_{nl}^{\frac{1}{2}} v_n^{\frac{1}{2}} s_n^{\frac{3}{4}})$.

Define the ball $\Theta_{nl} = \{(\theta_l^{\mathrm{T}}, 0^{\mathrm{T}})^{\mathrm{T}} : \theta_l - \theta_l^* = v_n u\}$ with $v_n = C_0(n^{-1} p_n)^{\frac{1}{2}}$, where $u \in \mathbb{R}^{M+s_n}$ and C_0 is a positive constant. Given that $v_n = C_0(n^{-1} p_n)^{\frac{1}{2}}$, $\lambda_n d_{nl}^{\frac{1}{2}} = O(n^{-1} p_n^{\frac{1}{2}})$, and $0 < \alpha_0 < \alpha_1 < \frac{1}{6}$, we see that $n^{\frac{1}{2}} s_n^{\frac{3}{4}} v_n^{\frac{3}{2}} \leq C n^{-\frac{1}{4}} s_n^{\frac{3}{4}} p_n^{\frac{3}{4}} = o_p(1)$, $n\sqrt{s_n} v_n^3 \leq C n^{-\frac{1}{2}} s_n^{\frac{1}{2}} p_n^{\frac{3}{2}} = o(n^{-\frac{1}{4}} s_n^{\frac{3}{4}} p_n^{\frac{3}{4}}) = o_p(1)$ and $n\lambda_n d_{nl}^{\frac{1}{2}} v_n^{\frac{1}{2}} s_n^{\frac{3}{4}} \leq \lambda_n d_{nl}^{\frac{1}{2}} n^{\frac{3}{4}} s_n^{\frac{3}{4}} p_n^{\frac{1}{4}} = O(n^{-\frac{1}{4}} s_n^{\frac{3}{4}} p_n^{\frac{3}{4}}) = o_p(1)$. Hence, considering (31), for all $(\theta_l^{\mathrm{T}}, 0^{\mathrm{T}})^{\mathrm{T}} \in \Theta_{nl}$ we have $L_n(\theta_l, 0) - L_n(\theta_l^*, 0) = \mathbf{A}_{nl}^{\mathrm{T}}(\theta_l - \theta_l^*) + \frac{1}{2}(\theta_l - \theta_l^*)^{\mathrm{T}} \mathbf{B}_{nl} (\theta_l - \theta_l^*) + o_p(1)$, which implies that for all $(\theta_l^{\mathrm{T}}, 0^{\mathrm{T}})^{\mathrm{T}} \in \Theta_{nl}$,

$$L_n(\theta_l, 0) - L_n(\theta_l^*, 0) + \frac{1}{2} \mathbf{A}_{nl}^{\mathrm{T}} \mathbf{B}_{nl}^{-1} \mathbf{A}_{nl} = \frac{1}{2} \left(\mathbf{B}_{nl}^{-\frac{1}{2}} \mathbf{A}_{nl} + \mathbf{B}_{nl}^{\frac{1}{2}}(\theta_l - \theta_l^*)\right)^{\mathrm{T}} \left(\mathbf{B}_{nl}^{-\frac{1}{2}} \mathbf{A}_{nl} + \mathbf{B}_{nl}^{\frac{1}{2}}(\theta_l - \theta_l^*)\right) + o_p(1). \tag{32}$$

By Theorems 1 and 2, we know that in probability a local minimizer $\hat{\theta}$ of $L_n(\theta) - L_n(\theta^*)$ lies in the ball Θ_{nl}, which implies that $\hat{\theta} = (\hat{\theta}_l^{\mathrm{T}}, 0^{\mathrm{T}})^{\mathrm{T}} \in \Theta_{nl}$ in probability. Hence, from (32), we have

$$\mathbf{B}_{nl}^{\frac{1}{2}}(\hat{\theta}_l - \theta_l^*) = -\mathbf{B}_{nl}^{-\frac{1}{2}} \mathbf{A}_{nl} + o_p(1)t, \tag{33}$$

where $t \in \mathbb{R}^{M+s_n}$ is an unit vector. Since $t^{\mathrm{T}} \mathbf{B}_{nl} t \leq \lambda_{\max}(\mathbf{B}_{nl}) \leq \lambda_{\max}(\mathbf{B}_n) \leq Cn$ by condition (L5), we have $\mathbf{B}_{nl}^{\frac{1}{2}} t = O_p(n^{\frac{1}{2}})t$. Multiplying both sides of (33) by a vector $b^{\mathrm{T}} \mathbf{B}_{nl}^{\frac{1}{2}}$, where $b \in \mathbb{R}^{M+s_n}$ is any unit vector, we obtain

$$b^{\mathrm{T}} \mathbf{B}_{nl}(\hat{\theta}_l - \theta_l^*) = -b^{\mathrm{T}} \mathbf{A}_{nl} + o_p(n^{1/2}). \tag{34}$$

Let $\xi_n = b^{\mathrm{T}} \mathbf{A}_{nl}$, and write $b = (b_1^{\mathrm{T}}, \ldots, b_M^{\mathrm{T}})^{\mathrm{T}}$ where b_m is the subvector of b corresponding to the subvector θ_{ml}^* of θ_l^*. By the definition of $\mathbf{A}_{nl}$ in (30), we see that

$$\xi_n = \sum_{m=1}^{M} \sum_{i=1}^{n} \left(\psi_{mi}^* U_{iml}^{\mathrm{T}} b_m\right) = \sum_{i=1}^{n} \zeta_i,$$

where $\zeta_i = \sum_{m=1}^{M} \psi_{mi}^* U_{iml}^{\mathrm{T}} b_m$ with $\psi_{mi}^* = I(Y_i < U_{iml}^{\mathrm{T}} \theta_{ml}^*) - \tau_m$. Clearly, $\{\zeta_i, \ i = 1, \ldots, n\}$ is an independent sequence. Next, we will verify that ξ_n satisfies the Lindeberg's condition

$$\sigma_n^{-2} \sum_{i=1}^{n} E\left(\zeta_i^2 I(|\zeta_i| \geq \sigma_n)\right) \to 0, \tag{35}$$

where $\sigma_n^2 = \text{Var}(\xi_n)$.

For $\zeta_i = \sum_{m=1}^{M} \psi_{mi}^* U_{iml}^{\mathrm{T}} b_m$, it is easy to see that $E(\zeta_i) = \sum_{m=1}^{M} E\left(E(\psi_{mi}^* | Z_i) U_{iml}^{\mathrm{T}} b_m\right) = 0$ from the fact $E(\psi_{mi}^* | Z_i) = F(U_{iml}^{\mathrm{T}} \theta_{ml}^* | Z_i) - \tau_m = F(U_i^{\mathrm{T}} \theta_m^* | Z_i) - \tau_m = 0$, and that

$$E(\zeta_i^2) = E\left(\sum_{m=1}^{M} \psi_{mi}^* U_{iml}^{\mathrm{T}} b_m\right)^2 = \sum_{m=1}^{M} \sum_{l=1}^{M} E\left(E(\psi_{mi}^* \psi_{li}^* | Z_i) b_m^{\mathrm{T}}(U_{iml} U_{ill}^{\mathrm{T}}) b_l\right)$$

$$= \sum_{m=1}^{M} \sum_{l=1}^{M} E\left((\min(\tau_m, \tau_l) - \tau_m \tau_l) b_m^{\mathrm{T}}(U_{iml} U_{ill}^{\mathrm{T}}) b_l\right) = b^{\mathrm{T}} \Sigma_n b,$$

where Σ_n is given in (3). Hence, we obtain that $E(\xi_n) = \sum_{i=1}^{n} E(\zeta_i) = 0$ and that, by independence $\text{Var}(\xi_n) = \sum_{i=1}^{n} E(\zeta_i^2) = nb^{\mathrm{T}} \Sigma_n b$. Under condition (L6), we obtain

$$\sigma_n^2 = nb^{\mathrm{T}} \Sigma_n b \geq n\lambda_{\min}(\Sigma_n) b^{\mathrm{T}} b \geq Cn. \tag{36}$$

Note that $(U_{iml}^{\mathrm{T}} b_m)^2 \leq \|U_{iml}\|_2^2 \|b_m\|_2^2 \leq Cs_n \|b\|_2^2 \leq Cs_n$ under condition (L2). By Cauchy–Schwarz's inequality, $\zeta_i^2 = (\sum_{m=1}^{M} \psi_{mi}^* U_{iml}^{\mathrm{T}} b_m)^2 \leq M \sum_{m=1}^{M} (\psi_{mi}^* U_{iml}^{\mathrm{T}} b_m)^2 \leq Cs_n \sum_{m=1}^{M} (\psi_{mi}^*)^2$, which implies that $\zeta_i^2 \leq Cs_n$ from the fact $|\psi_{mi}^*| \leq$

$1 + \tau_m \leq 2$. Hence, we have

$$\sum_{i=1}^{n} E\left(\zeta_i^2 I(|\zeta_i| \geq \sigma_n)\right) \leq Cs_n \sum_{i=1}^{n} E\left(I(\zeta_i^2 \geq \sigma_n^2)\right) \leq Cs_n \sum_{i=1}^{n} P\left(Cs_n \sum_{m=1}^{M} (\psi_{mi}^*)^2 \geq \sigma_n^2\right)$$

$$\leq Cs_n \sum_{i=1}^{n} E\left(Cs_n \sum_{m=1}^{M} (\psi_{mi}^*)^2\right) \sigma_n^{-2} \leq Cns_n^2 \sigma_n^{-2}.$$

Together with (36), this implies that, as $n \to \infty$,

$$\sigma_n^{-2} \sum_{i=1}^{n} E\left(\zeta_i^2 I(|\zeta_i| \geq \sigma_n)\right) \leq Cns_n^2 \sigma_n^{-4} \leq C\frac{s_n^2}{n} \leq Cn^{-1+2\alpha_0} \to 0,$$

which shows that Lindeberg's condition (35) holds. Hence,

$$(nb^{\mathrm{T}}\mathbf{\Sigma}_n b)^{-\frac{1}{2}} b^{\mathrm{T}} \mathbf{A}_{nl} = \frac{b^{\mathrm{T}}\mathbf{A}_{nl}}{\sigma_n} = \frac{\xi_n - E(\xi_n)}{\sigma_n} \to N(0, 1).$$

Together with (34) and (36), this implies that

$$(nb^{\mathrm{T}}\mathbf{\Sigma}_n b)^{-\frac{1}{2}} b^{\mathrm{T}} \mathbf{B}_{nl}(\hat{\theta}_l - \theta_l^*) = -(nb^{\mathrm{T}}\mathbf{\Sigma}_n b)^{-\frac{1}{2}} b^{\mathrm{T}} \mathbf{A}_{nl} + o_p(1) \to N(0, 1).$$

This completes the proof of the theorem. □

References

Avery, C.L., He, Q., North, K.E., Ambite, J.L., Boerwinkle, E., Fornage, M., Hindorff, L.A., Kooperberg, C., Meigs, J.B., Pankow, J.S., et al., 2011. A phenomics-based strategy identifies loci on APOC1, BRAP, and PLCG1 associated with metabolic syndrome phenotype domains. PLoS Genet. 7, e1002322.

Fan, J., Li, R., 2001. Variable selection via nonconcave penalized likelihood and its oracle properties. J. Amer. Statist. Assoc. 96, 1348–1360.

Gasso, G., Rakotomamonjy, A., Canu, S., 2009. Recovering sparse signals with a certain family of nonconvex penalties and DC programming. IEEE Trans. Signal Process. 57, 4686–4698.

Holland, E.C., 2000. Glioblastoma multiforme: the terminator. Proc. Natl. Acad. Sci. 97, 6242–6244.

Huang, J., Ma, S., Xie, H., Zhang, C.H., 2009. A group bridge approach for variable selection. Biometrika 96, 339–355.

Jiang, L., Bondell, H.D., Wang, H.J., 2014. Interquantile shrinkage and variable selection in quantile regression. Comput. Statist. Data Anal. 69, 208–219.

Jiang, L., Wang, H.J., Bondell, H.D., 2013. Interquantile shrinkage in regression models. J. Comput. Graph. Statist. 22, 970–986.

Koenker, R., 2004. Quantile regression for longitudinal data. J. Multivariate Anal. 91, 74–89.

Koenker, R., 2015. quantreg: Quantile Regression. URL: http://CRAN.R-project.org/package=quantreg R package version 5.11.

Koenker, R., Bassett, J.G., 1978. Regression quantiles. Econometrica 46, 33–50.

Landmark-Høyvik, H., Dumeaux, V., Nebdal, D., Lund, E., Tost, J., Kamatani, Y., Renault, V., Børresen-Dale, A.L., Kristensen, V., Edvardsen, H., 2013. Genome-wide association study in breast cancer survivors reveals SNPs associated with gene expression of genes belonging to MHC class I and II. Genomics 102, 278–287.

Li, Y., Zhu, J., 2008. L_1-norm quantile regression. J. Comput. Graph. Statist. 17, 163–185.

Mazumder, R., Friedman, J.H., Hastie, T., 2011. Sparsenet: Coordinate descent with nonconvex penalties. J. Amer. Statist. Assoc. 106, 1125–1138.

Peng, L., Xu, J., Kutner, N., 2014. Shrinkage estimation of varying covariate effects based on quantile regression. Stat. Comput. 24, 853–869.

Puputti, M., Tynninen, O., Sihto, H., Blom, T., Mäenpää, H., Isola, J., Paetau, A., Joensuu, H., Nupponen, N.N., 2006. Amplification of KIT, PDGFRA, VEGFR2, and EGFR in gliomas. Mol. Cancer Res. 4, 927–934.

Tibshirani, R., 1996. Regression shrinkage and selection via the lasso. J. R. Stat. Soc. Ser. B 58, 267–288.

Tibshirani, R., Saunders, M., Rosset, S., Zhu, J., Knight, K., 2005. Sparsity and smoothness via the fused lasso. J. R. Stat. Soc. Ser. B 67, 91–108.

Wang, S., Nan, B., Zhu, N., Zhu, J., 2009. Hierarchically penalized Cox regression with grouped variables. Biometrika 96, 307–322.

Wang, L., Wu, Y., Li, R., 2012. Quantile regression for analyzing heterogeneity in ultra-high dimension. J. Amer. Statist. Assoc. 107, 214–222.

Wu, Y., Liu, Y., 2009. Variable selection in quantile regression. Statist. Sinica 19, 801–817.

Zhao, Z., Xiao, Z., 2014. Efficient regressions via optimally combining quantile information. Econometric Theory 30, 1272–1314.

Zou, H., 2006. The adaptive Lasso and its oracle properties. J. Amer. Statist. Assoc. 101, 1418–1429.

Zou, H., Yuan, M., 2008. Regularized simultaneous model selection in multiple quantiles regression. Comput. Statist. Data Anal. 52, 5296–5304.