PART 1

Review of DSP

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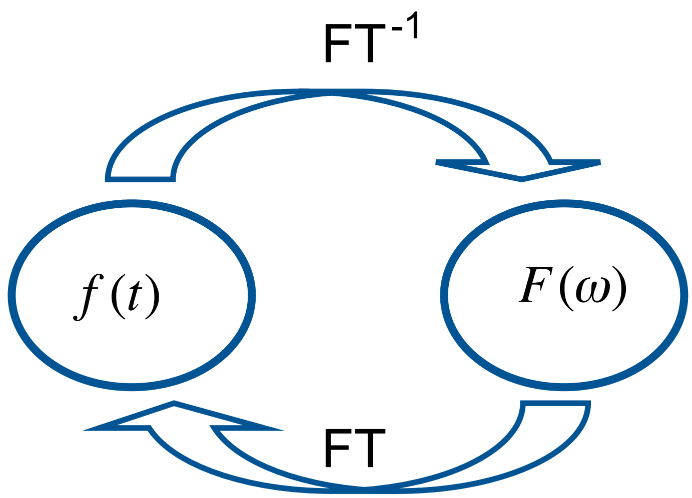
The Fourier Transform

$$F(\omega) = \int f(t)\, e^{-i\omega t}\, dt$$ Fourier Transform

$$f(t) = \frac{1}{2\pi} \int F(\omega)\, e^{i\omega t}\, d\omega$$ Inverse Transform

$$f(t) \longleftrightarrow F(\omega)$$

Forward / Inverse Pairs



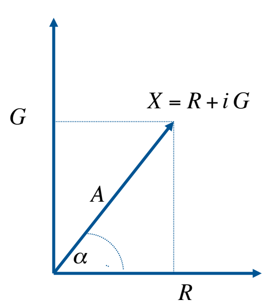
Amplitude and Phase

- Amplitude and Phase of a complex number

$$X = R + i\,G$$

$$\alpha = \tan^{-1}\frac{G}{R}$$

$$A = \sqrt{(R^2 + G^2)}$$



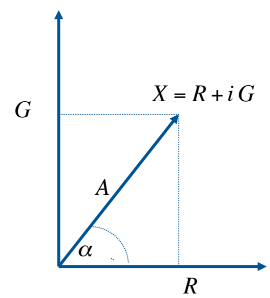
Amplitude and Phase

- Amplitude and Phase of the FT

$$F(\omega) = R(\omega) + i\,G(\omega)$$

$$\alpha(\omega) = \tan^{-1}\frac{G(\omega)}{R(\omega)}$$

$$A(\omega) = \sqrt{(R(\omega)^2 + G(\omega)^2)}$$

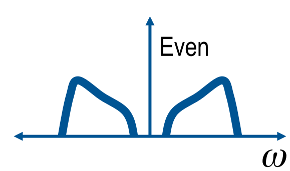


Symmetries of the FT

- If the signal is ***real***, then

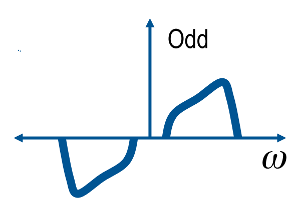
$$R(\omega) = R(-\omega)$$

$$A(\omega) = A(-\omega)$$



$$G(\omega) = -G(-\omega)$$

$$\alpha(\omega) = -\alpha(\omega)$$



FT: a simple physical interpretation

A signal can be represented as a superposition of elementary signals (complex exponentials) of frequency ω_k scaled by a *complex amplitude* $F(\omega_k)$

$$f(t) = \frac{1}{2\pi}\int F(\omega)e^{i\omega t}\,d\omega \approx \frac{\Delta\omega}{2\pi}\sum_k F(\omega_k)e^{i\omega_k t}$$

$$f(t) \approx \sum_k f_k(t), \quad f_k(t) = \frac{\Delta\omega}{2\pi}F(\omega_k)e^{i\omega_k t}$$

Properties of the FT

$$f(t) \leftrightarrow F(\omega)$$
$$g(t) \leftrightarrow G(\omega)$$

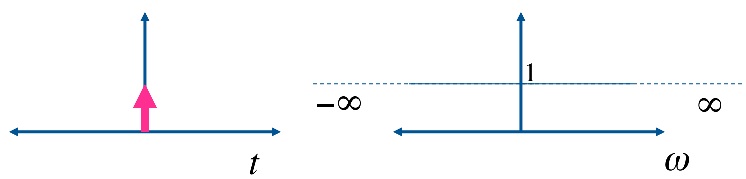
- Linearity $f(t)+g(t) \leftrightarrow F(\omega)+G(\omega)$
- Scale $f(ta) \leftrightarrow \frac{1}{|a|}F(\frac{\omega}{a})$
- Shifting $f(t-\tau) \leftrightarrow F(\omega)e^{-i\omega\tau}$
- Modulation in time $f(t)e^{i\varphi t} \leftrightarrow F(\omega-\varphi)$
- Convolution $\int f(\tau)g(t-\tau)d\tau \leftrightarrow F(\omega)G(\omega)$
- Convolution in Frequency $f(t)g(t) \leftrightarrow \frac{1}{2\pi}\int F(\varphi)G(\omega-\varphi)d\varphi$

Delta function

$$f(t) \longleftrightarrow F(\omega)$$

$$\delta(t) \longleftrightarrow 1$$

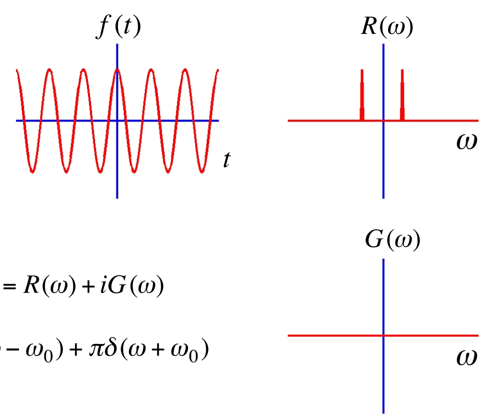
$$f(t) = \delta(t) \qquad F(\omega) = 1$$



represents the delta function which cannot be drawn

Cosine

$$f(t) = \cos(\omega_0 t), \quad -\infty < t < \infty$$

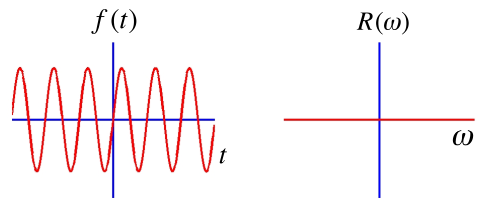


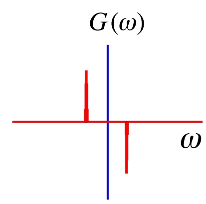
$$f(t) \longleftrightarrow F(\omega) = R(\omega) + iG(\omega)$$

$$\cos(\omega_0 t) \longleftrightarrow \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

Sine

$$f(t) = \sin(\omega_0 t), \quad -\infty < t < \infty$$

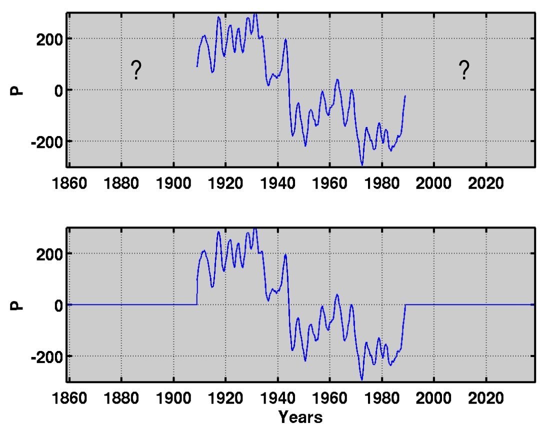




$$f(t) \longleftrightarrow F(\omega) = R(\omega) + iG(\omega)$$

$$\sin(\omega_0 t) \longleftrightarrow -i\,\pi\delta(\omega - \omega_0) + i\,\pi\delta(\omega + \omega_0)$$

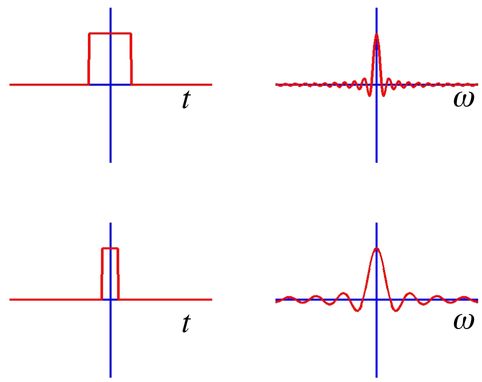
Truncation: zeros or ?



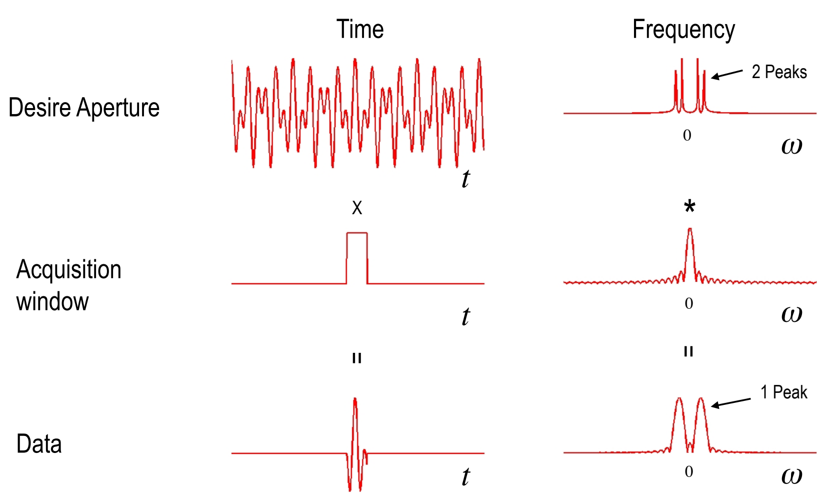
Atmospheric pressure FCAG - UNLP - La Plata (1909 - 1989)

Boxcar

- FT of the truncation operator (Boxcar)

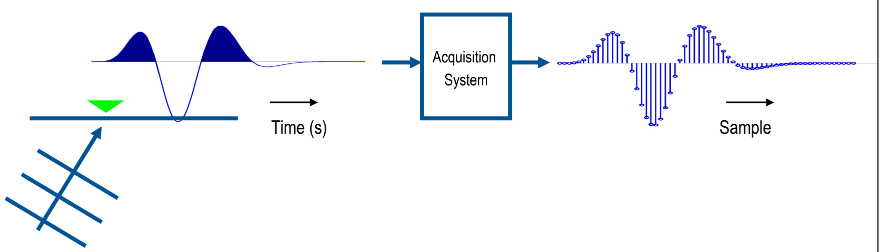


The truncation problem



The discrete world

- Analog signals (waveforms) are transformed into digital signals by acquisition systems
- How the FT of the true underlying continue signal/process relates to its discrete version??
  - This is answered by Nyquist theorem



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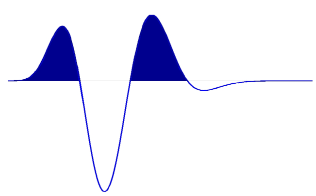
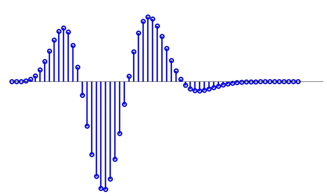
The discrete world

Analog signal



$$s(t) \longleftrightarrow S(\omega)$$

Digital Signal sampled every Δt secs



$$s(t_n) \longleftrightarrow S_d(\omega)$$

$$t_n = (n-1)\,\Delta t$$

Nyquist theorem

The FT of the discrete signal is a distorted version of the FT of the analog signal. The distortion is given by Poisson Formula:

This formula can be found in any book on harmonic analiyis

$$S_d(\omega) = \frac{1}{\Delta t}\sum_{k=-\infty}^{\infty} S(\omega - k\omega_0), \qquad \omega_0 = \frac{2\pi}{\Delta t}$$

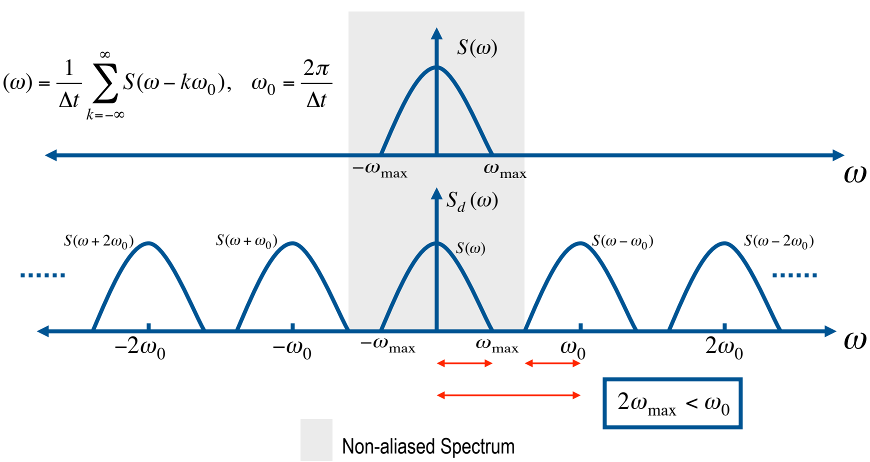
What you can measure ($S_d(\omega)$)

$S(\omega)$ is what you would have liked to measure

Nyquist theorem

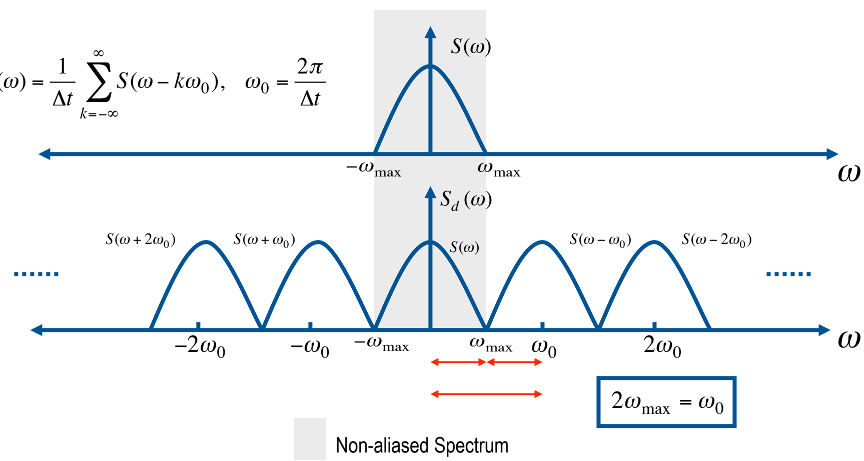
- Nyquist theorem or formula provides the sampling condition to compute the FT of the discrete signal is such a way that it is a perfect representation of the FT of the analog signal. The theorem is derived by simple inspection of Poisson formula. In a graphical manner:

$$S_d(\omega) = \frac{1}{\Delta t}\sum_{k=-\infty}^{\infty} S(\omega - k\omega_0), \quad \omega_0 = \frac{2\pi}{\Delta t}$$



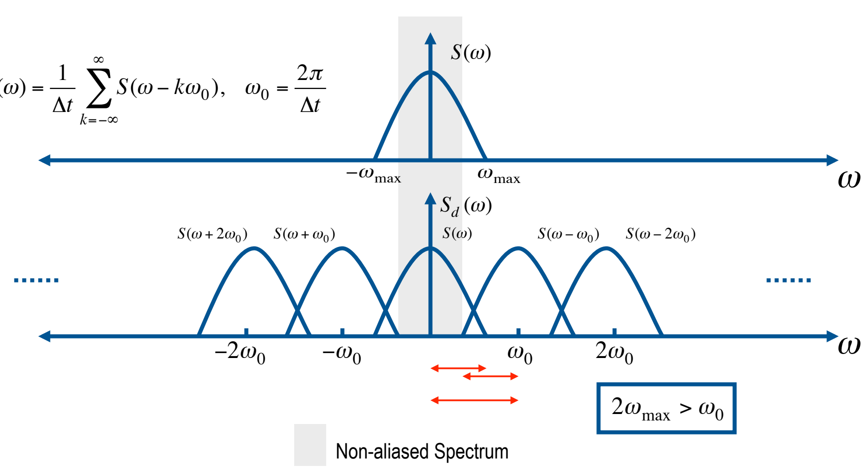
Nyquist theorem

$$S_d(\omega) = \frac{1}{\Delta t} \sum_{k=-\infty}^{\infty} S(\omega - k\omega_0), \quad \omega_0 = \frac{2\pi}{\Delta t}$$



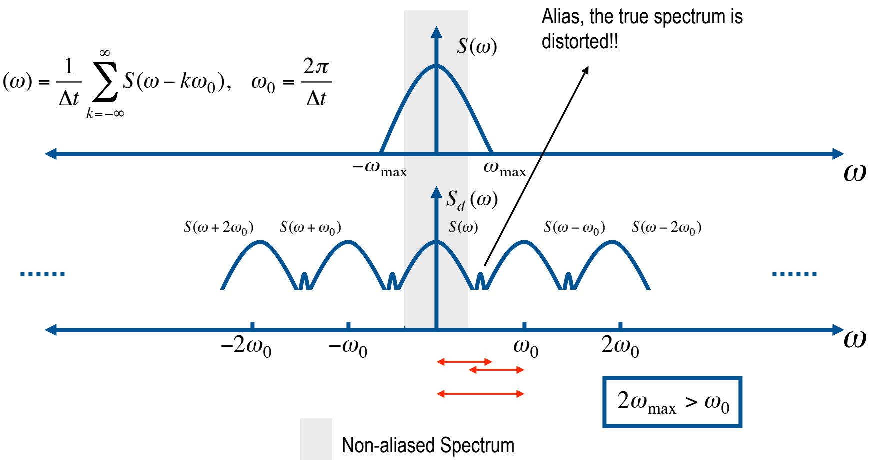
Nyquist theorem

$$S_d(\omega) = \frac{1}{\Delta t} \sum_{k=-\infty}^{\infty} S(\omega - k\omega_0), \quad \omega_0 = \frac{2\pi}{\Delta t}$$



Nyquist theorem

$$S_d(\omega) = \frac{1}{\Delta t}\sum_{k=-\infty}^{\infty} S(\omega - k\omega_0), \quad \omega_0 = \frac{2\pi}{\Delta t}$$



Nyquist theorem

From the previous figures we have found the condition to avoid aliasing:

$$2\omega_{max} < \omega_0$$

$$\omega_0 = 2\pi / \Delta t$$

$$\Rightarrow \quad \omega_{max} < \pi / \Delta t$$

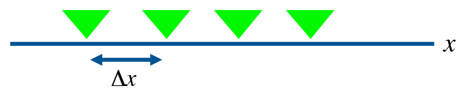
If we prefer to use frequency (Hz) rather than angular frequency (rad/sec):

$$2\pi f_{max} < \pi / \Delta t \quad \Rightarrow \quad \boxed{\Delta t < \frac{1}{2 f_{max}}}$$

Famous Nyquist condition

Nyquist theorem

- From now on, we consider signals where the sampling interval satisfies the Nyquist condition.
- It is clear that discrete signals must arise from the discretization of a band-limited analog signal. Electronic filters are often placed prior to discretization to guarantee that the signal to sample does contain not energy above a maximum frequency.
- Nyquist condition is easy to satisfied in the time domain (temporal sampling)
- <u>Spatial sampling is often dictated by cost & logistics not by hardware!!</u>



- Multi-dimensional sampling in space is a problem of current research since prestack seismic data are often under sampled in one or more coordinates (4 spatial coordinates)

DFT

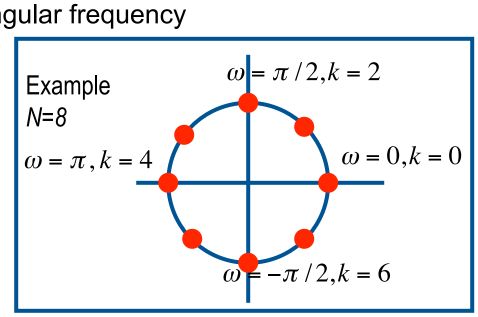
- When dealing with discrete time series or evenly sample data along the spatial domain we will use the *Discrete Fourier Transform (DFT)*

$$S(\omega) = \sum_{k=0}^{N-1} s_k \, e^{-i\omega k}$$

ω : angular frequency [rads, no dimensions]

$$\omega_l = \frac{2\pi l}{N}, \; l = 0,...N-1 \quad \text{discrete angular frequency}$$

$$S(\omega_l) = \sum_{k=0}^{N-1} s_k \, e^{-i\omega_l k}$$



IDFT

- We also need a transform to come back *Inverse Discrete Fourier Transform (IDFT)*

$$s_k = \frac{1}{N}\sum_{k=0}^{N-1} S(\omega_l)\, e^{i\omega_l k}$$

- *A note about frequency*

$$\omega_l = \frac{2\pi l}{N},\ l = 0,...N-1$$ discrete angular frequency

$$\omega_l = \frac{2\pi l}{N\ \Delta t},$$ radians/secs

$$f_l = \frac{\omega_l}{2\pi} = \frac{l}{N\ \Delta t},$$ Hertz

Notes

- Wrong wording →The FFT Spectrum,
- You should say the DFT Spectrum because the FFT is just the tool that is used to compute the DFT in a fast way
- Remember that to apply the DFT is equivalent to multiply a Matrix times a Vector (N^2 operations)
- FFT is a simple matrix multiplication via a faster algorithm ($N \log_2 N$ operations)

Linear systems

- Linear systems
  - An easy way of describing physical phenomena
  - A good approximation to some inverse problems in geophysics
  - Given

$$x_1(t) \rightarrow y_1(t)$$

$$x_2(t) \rightarrow y_2(t)$$

The system is linear if

$$\alpha x_1(t) + \beta x_2(t) \rightarrow \alpha y_1(t) + \beta y_2(t)$$

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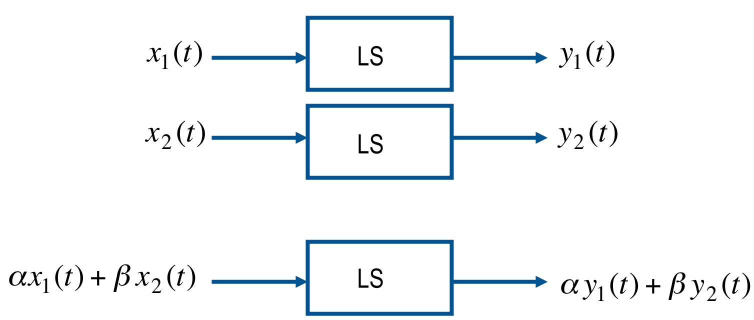
Linear systems



LS: Linear System (the Earth, if you do not consider important phenomena!)

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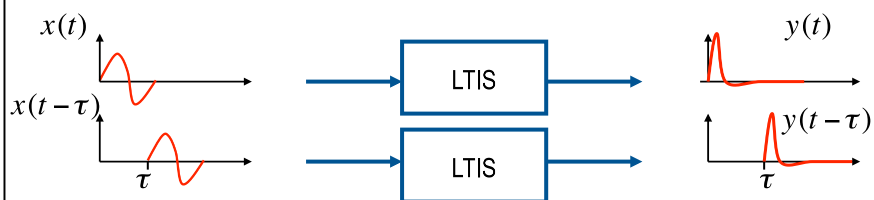
Linear systems and Invariance

- Invariance [Linear Time Invariant System]
  - Consider a system that is linear and also impose the condition of invariance:

$$x\ \ (t) \rightarrow y(t)$$

$$x(t-\tau) \rightarrow y(t-\tau)$$



Example: *deconvolution operator*

Linear systems and Invariance

- If the system is linear and time invariant, input and output are related by the following expression (it can be proven)

$$y(t) = \int h(t-\tau)x(\tau)d\tau = h(t) * x(t)$$

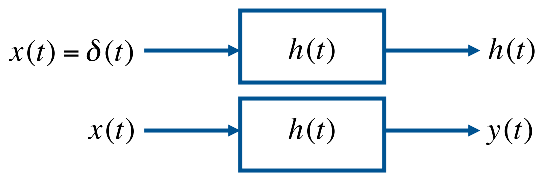
Convolution Symbol

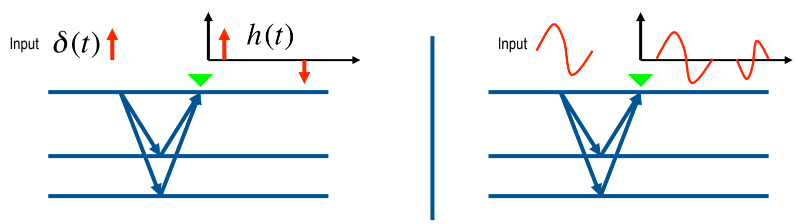
- We are saying that if our process is represented by an LTIS then the I/O can be represented via a convolution integral
- The new signal $h(t)$ is called the impulse response of the system

Linear systems and Invariance

- Impulse response (hitting the system with an impulse)

$$y(t) = \int h(t-\tau)x(\tau)d\tau = h(t) * x(t)$$



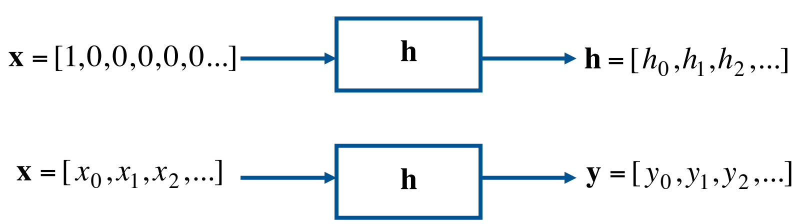


Linear systems and Invariance - Discrete case

- Convolution Sum

$$y_n = \sum_k h_{k-n} x_k = h_n * x_n$$

- Signals are time series or vectors



Discrete convolution

- Formula

$$y_n = \sum_k h_{n-k} x_k = h_n * x_n$$

- Finite length signals

$$x_k, \quad k = 0, NX - 1$$

$$y_k, \quad k = 0, NY - 1$$

$$h_k, \quad k = 0, NH - 1$$

- How do we do the convolution with finite length signals?
  - With paper and pencil
  - Computer code
  - Matrix times vector
  - Polinomial multiplication
  - DFT

Discrete convolution

```
% Initialize output
 y(1:NX+HH-1)=0
% Do convolution sum
 for i=1:NX
  for j = 1:NH
   y(i+j-1) = y(i+j-1) + x(i)h(j)
  end
 end
```

Discrete convolution

Example:

$$x = [x_0, x_1, x_2, x_3, x_4], \quad NX = 5$$

$$h = [h_0, h_1, h_2], \quad NH = 3$$

$$y_n = \sum_k h_{k-n} x_k = h_n * x_n$$

$$y_0 = x_0 h_0$$
$$y_1 = x_1 h_0 + x_0 h_1$$
$$y_2 = x_2 h_0 + x_1 h_1 + x_0 h_2$$
$$y_3 = x_3 h_0 + x_2 h_1 + x_1 h_2$$
$$y_4 = x_4 h_0 + x_3 h_1 + x_2 h_2$$
$$y_5 = x_4 h_1 + x_3 h_2$$
$$y_6 = x_4 h_2$$

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} x_0 & 0 & 0 \\ x_1 & x_0 & 0 \\ x_2 & x_1 & x_0 \\ x_3 & x_2 & x_1 \\ x_4 & x_3 & x_2 \\ 0 & x_4 & x_3 \\ 0 & 0 & x_4 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix}$$

Transient-free Convolution Matrix

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = \begin{pmatrix} x_0 & 0 & 0 \\ x_1 & x_0 & 0 \\ x_2 & x_1 & x_0 \\ x_3 & x_2 & x_1 \\ x_4 & x_3 & x_2 \\ 0 & x_4 & x_3 \\ 0 & 0 & x_4 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} \longrightarrow \begin{pmatrix} y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_2 & x_1 & x_0 \\ x_3 & x_2 & x_1 \\ x_4 & x_3 & x_2 \end{pmatrix} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix}$$

Classical convolution

Transient-free convolution

Discrete convolution and the z-transform

- Z-transform: a compact way of dealing with time series

The z-transform of $x = [x_0, x_1, x_2, x_3, x_4]$, $NX = 5$

is given by $X(z) = x_0 + x_1 z + x_2 z^2 + x_3 z^3 + x_4 z^4$

- Example:

Casual

$x = [\underset{\uparrow}{2}, -1, 3]$

$X(z) = 2 - 1z + 3z^2$

Non-causal

$x = [-1, \underset{\uparrow}{2}, -1, 3]$

$X(z) = -1z^{-1} + 2 - 1z + 3z^2$

$\uparrow$ Indicates sample *n=0*

What can we do with the z-transform?

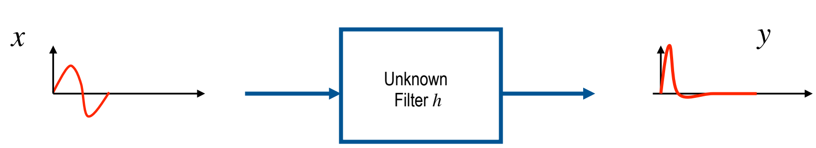
- **Convolve series**

$x = [x_0, x_1, x_2, x_3, x_4]$ → $X(z) = x_0 + x_1 z + x_2 z^2 + x_3 z^3 + x_4 z^4$

$h = [h_0, h_1, h_2]$ → $H(z) = h_0 + h_1 z + h_2 z$

$y = x * h$ → $Y(z) = X(z).H(z)$

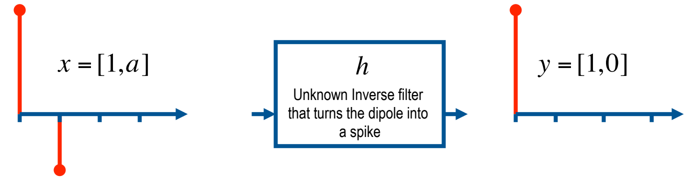
- **Design inverse filters** (*finally some seismology…*)



- Let's see how one can use the z-transform to find "Inverse Filters" of simple signals

Dipoles and inverse of a dipole

Dipole: a signal made of two elements



Find the inverse filter with the z-transform:

$$X(z)=1+az, \qquad Y(z)=1$$

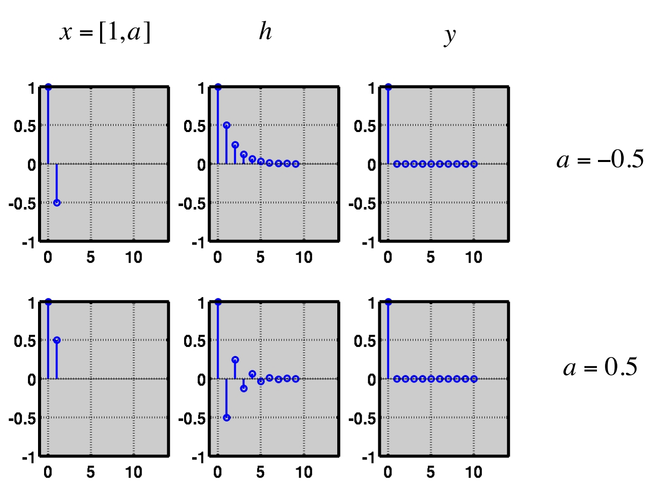
$$y=x*h \leftrightarrow Y(z)=X(z).H(z)$$

Geometric Series

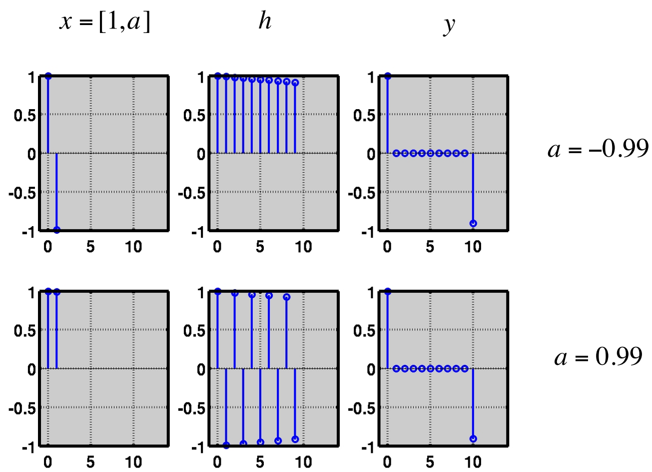
$$H(z)=\frac{Y(z)}{X(z)}=\frac{1}{1+az}=1-az+a^2z^2-a^3z^3+a^4z^4\ldots\ldots\ldots\ldots\ldots\ldots$$

$$h=[1,-a,a^2,-a^3,a^4,\ldots\ldots\ldots\ldots]$$

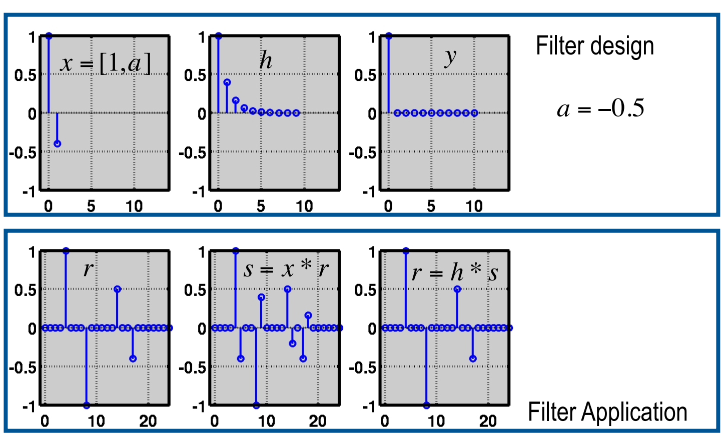
Inversion of a dipole using geometric series



Inversion of a dipole using geometric series: Truncation of the operator

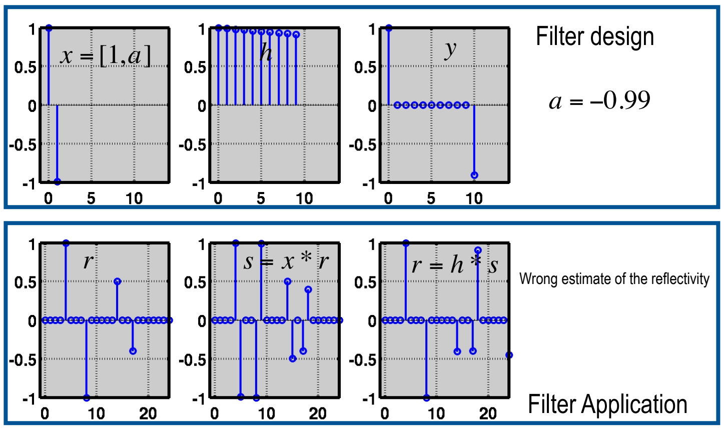


Inversion of a dipole using geometric series: Deconvolution of a simple reflectivity series



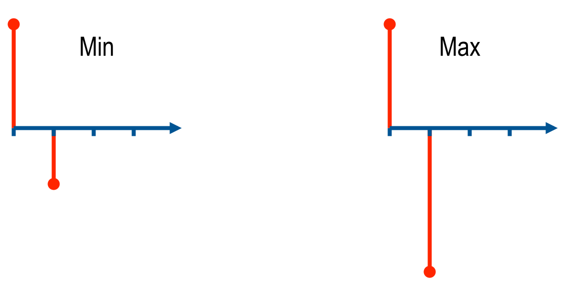
Inversion of a dipole using geometric series: Deconvolution of a simple reflectivity series

- Truncation in the operator introduces false reflections in the deconvolution output



Minimum and Maximum Phase dipoles

- *In simple terms*
- Minimum Phase $x = [1, a], \quad |a| < 1$
- Maximum Phase $x = [1, a], \quad |a| > 1$



Dipoles and Phase duality

- Take two dipoles

$$x_{MIN} = [1, a], \quad |a| < 1$$

$$x_{MAX} = [a, 1] = a[1, 1/a] = a[1, b], \quad |b| > 1$$

- You can show that

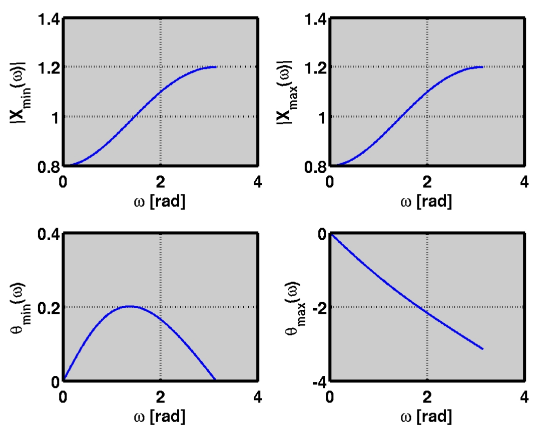
$$|X_{MAX}(\omega)| = |X_{MIN}(\omega)|$$

$$\theta_{MAX}(\omega) \neq \theta_{MIN}(\omega)$$

- *Same amplitude spectrum*
- *Different phase spectrum*
- If only the amplitude spectrum is measured, one cannot uniquely determine the dipole (two dipoles produce the same amplitude)

Dipoles and Phase duality

$$x_{MIN} = [1, a], \quad |a| < 1 \qquad x_{MAX} = [a, 1] = a[1, 1/a] = a[1, b], \quad |b| > 1$$

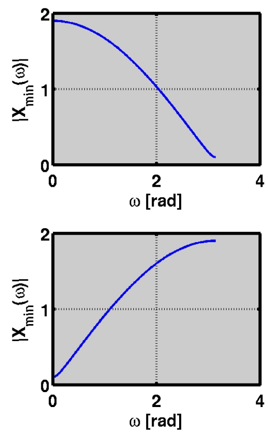


Dipole filters - careful here

- Some signal processing schemes attempt o increase BW by convolution with dipole filters. The amplitude spectrum of the dipole filter can be Low Pass or High Pass according to the sign of *a*

Examples:

- *Low Pass* $x_{MIN} = [1, a], \quad a = 0.9$
- *High Pass* $x_{MIN} = [1, a], \quad a = -0.9$



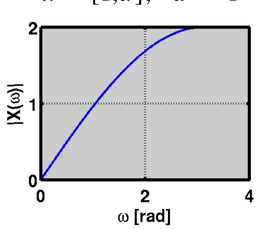
Dipole filters - careful here

- Differentiator (Extreme High Pass dipole)

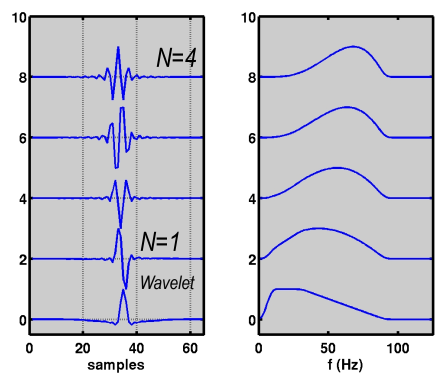
Examples:

- *High Pass*

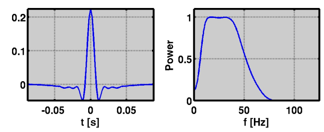
$x \quad = [1, a], \quad a = -1$



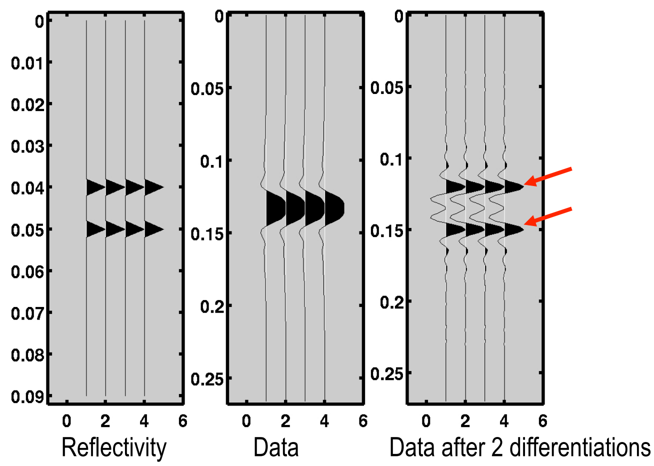
Wavelet convolved *n* times with differentiator - Cosmetic freq. enhancement ??



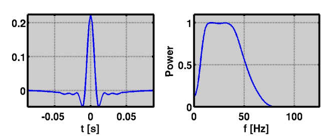
Dipole filters - careful here



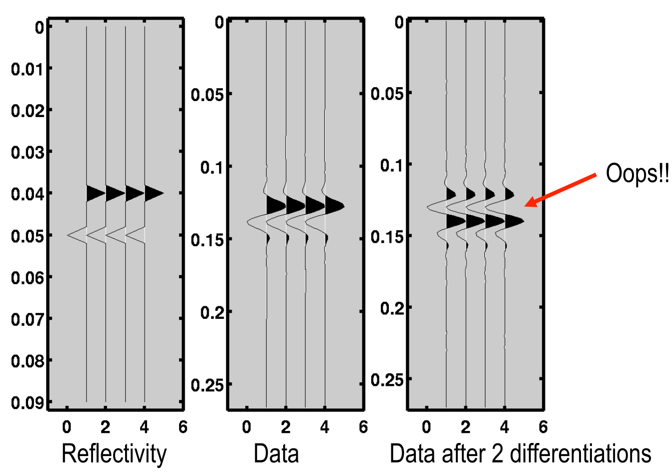
- *N=2* (two differentiations)



Dipole filters - careful here

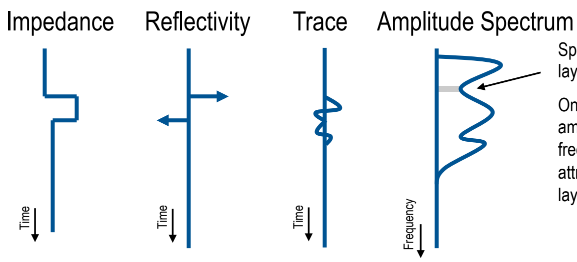


- *N=2* (two differentiations)



More about dipoles: Spectral Decomposition

- Some modern seismic interpretation methods are based on properties of dipoles filters
- Spectral Decomposition attempts to image thing layers by the spectral behaviour of signals similar to dipoles



Spectral notch is proportional to layer thickness

One, for instance, can map the amplitude at that particular frequency. This will provide an attribute for the *x-y* variability of layer thickness

Interesting to point out that rather than whitening (flattening) the spectrum like in conventional decon, spectral decomposition attempts to track spectral features/attributes

This is the basis of spectral decomposition.

*Partyka, G., 2005, Spectral Decomposition: Recorder, **30** (www.cseg.ca).*

More about dipoles: Spectral Decomposition

Thin Layer

$$r = [\overbrace{a,0,0,0,0,b}^{\tau = 4.\Delta t},0,0,0...]$$

$$R(\omega) = a + be^{-i\omega\tau}$$

Spectrum

$$|R(\omega)|^2 = a^2 + b^2 + 2\,a\,b\cos(\omega\tau)$$

Min/Max condition

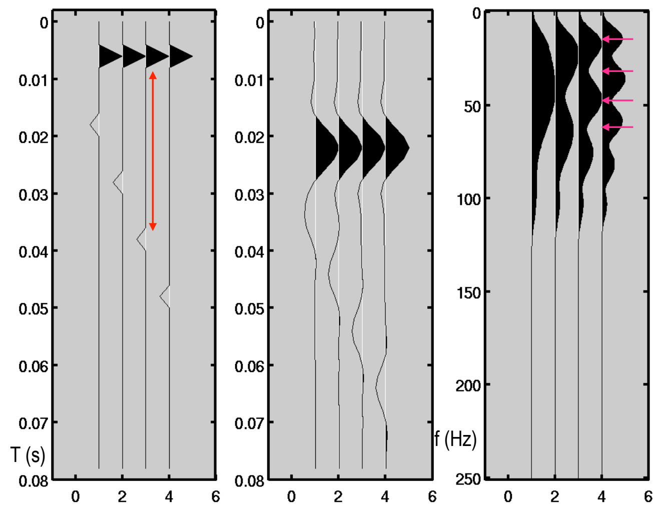
$$\frac{d\,|R(\omega)|^2}{d\omega} = 0 \Rightarrow \quad \sin(\omega\tau) = 0 \Rightarrow \omega\tau = \pi k,\; k = 0,1,2,3,4$$

Frequency at stationary point

$$\boxed{f_s = k/(2\tau)}$$

The second derivative can be used to determine if the stationary point is a min or max. Min or max depends on the signs of the reflection coefficients *a* and *b*.

More about dipoles: Spectral Decomposition

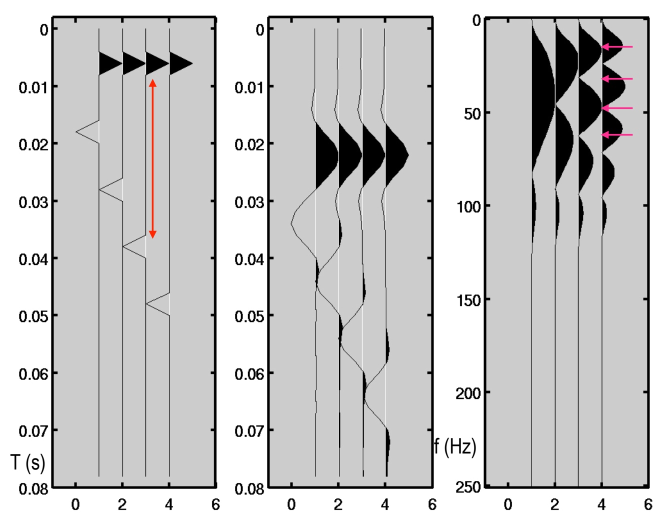


For trace #3:

$\tau = 0.028s$

$f_s = 17.8, 35.7, 53.6, 71.4Hz$

More about dipoles: Spectral Decomposition



For trace #3:

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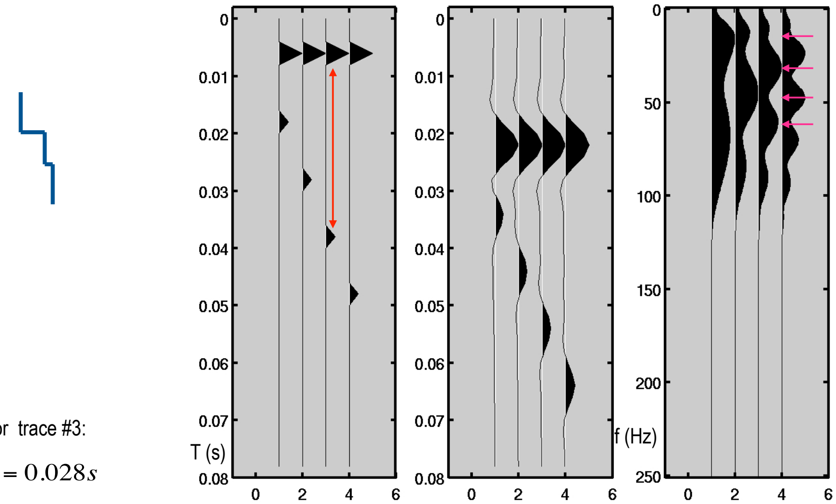
More about dipoles: Spectral Decomposition



For trace #3:

$\tau = 0.028s$

$f_s = 17.8, 35.7, 53.6, 71.4Hz$