Quantum Dimensions in Logarithmic CFT

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Logarithmic vertex operator algebras possess at least one module that is not completely reducible. In these cases not much is known about the relation between torus one-point functions and the representation category. I will review recent progress and discuss a conjecture that makes such a connection.

The rational modular story. In 1988 Verlinde observed that modular properties of characters of rational conformal field theories (CFTs) and the fusion ring of modules are closely connected [17]. For this let $\{M_0, M_1, \ldots, M_n\}$ be the set of inequivalent simple modules of a given CFT with M_0 the vacuum. Then the character of a module is

$$\mathrm{ch}[M](\tau) = \mathrm{tr}_M \left(q^{L_0 - \frac{c}{24}} \right), \qquad q = e(\tau),$$

and physics ensures that it converges on the upper half of the complex plane. Moreover these characters span a vector-valued modular form for the modular group which acts on functions of the upper half plane via Möbius transformations. Especially the transformation $\tau \mapsto -1/\tau$ is called the modular S-transformation and it defines a matrix, the S-matrix, via

$$\mathrm{ch}[M_i](-1/\tau) = \sum_{j=0}^{n} S_{ij} \mathrm{ch}[M_j](\tau) \quad \text{and numbers} \quad {N_{ij}}^{k} := \sum_{\ell=0}^{n} \frac{S_{i\ell} S_{j\ell} \left(S^{-1}\right)_{k\ell}}{S_{0\ell}}.$$

Verlinde's formula is that the ${N_{ij}}^k$ are the structure constants of the Grothendieck ring of the fusion ring. Verlinde gave a short physics argument for this conjecture and he verified it in examples. Not much later, Moore and Seiberg suggested that this conjecture is a consequence of the axioms of rational CFTs [15]. These axioms imply that the representation category of a rational CFT is a modular tensor category. A modular tensor category is especially a braided tensor category and the quantum dimensions, that is the traces over braiding isomorphisms, together with the action of the twist define a projective action of the modular group on the span of simple objects. The Verlinde formula for this second modular group action inside a modular tensor category is true, as for example explained in the book by Turaev [16]. The natural question is whether it is possible to translate the physics to the vertex algebra setting and around ten years ago Huang has finally succeeded to prove the Verlinde formula for rational vertex operator algebras satisfying a few additional conditions [13]. Note that much earlier Faltings came up with a very geometric and much shorter proof in the WZW case [11].

A weak Verlinde formula. Define the map $q_\ell(M_i) := \frac{S_{i\ell}}{S_{0\ell}}$. A corollary of Verlinde's formula is

$$\frac{S_{i\ell}}{S_{0\ell}} \frac{S_{j\ell}}{S_{0\ell}} = \sum_{k=0}^{n} {N_{ij}}^{k} \frac{S_{k\ell}}{S_{0\ell}}.$$

Meaning that q_ℓ for every $\ell = 0, \ldots, n$ is a one-dimensional representation of the fusion ring. If there is one special simple module M_r with lowest conformal weight, then the quantum character q_r is related to the asymptotics of characters of modules. The reason is that for large imaginary part of τ this character dominates the other ones impliying that

$$\mathrm{qdim}(M_i) := \lim_{\tau\to 0} \frac{\mathrm{ch}[M_i](\tau)}{\mathrm{ch}[M_0](\tau)} = \lim_{\tau\to i\infty} \frac{\mathrm{ch}[M_i](-1/\tau)}{\mathrm{ch}[M_0](-1/\tau)} = \frac{S_{ir}}{S_{0r}} = q_r(M_i).$$

A weak Verlinde formula is that these asymptotic dimensions give a one-dimensional representation of the fusion ring. Our question is whether this weak version is true beyond rationality.

Beyond rationality. After this review of well-known results I want to turn to logarithmic CFTs. The name logarithmic is due to the appearance of logarithmic singularities in the operator product expansion of fields involving reducible but indecomposable modules. There are two cases, logarithmic rational VOAs are those which still only have finitely many simple modules in contrast to logarithmic non-rational ones. In the first case, Miyamoto [14] has proven convergence of characters on the upper half of the complex plane provided a condition called C_2 cofiniteness is satisfied. He has also shown that characters are elements of a vector-valued modular form of mixed weight, but not all elements of this representation are characters. The by far best understood example of a logarithmic rational vertex algebra is the family of (p, q)-triplet algebras. In this case, a Verlinde formula has been conjectured in [12]. Verifying that the quantum dimensions as defined in last section respect fusion in these cases is an exercise. It reveals that quantum dimensions of modules that are elements of the maximal non-trivial ideal of the fusion ring vanish. They are in some sense negligible objects. So that the quantum dimensions capture fusion in the quotient and coincide with theose of some well-known rational vertex algebra.

In the non-rational logarithmic setting the situation of quantum dimensions becomes much richer. Also in this case characters become objects of very modern interest in number theory/modular forms. Characters are of course not modular anymore, but they are sometimes mock modular, sometimes false theta and sometimes expansions of meromorphic Jacobi forms. David Ridout and I, we have developped a conjecture extending the Verlinde formula to this setting [6]. Our first toy example has been the affine vertex superalgebra of gl(1|1) [7], since then it has been applied to many other examples including cases with mock modular forms [2] and most importantly the simple affine vertex algebra of sl(2) at admissible but non-integer level [8, 9].

The triplet vertex algebra has a non-rational subalgebra, the singlet vertex algebra. There characters are built out of partial (or false) theta functions (they are partial sums over lattices). These partial theta functions are usually not modular, but Antun Milas and I, we find modular-like behaviour if we regularize the partial theta function [4]. This means, we have a one-parameter family of objects, parameterized by ϵ, that for $\epsilon = 0$ specialize to the partial theta function and that have modular-like properties if ϵ is not purely imaginery, especially not zero. This regularization allows us to define regularized characters and their asymptotics, the regularized quantum dimensions, follow easily from the modular-like behaviour.

Together with Simon Wood and Antun Milas, we find [5] that depending on the sign of the real part of ϵ we either get continuous functions of ϵ or stripwise constant ones. The continuous part gives the Grothendieck ring, while the stripwise constant one captures fusion on some quotient. The fusion ring on this quotient is again the same as the one of some well-known rational vertex algebra. We take these findings as a strong hint that there is a categorical trace in the representation categories of the singlet algebras that reproduces our results. It is conjectured that the representation categories of the unrolled quantum group of sl(2) at $2p$-th root of unity and the one of the $(p, 1)$-singlet algebra are equivalent [3]. The singlet algebras are cosets of admissible non-integer level affine sl(2) and related algebras [1, 10] and hence should also help understanding these theories.

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