The Alberta High School Mathematics Competition
Part II, February 5th, 2020

Problem 1
Let \( P(x) \) be a polynomial with integer coefficients. Show that if \( P\left(\frac{1}{3}\right) \) is an even integer then \( P(3) \) will also be an even integer.

**Solution:** If \( P(x) = a_n x^n + \cdots + a_1 x + a_0 \), and \( P\left(\frac{1}{3}\right) = M \) where \( M \) is even, then
\[
a_n \left(\frac{1}{3}\right)^n + a_{n-1} \left(\frac{1}{3}\right)^{n-1} + \cdots + a_1 \frac{1}{3} + a_0 = M
\]
or equivalently
\[
a_n + a_{n-1} \cdot 3 + \cdots + a_1 \cdot 3^{n-1} + a_0 \cdot 3^n = M \cdot 3^n
\]
Thus,
\[
P(3) - 3^n M = a_n \cdot 3^n + a_{n-1} \cdot 3^{n-1} + \cdots + a_1 \cdot 3 + a_0 - (a_n + a_{n-1} \cdot 3 + \cdots + a_1 \cdot 3^{n-1} + a_0 \cdot 3^n)
\]
\[
= a_n (3^n - 1) + a_{n-1} (3^{n-1} - 3) + \cdots + a_1 (3 - 3^{n-1}) + a_0 (1 - 3^n)
\]
Since the \( 3^n M \) and \( 3^n - 1, 3^{n-1} - 3, \ldots, 1 - 3^n \) are all even numbers, one obtains that \( P(3) \) is an even number.

**Alternative Solution:** Since \( 3 \equiv 1 \) (mod 2) then \( P(3) \equiv a_0 + a_1 + \cdots + a_n \equiv M3^n \equiv M \) (mod 2) and hence \( P(3) \) is even.

Problem 2
Find all functions \( f(x) = \frac{47}{ax+b} \) where \( a \) and \( b \) are integers and \( a > 0 \), so that \( f(4) \) and \( f(7) \) are both integers.

**Solution:** Since \( f(4) = \frac{47}{4a+b} \) and \( f(7) = \frac{47}{7a+b} \) are both integers, and 47 is prime, \( 4a+b \) and \( 7a+b \) must each be equal to one of \( \pm 1, \pm 47 \). By subtraction, \( 3a \) (being positive) must be one of 2, 46, 48 or 94. The only possibility for integer \( a \) is 3a = 48, which arises from either \( 7a+b = 47, 4a+b = -1 \) or \( 7a+b = 1, 4a+b = -47 \). With \( a = 16 \), the former gives \( b = -65 \) and the latter gives \( b = -111 \). So the possible functions are
\[
f(x) = \frac{47}{16x-65} \quad \text{and} \quad f(x) = \frac{47}{16x-111}.
\]

Problem 3
Consider all the subsets of \( \{5, 6, 7, \ldots, 15\} \) having at least two elements. How many of these subsets have the property that the sum of the smallest and the largest element in the subset is 20?

**Solution:** Let \( A_k \) be a non-empty subset of \( \{5, 6, 7, \ldots, 15\} \) having the required property, and with smallest element \( k \), hence \( 20 - k \) is the largest element of \( A_k \). For the remaining elements in \( A_k \) it is possible to choose any (or none) of \( k+1, k+2, \ldots, 19-k \). Therefore, the number of such subsets \( A_k \) is \( 2^{19-2k} \). On the other hand since \( 20 - k > k \) and \( k \geq 5 \), i.e., \( 5 \leq k \leq 9 \), the total number of requested subsets is
\[
2^{19-25} + 2^{19-26} + 2^{19-27} + 2^{19-28} + 2^{19-29} = 2(1 + 4 + 16 + 64 + 256) = 682.
\]

Problem 4
\( \triangle ABC \) has right angle at \( A \). Point \( D \) lies on \( AB \), between \( A \) and \( B \), such that \( 3\angle ACD = \angle ACB \) and \( BC = 2BD \). Find the ratio \( \frac{DB}{DA} \).

**Solution:**
Let \( DA = m, DB = n \), and \( 3\angle ACD = \angle ACB \). Then \( BC = 2n \), and \( \angle DCA = 2a \). If the Sine Law is applied in \( \triangle DCA \) then
\[
\frac{n}{\sin(2a)} = \frac{2n}{\sin(90+a)} \quad \iff \quad \frac{n}{2 \sin(a) \cos(a)} = \frac{2n}{\cos(a)} \quad \iff \quad \sin(a) = \frac{1}{4}
\]
and since \( CD = \frac{m}{\sin a} \) one obtains \( CD = 4m \). Now
\[
AC^2 = (4m)^2 - m^2 = (2n)^2 - (m+n)^2 \quad \Rightarrow \quad 3n^2 - 2mn - 16m^2 = 0 \quad \iff \quad (3n-8m)(n+2m) = 0
\]
and hence \( \frac{DB}{DA} = \frac{n}{m} = \frac{4}{3} \).
Alternative Solution: Take $D'$ the symmetrical of $D$ with respect to $AC$, then $D'A = m$ and $CD$ is the bisector of $\angle D'CB$. Also, $\triangle D'CD$ is isosceles ($CA$ is the bisector and the altitude) hence $CD' = CD$. In $\triangle D'CB$ by using the Bisector Theorem one obtains

$$\frac{CD'}{CB} = \frac{DD'}{DB} \iff \frac{CD'}{2m/n} = \frac{2m}{n} \iff CD' = 4m$$

and hence $CD = CD' = 4m$. The solution continues as above.

Problem 5
Let $b_0 < c_0$ be real numbers so that the polynomial $f_0(x) = x^2 + b_0x + c_0$ has two real roots $b_1 < c_1$ (that is, $f_0(b_1) = f_0(c_1) = 0$) and let $f_1(x) = x^2 + b_1x + c_1$. If $f_1(x)$ has two real roots $b_2 < c_2$, a new quadratic polynomial $f_2(x) = x^2 + b_2x + c_2$ is constructed. The process is continued until the quadratic polynomial $f_n(x) = x^2 + b_{n-1}x + c_{n-1}$, $b_{n-1} < c_{n-1}$ has two real roots $b_n < c_n$, but $f_n(x) = x^2 + b_nx + c_n$, $n \geq 1$ has no real roots.

(a) Show that $n \leq 3$.

(b) Show that $n = 3$ is a possible value.

Solution:
Notice that if $x_1, x_2$ are the solutions of the quadratic equation $x^2 + Sx + P = 0$ then $x_1 + x_2 = -S$ and $x_1x_2 = P$; this known fact, which can be immediately justified by using the identity $x^2 + Sx + P = (x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1x_2$, will systematically be used in the solution of this problem.

(a) Note that one of the inequalities $b_1 \leq 0 \leq c_1, b_1 < c_1 < 0 < b_1 < c_1$ must hold. We prove the following statements:

(A) If $b_1 \leq 0 \leq c_1$ then $f_1(x)$ has no real roots (hence $n = 1$).

Proof of the statement (A):
Since $b_0 = -(b_1 + c_1)$, $c_0 = b_1c_1$ and $b_0 < c_0$ one obtains $b_1 + c_1 + b_1c_1 > 0$ and thus $0 \geq b_1 > -\frac{c_1}{1+c_1}$. Hence

$$\Delta_1 = b_1^2 - 4c_1 < \frac{c_1^2}{(c_1+1)^2} - 4c_1 = -c_1 \frac{4c_1^2 + 7c_1 + 4}{(c_1+1)^2} \leq 0,$$

that is $\Delta_1 < 0$, and therefore $f_1(x) = x^2 + b_1x + c_1$ has no real roots.

(B) If $b_1 < c_1 < 0$ and $f_1(x)$ has real roots $b_2 < c_2$ then $f_2(x) = x^2 + b_2x + c_2$ has no real roots (hence $n = 2$).

Proof of the statement (B):
Since $b_2c_2 = c_1 < 0$ we get $b_2 < 0 < c_2$, consequently, following the statement (A) and its proof (by replacing the subscript 0 with 1 and 1 with 2) one obtains that $f_2(x) = x^2 + b_2x + c_2$ has no real roots.

(C) If $0 < b_1 < c_1$ and $f_1(x), f_2(x)$ have real roots $b_2 < c_2$ and respectively $b_3 < c_3$, then $f_3(x) = x^2 + b_3x + c_3$ has no real roots (hence $n = 3$).

Proof of the statement (C):
From $b_2 + c_2 = -b_1 < 0$ and $b_2c_2 = c_1 > 0$ we get $b_2 < c_2 < 0$. Now, according to (B) (by replacing the subscript 1 with 2 and 2 with 3), one obtains that $f_3(x) = x^2 + b_3x + c_3$ has no real roots.

Thus $n \leq 3$ in all three cases.

(b) The maximum value of $n$ may be obtained if we can find two real numbers $b_1, c_1$ so that the conditions in the statement (C) are verified, i.e., $0 < b_1 < c_1$ and both polynomials $f_1(x), f_2(x)$ have two real roots $b_2 < c_2$ and respectively $b_3 < c_3$. We will find all these possibilities. Notice that for a selection of $b_1, c_1$ as above, we have $b_0 = -(b_1 + c_1)$ and $c_0 = b_1c_1$.

Put $b_1 = a, c_1 = a + h$, such that $a, h > 0$. The quadratic polynomial $f_1(x)$ has distinct real roots if and only if

$$\Delta_1 = b_1^2 - 4c_1 > 0 \iff a^2 - 4a - 4h > 0 \iff (a - 2)^2 > 4(h + 1) \iff a > 2 + 2\sqrt{1 + h},$$

while $f_2(x)$ has real distinct roots if and only if $\Delta_2 = b_2^2 - 4c_2 > 0$. Since $b_2 < c_2$ are the roots of $f_1(x) = x^2 + b_1x + c_1$, we know that $b_2 = -b_1 - \sqrt{b_1^2 - 4c_1} = -a - 2\sqrt{1 + h}$ and $c_2 = -a + \sqrt{1 + h}$ and hence

$$\Delta_2 = b_2^2 - 4c_2 > 0 \iff \left(\frac{-a - 2\sqrt{1 + h}}{2}\right)^2 > 2(-a + \sqrt{1 + h}) \iff a^2 + 2(a - 4)\sqrt{1 + h} + 2(a - 4) > 0.$$