Bayesian Priors for Sparse Inversion

Mauricio D. Sacchi and Tadeusz J. Ulrych

Consortium for the Development of Specialized Seismic Techniques
Department of Geophysics and Astronomy, UBC

Outline

- Sparse inversion of seismograms
- Priors for sparse inversion
- Hyperparameter selection
- Applications
- Discussion and Conclusions

The problem

$$\mathbf{W}\mathbf{x} = \mathbf{y} + \mathbf{n}$$

$$\mathbf{C}\mathbf{x} = \boldsymbol{\xi} + \boldsymbol{\epsilon}$$

The linear programming approach

Minimize $\quad J = \alpha|\mathbf{x}|_1 + |\mathbf{e}|_1$

subject to $\quad \mathbf{W}\mathbf{x} = \mathbf{y} + \mathbf{e}$

and $\quad \boldsymbol{\xi}_l < \mathbf{C}\mathbf{x} < \boldsymbol{\xi}_\mathbf{u}\,.$

Bayes' rule

$$p(m|d) \propto p(m) \times p(d|m)$$

$$\boxed{\text{Posterior} \propto \text{Prior} \times \text{Likelihood}}$$

MAP solution

$$J = -\ln[p(m|d)]$$

MAP - Objective Function

$$J = \alpha \underbrace{J_x}_{1} + \underbrace{\frac{1}{2}||\frac{1}{\sigma}(\mathbf{Wx} - \mathbf{y})||^2}_{2} + \underbrace{\frac{1}{2}||\mathbf{S}^{-1}(\mathbf{Cx} - \boldsymbol{\xi})||^2}_{3}$$

- 1 - The solution must be sparse.
- 2 - The solution must honour the seismic trace.
- 3 - The solution must honour a set of impedance constraints.

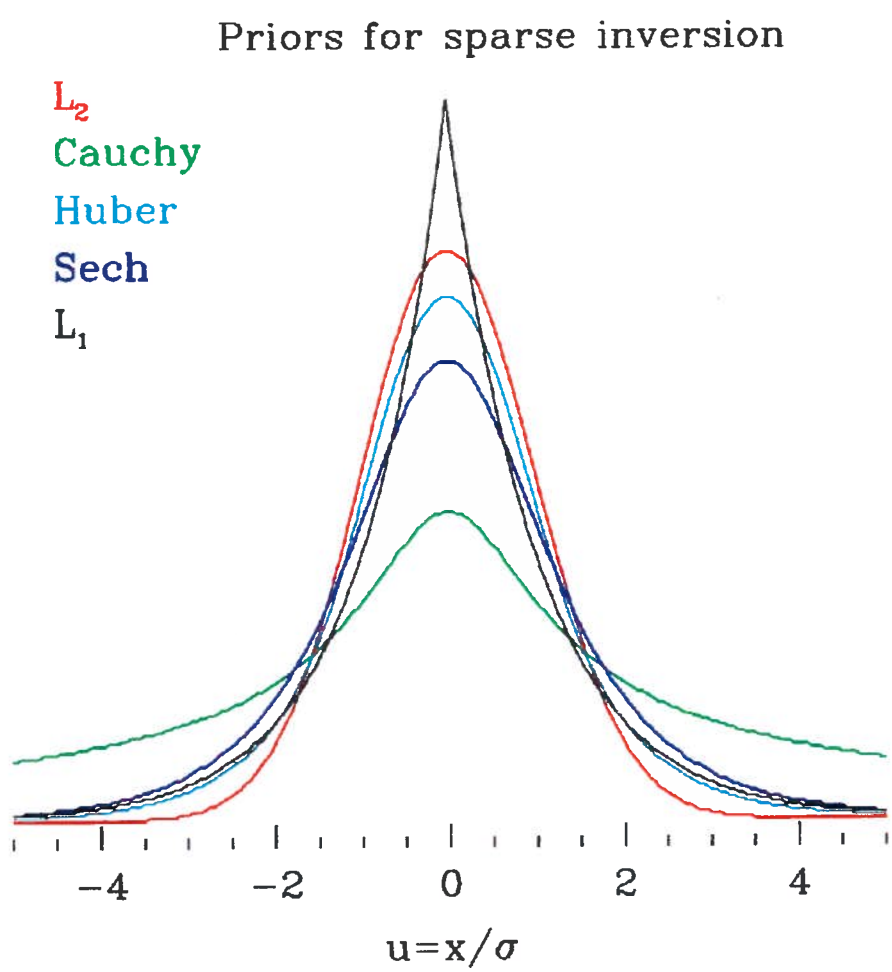
Regularization - Sparseness criteria

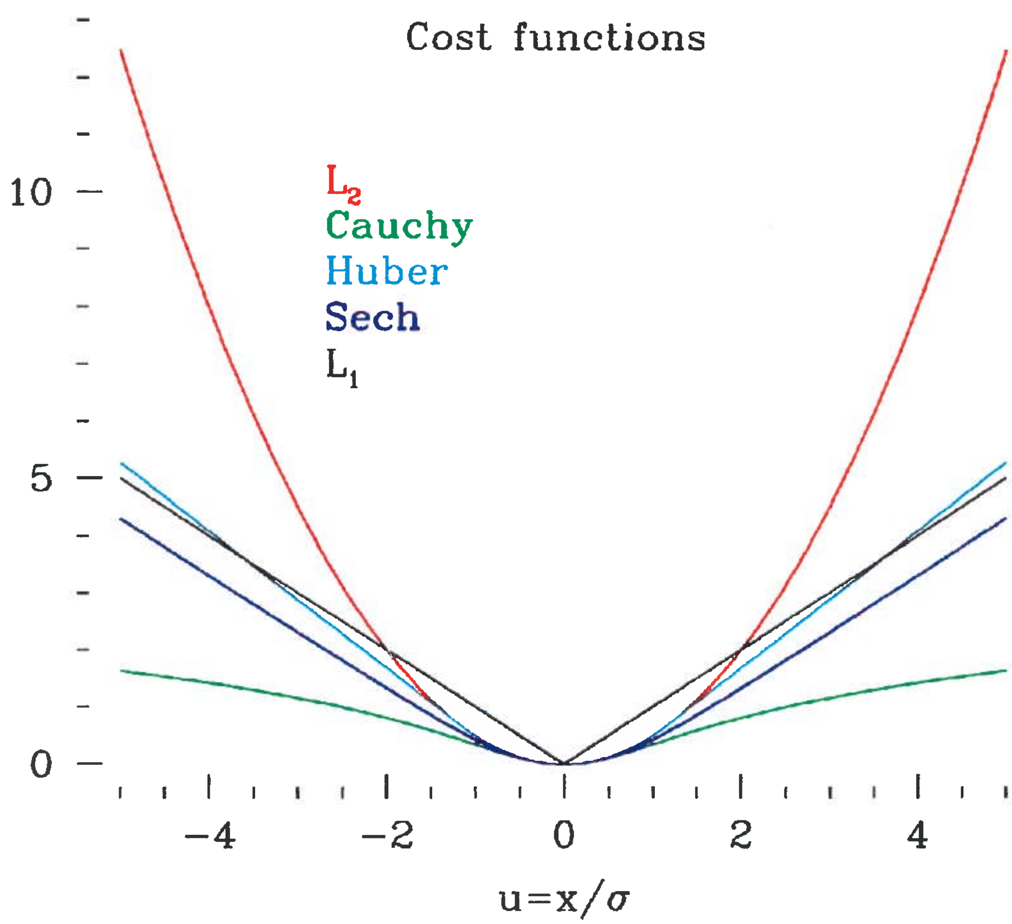
$$J_p = \frac{1}{p} \sum_i |x_i|^p$$

$$J_{Cauchy} = \frac{1}{2} \sum_i \ln(1 + \frac{x_i^2}{\sigma_x^2})$$

$$J_{Sech} = \sum_i \ln(\cosh \frac{x_i}{\sigma_x})$$

$$J_{Huber} = \sum_i \begin{cases} x_i^2/2 & \text{if } |x_i| \leq x_c \\ a\,|x_i| - x_c^2/2 & \text{if } |x_i| > x_c \end{cases}$$


Priors for sparse inversion
L2
Cauchy
Huber
Sech
L1
−4
−2
0
2
4
u=x/σ


Cost functions
L2
Cauchy
Huber
Sech
L1
10
5
0
−4
−2
0
2
4
u=x/σ

Minimization of J - CG algorithm

1. Given $\mathbf{x}_0$ compute $\mathbf{g}_0 = \nabla J(\mathbf{x})$ and set $\mathbf{d}_0 = -\mathbf{g}_0$

2. for $k = 0, 1, 2, \ldots n-1$

3. $\mathbf{x}_{k+1} = \mathbf{x}_k + a\mathbf{d}_k$ where a minimizes $J(\mathbf{x}_k + a\mathbf{d}_k)$ (line search)

4. compute $\mathbf{g}_k = \nabla J(\mathbf{x}_{k+1})$

5. set $\mathbf{d}_{k+1} = -\mathbf{g}_{k+1} + b_k \mathbf{d}_k$ where

$$b_k = (\mathbf{g}_{k+1} - \mathbf{g}_k)^T \mathbf{g}_k / \mathbf{g}_k^T \mathbf{g}_k$$

6. Stop if

$$\frac{|J_{k+1} - J_k|}{1/2(|J_{k+1} + |J_k|)} < \text{tolerance}$$

Hyperparameter estimation

- Cost function

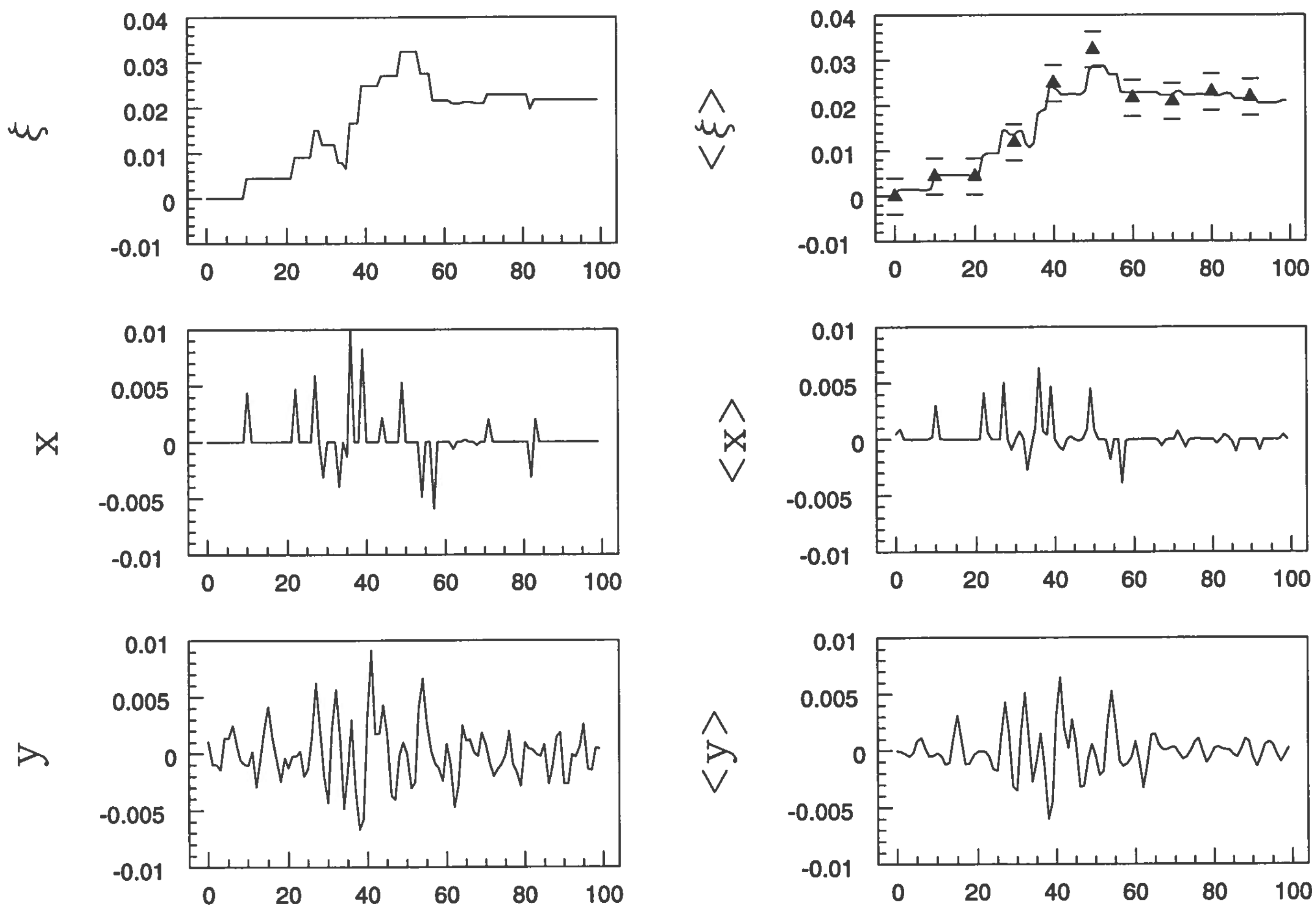
$$J = \alpha J_x + \frac{1}{2}||\frac{1}{\sigma}(\mathbf{Wx} - \mathbf{y})||^2 + \frac{1}{2}||\mathbf{S^{-1}}(\mathbf{Cx} - \boldsymbol{\xi})||^2$$

- Discrepancy principle

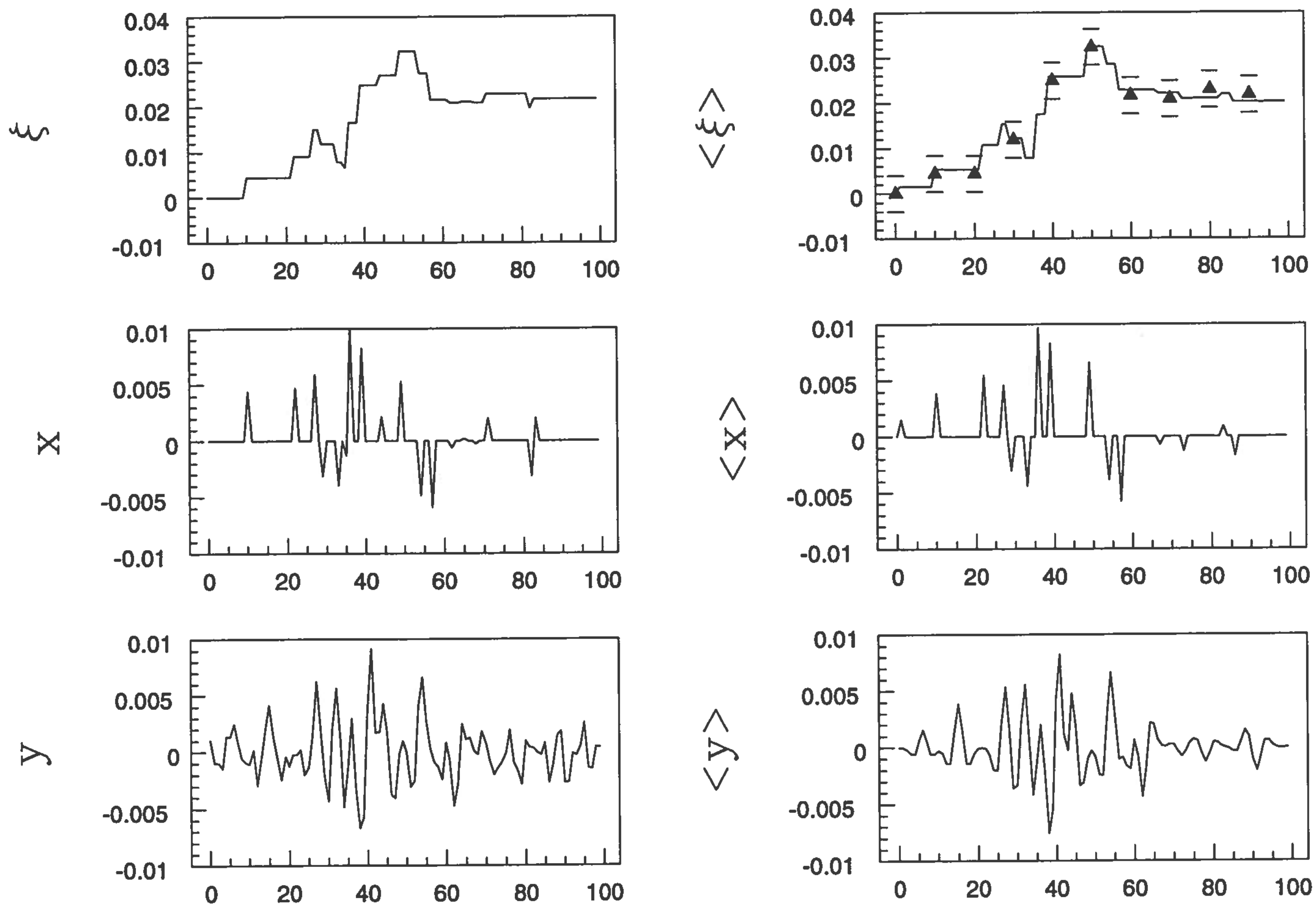
$$\chi^2 = ||\frac{1}{\sigma}(\mathbf{Wx} - \mathbf{y})||^2 + ||\mathbf{S^{-1}}(\mathbf{Cx} - \boldsymbol{\xi})||^2$$

$$E[\chi^2] = n \pm \sqrt{2\,n}$$

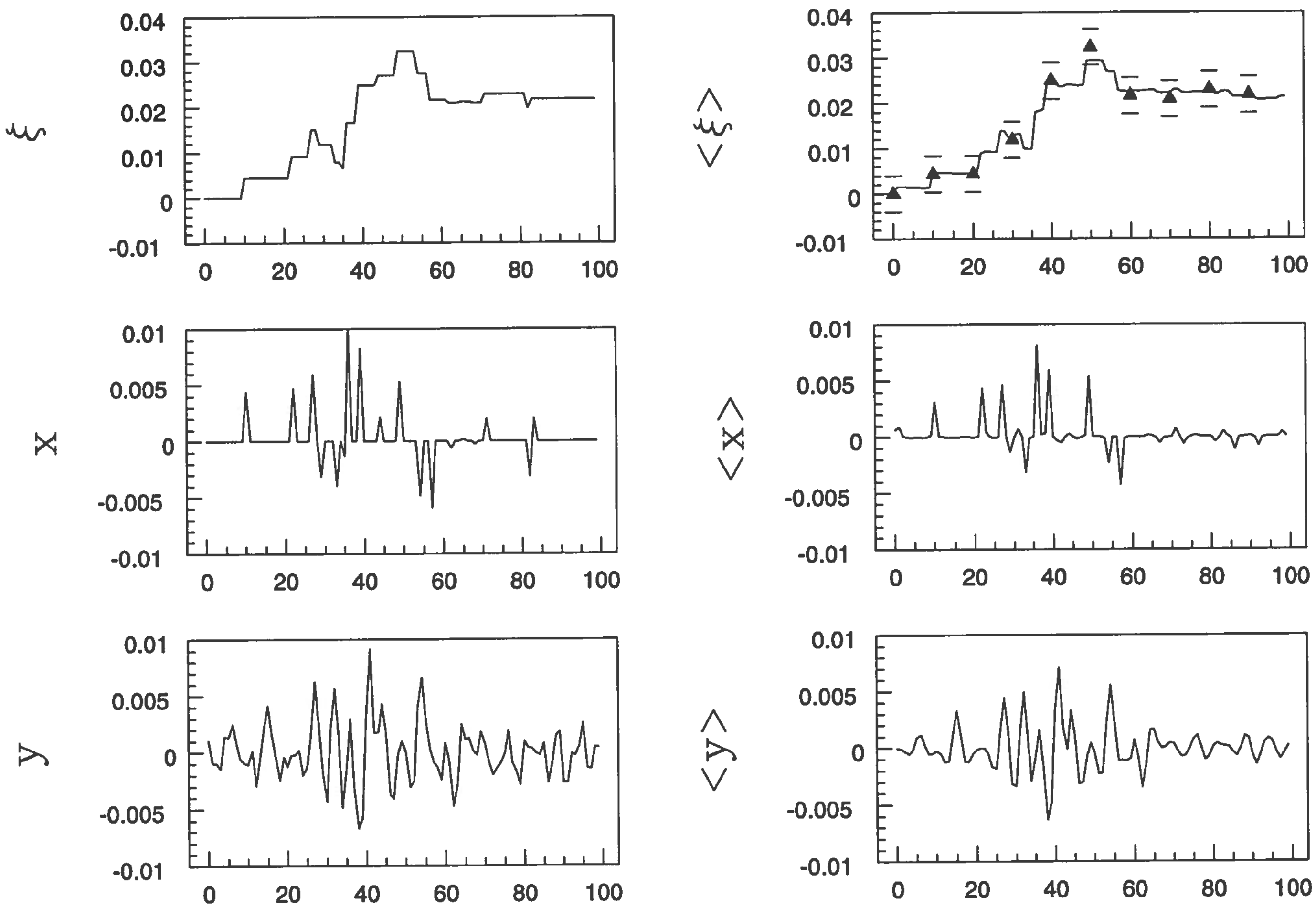
L_1 prior, Gaussian Likelihood



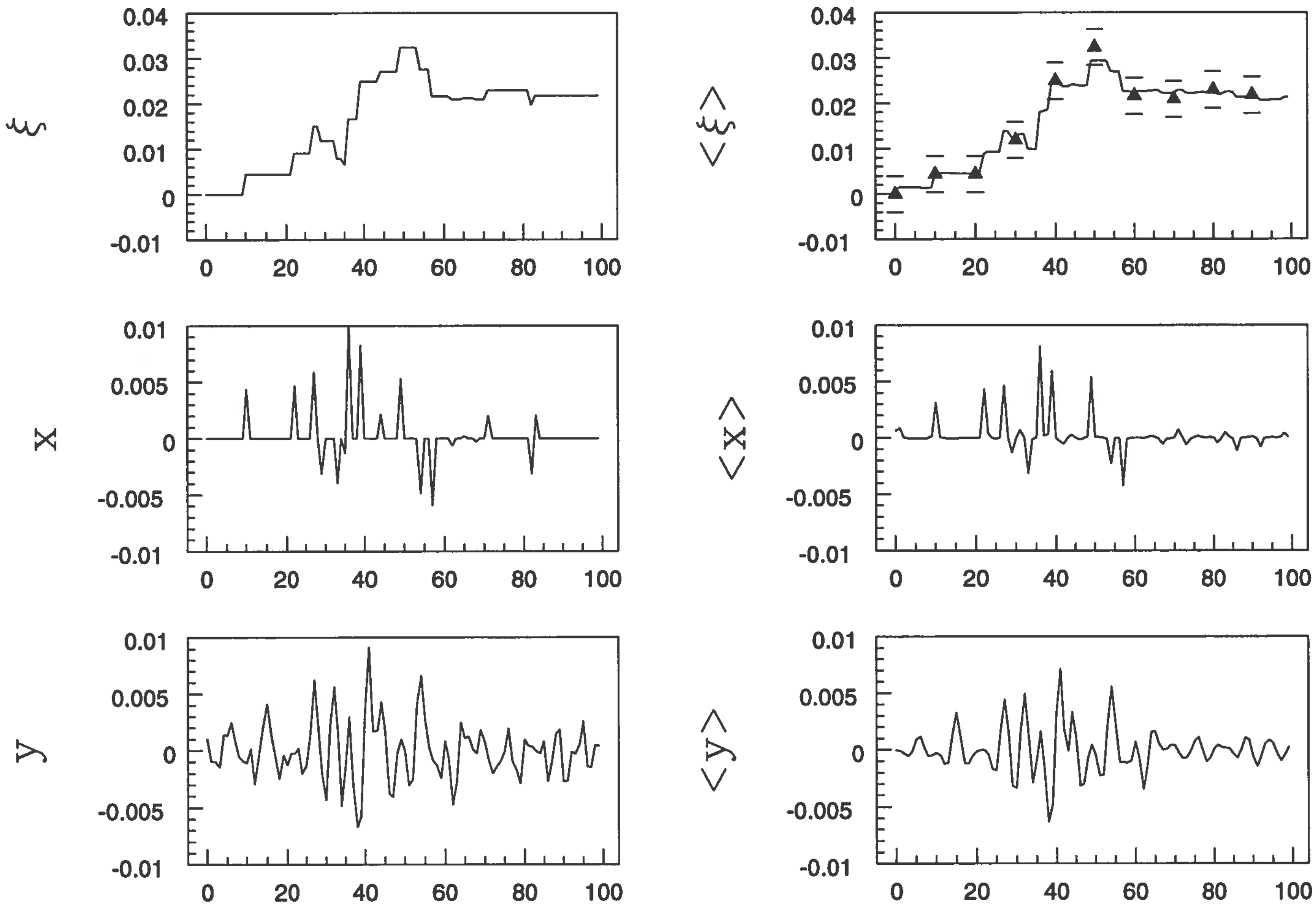
Cauchy prior, Gaussian Likelihood



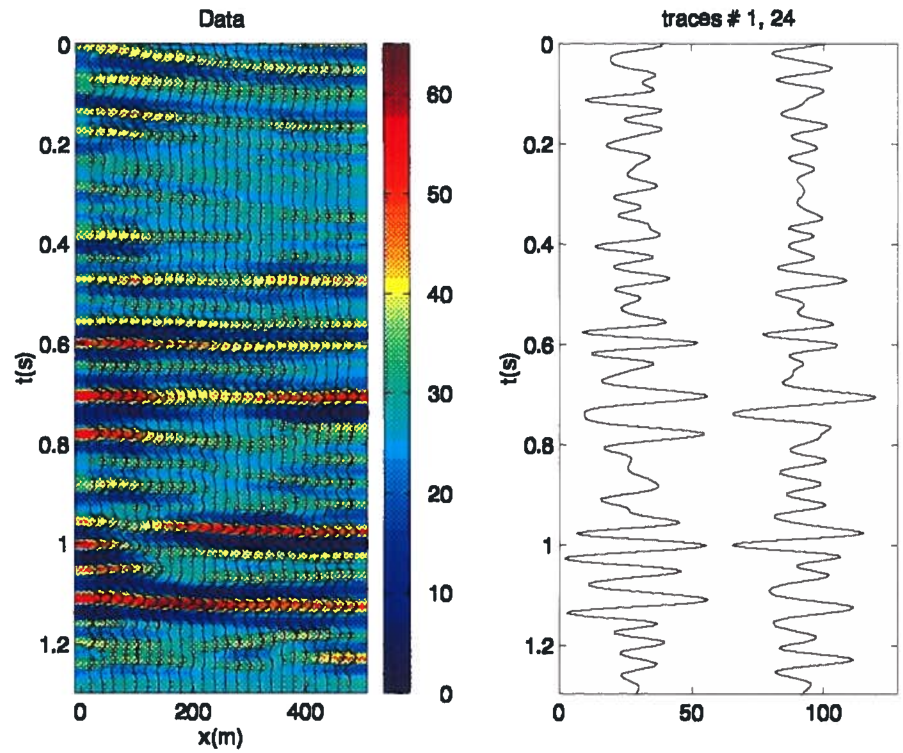
Sech prior, Gaussian Likelihood



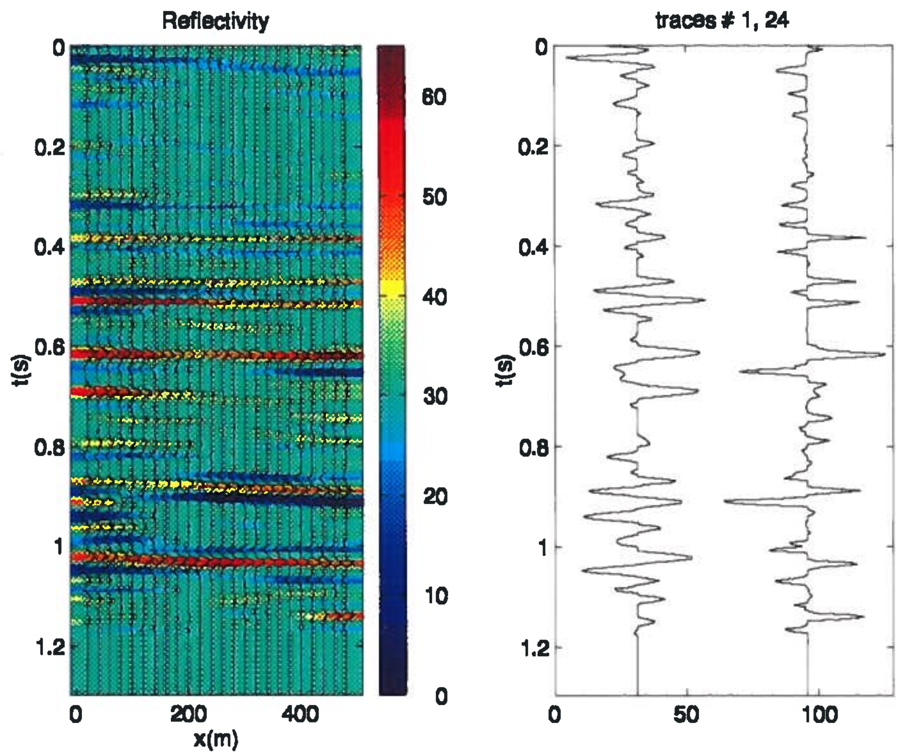
Huber prior, Gaussian Likelihood



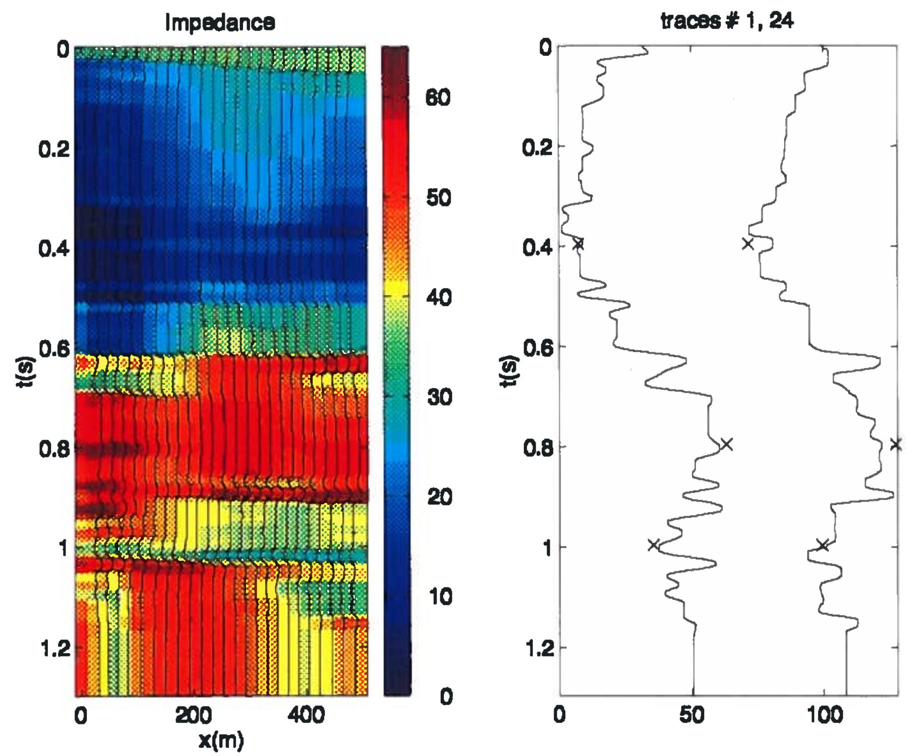
Field data example



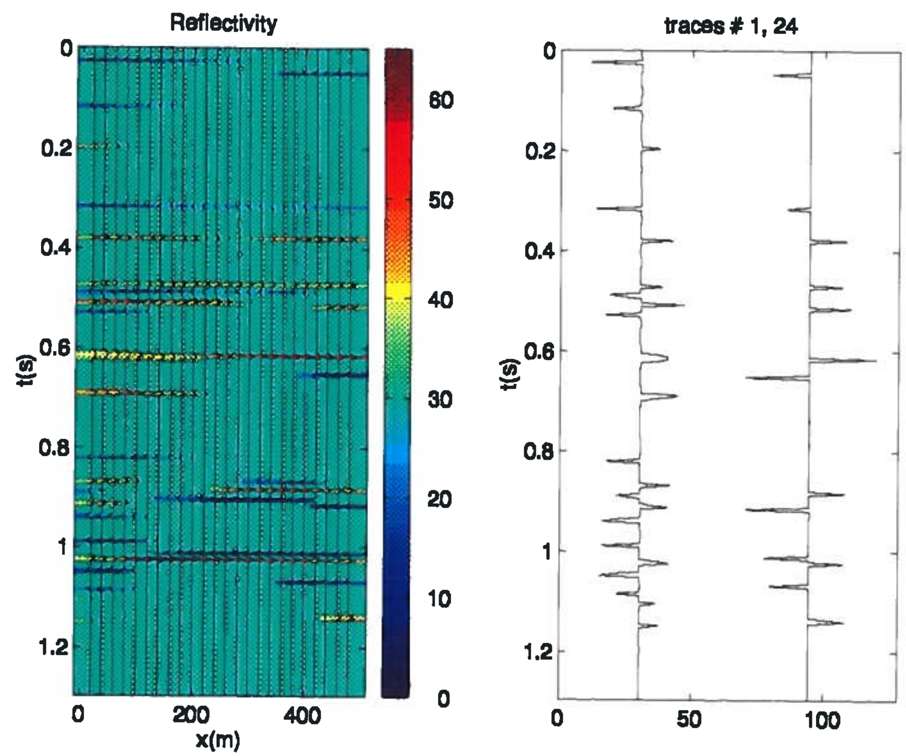
L_1 prior, Gaussian likelihood



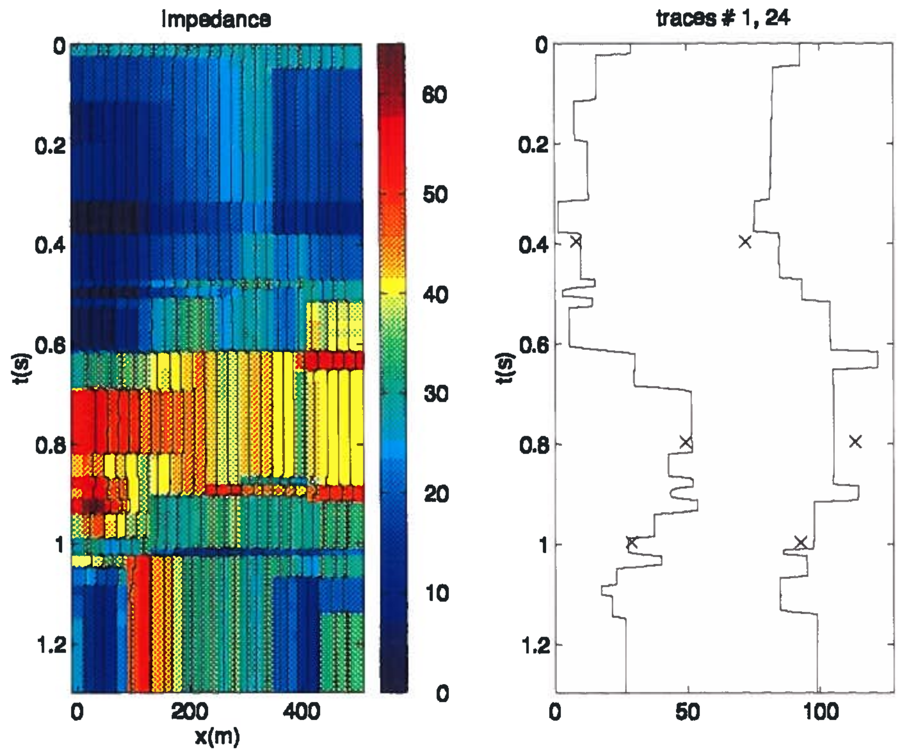
L_1 prior, Gaussian likelihood



Cauchy prior, Gaussian likelihood



Cauchy prior, Gaussian likelihood



Computations time for Bayes (CG) and Linear Programming

	N	NC	Time (s) (*)
Bayes (CG)	100	5	1.8
	300	15	6.3
L-prog. (BR)	100	5	13.3
	300	15	225.

(*) Sparc 5.

CG: Conjugate gradients

BR: Barrodale & Roberts

Conclusions

- Priors for sparse inversion

  The *Huber* and the *Sech* criteria tend to treat large amplitudes like the L_1 criterion and small amplitudes like the L_2 criterion.

  The Cauchy criterion treats small amplitudes like the L_2 criterion. Large amplitudes are emphasized.

- The algorithm

  The cost function derived with Bayes' rule is minimized using a CG procedure. The computational cost of the linear programming approach is avoided.

  Human intervention can be minimized using diverse strategies for hyperparameter selection.

Acknowledgment

This work was supported by the Consortium for the Development of Specialized Seismic Techniques at UBC.

- Participant companies

  AGIP

  ARCO

  JNOC

  PanCanadian

  Pulsonic