

Aspects of Vertex Algebras in Geometry and Physics

by

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Abstract

This thesis studies various aspects of the theory of vertex algebras.

It has been shown that the moonshine module for Conway's group C_0 has close ties to the equivariant elliptic genera of sigma models with a K3 surface as target space. This is taken as a motivation to investigate conditions under which a self-dual vertex operator superalgebra and the bulk Hilbert space of a superconformal field theory may be identified. To that end a classification of self-dual vertex operator superalgebras with central charge less than or equal to 12 is given and several examples of how these vertex algebras can be related to bulk superconformal field theories are provided. This includes field theories which arise from sigma models where the target space is a torus or a K3 surface.

Following this, we study orbifolds and cosets of the small $\mathcal{N} = 4$ superconformal algebra. Minimal strong generators for generic and specific levels are found and as a corollary we obtain the vertex algebra of global sections of the chiral de Rham complex on any complex Enriques surface. The commutant $\text{Com}(V^\ell(\mathfrak{sl}_2), V^{\ell+1}(\mathfrak{sl}_2) \otimes \mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}}))$ is identified with orbifolds of cosets of the small $\mathcal{N} = 4$ superconformal algebra which, in addition, can be identified with Grassmannian cosets and principal \mathcal{W} -algebras of type A at special levels. We conclude by proving a new level-rank duality which includes Grassmannian supercosets.

Furthermore, we provide a constructive proof of existence of an embedding of the Odake vertex algebra into a lattice vertex algebra in any dimension.

In addition, we show that the elliptic genus of this family of lattice vertex algebras at hand is non-vanishing if and only if the dimension does not equal 1.

Finally, we investigate conformal embeddings of maximal affine vertex algebras into rectangular \mathcal{W} -algebras at admissible levels. We prove that such \mathcal{W} -algebras are conditionally isomorphic to affine vertex algebras at boundary admissible levels for cases of type A , B , C and D .

Preface

Chapter 3 is joint work with Thomas Creutzig and John Duncan and has been published in the *Journal of Physics A: Mathematical and Theoretical*, Volume 51, Number 3 by IOP Publishing Ltd ©2017. For an online version see

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The version printed here is almost identical; exceptions being the correction of equation 3.3.1, the correction of typographical errors, and a difference in formatting.

Chapter 4 is joint work with Thomas Creutzig and Andrew Linshaw and has been submitted for publication. A preprint of the article is available online

<https://arxiv.org/abs/1910.02033>

Chapter 4 differs to the submitted version as follows: The appendix as it appears in the article can be found in Appendix A, the format of some equations and expressions was altered to fit the format of this thesis, and some semantic and typographical errors have been corrected.

Chapter 5 is joint work with Thomas Creutzig and chapter 6 is joint work with Thomas Creutzig and Jinwei Yang. An altered and extended version of each of these chapters will be submitted for publication at a later date. The versions printed here should not be redistributed by the end-user.

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Table of Contents

1	Introduction	1
2	Background	11
2.1	Vertex algebras	11
2.2	The chiral de Rham complex and invariant theory	17
2.3	The quantum Drinfeld-Sokolov reduction and \mathcal{W} -algebras	23
3	Self-dual vertex operator superalgebras and superconformal field theory	27
3.1	Introduction	27
3.2	Background	32
3.2.1	Vertex Superalgebra	32
3.2.2	Modularity	36
3.2.3	Superconformal Algebras	38
3.2.4	Free Fields	41
3.3	Self-Dual Vertex Operator Superalgebras	43
3.4	Superconformal Field Theory	46
3.4.1	Potential Bulk Conformal Field Theory	46
3.4.2	Potential Bulk Superconformal Field Theory	49
3.4.3	Superconformal structure	52
3.5	Examples	55
3.5.1	Type D Conformal Field Theory	55
3.5.2	Type D Superconformal Field Theory	57
3.5.3	Type A Superconformal Field Theory	59
3.5.4	Gepner Type Superconformal Field Theory	60
4	Invariant subalgebras of the small $\mathcal{N} = 4$ superconformal algebra	65
4.1	Introduction	65
4.1.1	Global sections on Enriques surfaces	66
4.1.2	Invariant subalgebras of the small $\mathcal{N} = 4$ superconformal algebra	67
4.1.3	A new level-rank duality	68
4.1.4	Organization	69
4.2	Background	70
4.2.1	Vertex Algebras	70
4.2.2	Filtrations	72
4.2.3	Associative G -modules and orbifolds	73
4.2.4	Associated vertex algebra to \mathfrak{n}_4	74
4.3	Automorphisms of $V^k(\mathfrak{n}_4)$	75
4.4	Construction of the vertex algebras	76
4.4.1	Construction of $V^k(\mathfrak{n}_4)^{U(1)}$	76

4.5	Construction of the cyclic orbifold	83
4.6	Structure of the vertex algebras	87
4.7	Coset of $V^k(\mathfrak{n}_4)$ by its affine subalgebra	90
4.8	Level-rank dualities	98
5	Lattice VOAs and vertex algebras of Odake type	101
5.1	Embedding VOAs of Odake type	102
5.2	Elliptic genera	111
6	On conformal embeddings of affine VOAs into rectangular \mathcal{W}-algebras	120
6.1	Background	121
6.2	Obstructions to conformal embeddings of maximal affine sub vertex algebras	123
6.2.1	Type A	124
6.2.2	Type B	126
6.2.3	Type C	128
6.2.4	Type D	132
7	Conclusion	134
	References	137
	Appendix A Decoupling relations and singular fields	149
A.1	Decoupling relations	149
A.2	Singular fields in $V^k(\mathfrak{n}_4)^{U(1)}$	151
A.3	Singular fields in $V^k(\mathfrak{n}_4)^{\mathbb{Z}/2\mathbb{Z}}$	155
	Appendix B Characters of rectangular \mathcal{W}-algebras at boundary principal admissible level	157
B.1	Type A	157
B.2	Type B	159
B.3	Type C	165
B.3.1	Principal embedding into \mathfrak{so}_n	165
B.3.2	Principal embedding into \mathfrak{sp}_{2n}	167
B.4	Type D	175

List of Tables

1.1	Levels at which the central charges of the rectangular \mathcal{W} -algebra $\mathcal{W}_k(\mathfrak{g}, \iota(f))$ and the affine vertex algebra $\mathcal{V}_\ell(\mathfrak{s})$ coincide. Note that \mathfrak{s} is the maximal Lie algebra in $\mathfrak{g}^{\mathfrak{g}'}$. For the rectangular vertex algebra $\mathcal{W}_k(\mathfrak{g}, \iota(f))$, \mathfrak{sl}_2 is chosen to be principally embedded into \mathfrak{g}'	10
3.1	Dual Coxeter numbers of simple complex Lie algebras	45
5.1	Summary of the preceding examples. The letter χ denotes the Euler characteristic.	115
5.2	Euler Characteristic for the lowest dimensions.	117
5.3	Expansions of the conformal field theoretic Hodge-elliptic genus in lowest dimensions. Setting $y = 1$ yields the elliptic genus $\mathcal{E}(\tau, u)$ (cf. Table 5.1).	119
5.4	Changing all signs in $\lim_{\tau \rightarrow i\infty} (zy)^{\frac{c}{6}} \mathcal{E}_{\text{Hodge}}(\tau, u, v)$ to be positive yields the Poincaré polynomial (equivalently the Hodge diamond). Setting $z = y = 1$ yields the Euler characteristic.	119
5.5	The Hodge diamond in dimension 3.	119
6.1	Examples where \mathfrak{s} is the maximal Lie algebra in $\mathfrak{g}^{\mathfrak{g}'}$	121
6.2	Levels at which the central charges of the rectangular \mathcal{W} -algebra $\mathcal{W}_k(\mathfrak{g}, \iota(f))$ and the affine vertex algebra $\mathcal{V}_\ell(\mathfrak{s})$ coincide. Note that \mathfrak{s} is the maximal Lie algebra in $\mathfrak{g}^{\mathfrak{g}'}$. For the rectangular vertex algebra $\mathcal{W}_k(\mathfrak{g}, \iota(f))$, \mathfrak{sl}_2 is chosen to be principally embedded into \mathfrak{g}'	123

Chapter 1

Introduction

Mathematics and physics have enjoyed a fruitful cross-fertilization for numerous decades by now. Amongst the many areas which have ties to both disciplines is the theory of vertex algebras. This thesis investigates various aspects of these algebraic objects thereby drawing connections to geometry and physics. An algebraic object which lies in the intersection of most of the topics discussed in this thesis is the small $\mathcal{N} = 4$ superconformal extension of the Virasoro algebra, from here on referred to by \mathfrak{n}_4 . A definition of the associated small $\mathcal{N} = 4$ superconformal vertex algebra $\mathcal{V}^k(\mathfrak{n}_4)$ can be taken to be the minimal \mathcal{W} -superalgebra of $\mathcal{V}^{-k-1}(\mathfrak{psl}(2|2))$. The algebra \mathfrak{n}_4 and its associated vertex algebra prominently appear in physics, such as string theory on $K3$ surfaces [ET88b], the *AdS/CFT* correspondence [Mal99], and as chiral algebras of certain four-dimensional super Yang-Mills theories [BMR19].

As one amongst many moonshine phenomena, Mathieu moonshine continues the story that started with Conway and Norton's monstrous moonshine conjectures [CN79] which historically paved the way for the definition of vertex algebras. It is here where \mathfrak{n}_4 makes its first appearance: It was observed in [EOT11] that the elliptic genus of $K3$ surfaces decomposes into characters of modules of \mathfrak{n}_4 such that the appearing coefficients can be written as sums of dimensions of irreducible representations of the sporadic group M_{24} . Given that monstrous moonshine had already been established, one similarly wondered about the existence of a graded M_{24} -module. Its existence has by now

been proven [Gan16], however, a construction of it remains elusive to this day.

Monstrous moonshine shed light upon the fact that certain discrete subgroups of $SL(2, \mathbb{R})$ somehow "know" about the representation theory of the monster group \mathbb{M} . We summarize: Frenkel, Lepowsky and Meurman [FLM84] constructed an infinite dimensional graded \mathbb{M} -module $V^{\natural} = \sum_{n \geq 0} V_n^{\natural}$ such that the McKay-Thompson series

$$T_g(\tau) = q^{-1} \sum_{n \geq 0} \text{tr}_{V_n^{\natural}}(g) q^n$$

for $g \in \mathbb{M}$ and $q = e^{2\pi i \tau}$ with $\tau \in \mathbb{H}$ is the unique Γ_g -invariant holomorphic function on \mathbb{H} satisfying certain finiteness conditions with Γ_g being a discrete subgroup of $SL(2, \mathbb{R})$ and commensurable with $SL(2, \mathbb{Z})$, such that the series $T_g(\tau) = q^{-1} + \mathcal{O}(q)$ as $\Im(\tau)$ approaches ∞ . In particular, each McKay-Thompson series is a modular function of genus 0. A proof of this was first given in [Bor92]. As an example, taking g to be the identity e yields $\Gamma_e = SL(2, \mathbb{Z})$ and thus

$$T_e(\tau) = q^{-1} \sum_{n \geq 0} \dim(V_n^{\natural}) q^n = q^{-1} + 196884q + \dots = j(\tau) - 744.$$

where the function $j(\tau)$ appearing on the right hand side is the elliptic modular function.

It is known that the moonshine module V^{\natural} can be constructed as a $\mathbb{Z}/2\mathbb{Z}$ -orbifold of the lattice vertex algebra \mathcal{V}_{Λ} where Λ is the Leech lattice. Recall that Λ is the unique self-dual unimodular lattice without roots in \mathbb{R}^{24} . Among rational and C_2 -cofinite vertex algebras, the moonshine module can be conjecturally characterized (see [FLM84]) up to isomorphism as the unique self-dual vertex algebra of rank 24 with $V_1^{\natural} = \emptyset$. The automorphism group of V^{\natural} is isomorphic to the monster group \mathbb{M} . In analogy to this construction, Duncan [Dun07] constructed a vertex algebra $A^{f^{\natural}}$ which can be characterized uniquely up to isomorphism among rational and C_2 -cofinite vertex algebras as having rank 12, being self-dual, and the property $A_{\frac{1}{2}}^{f^{\natural}} = \emptyset$. In particular, the vertex algebra $A^{f^{\natural}}$ has the structure of a $N = 1$ super VOA and its automorphism group is isomorphic to Conway's largest sporadic group Co_1 . Following

his work, Duncan and Mack-Crane [DM15] considered a similar module V^{st} whose automorphism group is isomorphic to Co_0 . Recall that this is also the isomorphism group of the Leech lattice $Co_0 = Aut(\Lambda)$ and $Co_1 = Co_0/G$ with $G \cong \{\pm 1\}$ being the center. In fact, A^{st} is isomorphic (as a vertex algebra) to V^{st} over \mathbb{C} . The difference is that the group action of Co_0 is not faithful on A^{st} . Moonshine for Co_1 was already considered in [Dun07] where the McKay-Thompson series were computed, however, these do not satisfy all properties as mentioned previously in the case of monstrous moonshine. This is mentioned by the authors in [DM15] where they state that working with V^{st} is preferable under this consideration. One of their main results (see Theorem 4.9) is that the McKay-Thompson series

$$T_g^s(\tau) = q^{-\frac{1}{2}} \sum_{n \geq 0} tr_{V_{\frac{n}{2}}^{st}}(g) q^{\frac{n}{2}}$$

for $g \in Co_0$ satisfy all relevant properties in analogy to the functions $T_g(\tau)$ considered under monstrous moonshine. In [DM16] it was further shown that some (but not all) of the trace functions appearing in Mathieu moonshine can be recovered using the canonically twisted V^{st} -module. One of them being the $K3$ elliptic genus.

Moonshine, in its present form, also has ties to string theory and with it to sigma models. The foundation of the theory of sigma models as used in string theory is not built on mathematical rigor as of yet, but has shown to be of importance nonetheless and is continued to be used within the community. In this work we will treat the theory of sigma models as a black box. Witten showed [Wit87] that a (supersymmetric, non-linear) sigma model fixes a weak Jacobi form called the *elliptic genus* somewhat related¹ to the definition following Hirzebruch [Hir78, Lan88]. The closely connected *partition function* is a completion of a direct sum of modules for a tensor product of vertex algebras which is required to satisfy certain properties such as modular invariance and closure under fusion. This will be made precise in chapter 3.

¹ For a discussion on this see [Wen15] and references therein.

Superficially, such a partition function resembles the structure of a self-dual vertex algebra. The appearance of the $K3$ elliptic genus in the trace functions of moonshine for Conway's Co_0 could be a clue that it is somehow connected to Mathieu moonshine. Taking these as motivation we provide a framework in chapter 3 in which a self-dual vertex operator algebra may be identified with such a partition function. We prove a classification result of super VOAs of central charge up to 12 and provide examples of such vertex algebras which experience close ties to such partition functions.

Chapter 3 is the content of the publication [CDR17] which is joint work with Thomas Creutzig and John Duncan. The main results are:

Theorem 3.3.1. *If W is a self-dual C_2 -cofinite vertex operator superalgebra of CFT type with central charge $c \leq 12$ then either $W \cong F(n)$ for some $0 \leq n \leq 24$, or $W \cong V_{E_8} \otimes F(n)$ for $0 \leq n \leq 8$, or $W \cong V_{D_{12}^+}$.*

Theorem 3.5.6. *Let n be a positive integer. Then the vertex operator superalgebra $W = V_{D_{4n}^+} \otimes F(4n)$ is a potential bulk $N = (2, 2)$ superconformal field theory in the sense of §3.4, for $V' \cong V'' \cong V_{D_{2n}} \otimes F(2n)$, and the elliptic genus defined by this structure vanishes.*

Theorem 3.5.8. *The vertex operator superalgebra $V_{D_{12}^+}$ is a potential bulk $N = (4, 4)$ superconformal field theory in the sense of §3.4, for $V' \cong V'' \cong V_L$, and the elliptic genus defined by this structure is the $K3$ elliptic genus.*

Theorem 3.5.9. *The vertex operator superalgebra $V_{D_{12}^+}$ is a quasi potential bulk $N = (2, 2)$ superconformal field theory in the sense of §3.4, for $V' \cong V'' \cong V_K$, and the elliptic genus defined by this structure is the $K3$ elliptic genus.*

The vertex algebra $\mathcal{V}^k(\mathfrak{n}_4)$ is further investigated from a different viewpoint in chapter 4, where a connection of an orbifold to the chiral de Rham complex is drawn. Let X be a smooth scheme of finite type over \mathbb{C} and let Ω^{ch} be the

chiral de Rham sheaf. Examples of the cohomology vertex algebra $H^\bullet(X, \Omega^{ch})$ are as of yet still unknown. Moreover, even restricting to the sub vertex algebra of global sections, the only example in the literature so far has been given when X is a $K3$ surface and was constructed in [Son16] where it was shown that $H^0(X, \Omega^{ch})$ is isomorphic to the simple small $\mathcal{N} = 4$ vertex algebra at central charge 6. A qualitative statement can also be made in case of projective space. Let X be the projective line. For the chiral sheaf \mathcal{O}^{ch} , i.e. a purely even version of Ω^{ch} , it can be shown that the Wakimoto construction at the critical level appears in the transition functions on the intersection of an open covering $U_0 \cup U_1$ and that the space of global sections $H^0(X, \mathcal{O}^{ch})$ has the natural structure of an irreducible vacuum $\widehat{\mathfrak{sl}}_2$ -module at the critical level (see Theorem 5.7 in [MSV99]). This generalizes to higher dimensions: It was shown in §2 of [MS99a] that $H^0(\mathbb{P}^n, \mathcal{O}^{ch})$ has a natural $\widehat{\mathfrak{sl}}_{n+1}$ -action within a generalized Wakimoto module. Furthermore, the formal character of the space of global sections on even dimensional projective space $H^0(\mathbb{P}^{2n}, \Omega^{ch})$ equals the elliptic genus of \mathbb{P}^{2n} as was shown in [MS03]. In addition to the case of projective space another statement can be made when X is a compact Ricci-flat Kähler manifold: As shown in [Son18] this assumption on X is sufficient such that $H^0(X, \Omega^{ch})$ is isomorphic to a subspace of a $bc - \beta\gamma$ -system that is invariant under the action of a certain Lie algebra. The motivation for chapter 4 was to provide a further example to this list by constructing the vertex algebra of global sections of the chiral de Rham complex on any complex Enriques surface.

In chapter 4 we consider a more general problem and construct a $\mathbb{Z}/2\mathbb{Z}$ -orbifold of the vertex algebra associated to the small $\mathcal{N} = 4$ superconformal Lie algebra at any level $k \neq -2, 0$. The vertex algebra $H^0(X, \Omega^{ch})$ is obtained as a specific example thereof. In doing so we first construct a $U(1)$ -orbifold and give new proofs that the vertex algebras $Com(\mathcal{H}, \mathcal{V}^k(\mathfrak{sl}_2))$ and $Com(\mathcal{H}, \mathcal{V}^k(\mathfrak{n}_2))$ are both of type $\mathcal{W}(2, 3, 4, 5)$. The commutant

$$Com(V^\ell(\mathfrak{sl}_2), V^{\ell+1}(\mathfrak{sl}_2) \otimes \mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}}))$$

is identified with orbifolds of cosets of the small $\mathcal{N} = 4$ superconformal algebra.

In addition, these orbifolds of cosets can be identified with Grassmannian cosets and principal \mathcal{W} -algebras of type A at special levels. These findings culminate in a proof of a new level-rank duality which includes Grassmannian supercosets.

Chapter 4 has been submitted for publication and is joint work with Thomas Creutzig and Andrew Linshaw. The main results are:

Corollary 1.0.1. (cf. **Corollary 4.6.5** and **Remark 4.7.7**) *The vertex algebra of global sections of the chiral de Rham complex on a complex Enriques surface is of type $\mathcal{W}(1, \frac{3}{2}, 2, \frac{7}{2}, 4^4)$. Its strong generators are explicitly constructed in the main text, and it can be regarded as an extension of $\mathcal{H} \otimes N_{-4}(\mathfrak{sl}_2)$. Here \mathcal{H} denotes the Heisenberg vertex algebra, and $N_{-4}(\mathfrak{sl}_2)$ denotes the parafermion algebra of \mathfrak{sl}_2 at level -4 .*

Theorem 4.8.1 *Let r, n, m be positive integers, then there exist vertex algebra extensions $A^{-n}(\mathfrak{sl}_m)$ and $A^m(\mathfrak{sl}_{r|n})$ of homomorphic images $\tilde{V}^{-n}(\mathfrak{sl}_m)$ and $\tilde{V}^m(\mathfrak{sl}_{r|n})$ of $V^{-n}(\mathfrak{sl}_m)$ and $V^m(\mathfrak{sl}_{r|n})$ such that the level-rank duality*

$$\begin{aligned} \text{Com}\left(V^{-n+r}(\mathfrak{sl}_m), A^{-n}(\mathfrak{sl}_m) \otimes L_r(\mathfrak{sl}_m)\right) &\cong \\ &\text{Com}\left(V^{-m}(\mathfrak{sl}_n) \otimes L_m(\mathfrak{sl}_r) \otimes \mathcal{H}(1), A^m(\mathfrak{sl}_{r|n})\right) \end{aligned}$$

holds.

Chapter 5 shows a connection between vertex algebras of Odake type \mathcal{O}_d and a family of lattice VOAs and investigates further properties of the latter. It is here where (a specific instance of) $\mathcal{V}^k(\mathfrak{n}_4)$ makes its third and final appearance in this work, be it just as a bystander. In [Oda90] the symmetry algebra of the non-linear σ model on a complex d dimensional Calabi-Yau manifold was constructed. It is an extension of the $\mathcal{N} = 2$ algebra with central charge $3d$. The associated vertex algebra \mathcal{O}_d is strongly generated by 8 fields and is of type $\mathcal{W}(1, \frac{3}{2}, 2, \frac{d^2}{2}, \frac{d+1}{2})$. It has a free field realization via a $bc - \beta\gamma$ system.² The vertex algebra has a connection to specific instances of the chiral de Rham

²See [CL13] for another construction of \mathcal{O}_3 as a commutant.

complex: Note that \mathcal{O}_2 is isomorphic to the vertex algebra associated to the small $\mathcal{N} = 4$ extension of the Virasoro algebra at central charge 6. Thus, due to the main result in [Son16],

$$H^0(X, \Omega^{ch}) \cong \mathcal{O}_2$$

for a complex $K3$ surface X . Furthermore, it was shown in [EHKZ13] that the vertex algebra $H^0(X, \Omega^{ch})$ on a Calabi-Yau 3-fold X contains \mathcal{O}_3 .

Chapter 5 takes a look at a particular subset of a family of lattice vertex (super)algebras parameterized by $d \in \mathbb{N}$ containing a simple current. Denoting the underlying lattice by L , we will show that for any d there exists an embedding into the lattice vertex superalgebra

$$\mathcal{O}_d \hookrightarrow \mathcal{V}_L.$$

Moreover, it is shown that the elliptic genera vanish if and only if $d = 1$.

Chapter 5 is joint work with Thomas Creutzig. The main results are:

Theorem 5.1.8 *Let $d \in \mathbb{N}$. There exists a vertex algebra embedding*

$$\mathcal{O}_d \hookrightarrow \mathcal{V}_L.$$

Corollary 5.2.7 *The elliptic genus associated to \mathcal{V}_L vanishes if and only if $d = 1$.*

Proposition 5.2.9 *The Poincaré polynomial equals*

$$(y^d + z^d) + \sum_{k=0}^d \binom{3d}{3k} (yz)^{d-k}.$$

Finally, chapter 6 investigates isomorphisms between families of rectangular vertex algebras $\mathcal{W}^k(\mathfrak{g}, f)$ of type A , B , C , and D and affine vertex algebras. Historically, the first examples of $\mathcal{W}^k(\mathfrak{g}, f)$ that were discovered are examples

of so-called *principal* or *regular* \mathcal{W} -algebras. In this case, the nilpotent element f is chosen to be conjugate to a single Jordan block. It has been shown [FF90] that the principal \mathcal{W} -algebras $\mathcal{W}^k(\mathfrak{sl}_N, f)$ at non-critical level are isomorphic to the W_N -algebras as given by Fateev-Lukyanov [FL88]. The definition of a principal nilpotent element lends itself to a generalization where the nilpotent element is conjugate to Jordan blocks of equal size. The resulting \mathcal{W} -algebras are referred to as *rectangular* due to the shape of the associated Young diagram: For a nilpotent element conjugate to m Jordan blocks of size $n \times n$ its Young diagram is a rectangle of $n \times m$ boxes where nm equals the dimension of the standard representation of \mathfrak{g} . This definition was first stated in [AM17] where an explicit description of the free generators of $\mathcal{W}^k(\mathfrak{sl}_N, f)$ was given and the quantum Miura transformation of Fateev and Lukyanov was recovered when restricting to the case of a principal nilpotent element.

The motivation for chapter 6 is threefold: In [CH19a] a matrix version of the higher spin *AdS/CFT* correspondence is considered where the associated *CFT* is supposedly simultaneously a coset and a rectangular \mathcal{W} -algebra. Hence, the authors conjecture an isomorphism between a family of coset theories and the simple quotient of rectangular \mathcal{W} -algebras. For $\ell = kn + mn(n-1)$, the simplest of these conjectured isomorphisms is

$$\mathcal{W}_k(\mathfrak{sl}_{mn}, f) \cong \mathcal{V}_\ell(\mathfrak{sl}_m)$$

if either $\ell = 0$ or $\ell = -m + \frac{m}{n+1}$. We fully resolve this conjecture by showing that, among the levels considered, this isomorphism holds only under the condition that the level ℓ is boundary admissible, i.e. under the condition that m and $n + 1$ are co-prime. Note that ℓ being boundary admissible is equivalent to k being boundary admissible. Recently, further such families of isomorphisms have been conjectured (see Table 2 in [CHU19]). We similarly resolve the simplest of these isomorphisms in all cases considered for type *B*, *C* and *D*.

An additional motivation is that this work can be compared to work by Adamović et al. [AKM⁺18b, AKM⁺17] which considers simple minimal \mathcal{W} -algebras and conformal embeddings of its maximal affine sub vertex algebra.

We note that subsequent work [AKM⁺18a] showed that in case such a conformal embedding

$$\mathcal{V}_\ell(\mathfrak{g}^\natural) \hookrightarrow \mathcal{W}_k(\mathfrak{g}, f_{min})$$

is an isomorphism, the representation category of ordinary modules of the universal affine VOA $\mathcal{V}_k(\mathfrak{g})$ is semi-simple even at non-admissible levels. It would be interesting to know whether such a behaviour similarly appears in non-minimal cases, e.g. rectangular \mathcal{W} -algebras.

Finally, this work also has ties to four-dimensional quantum field theories. In [BLL⁺15] the authors showed existence of a map χ from four-dimensional $\mathcal{N} = 2$ superconformal field theories to vertex operator algebras. Examples of this have been studied for Argyres-Douglas theories [Cre17, Cre18]. These theories require a pair of Dynkin diagrams (X_m, Y_n) as input data and yield a vertex operator algebra associated to each Lie algebra corresponding to these Dynkin diagrams under the map χ . It has been observed that \mathcal{W} -algebras corresponding to certain Argyres-Douglas theories are of boundary admissible level [Cre17, Observation 1]. In chapter 6 the vertex algebras of interest are associated to a pair of Lie algebras $(\mathfrak{g}, \mathfrak{s})$ (see Table 1.1) at boundary admissible levels.

Chapter 6 is joint work with Thomas Creutzig and Jinwei Yang. The main results can be summarized as follows:

Theorem 6.2.1 *For any tuple of Lie algebras $(\mathfrak{g}, \mathfrak{s})$ at boundary principal admissible level k as stated in Table 1.1 there exists an isomorphism of vertex algebras*

$$\mathcal{W}_k(\mathfrak{g}, \iota(f)) \xrightarrow{\sim} \mathcal{V}_\ell(\mathfrak{s})$$

if ℓ is either boundary principal admissible or zero.

\mathfrak{g}	$\mathfrak{g}' \otimes \mathfrak{s}$	\mathfrak{g} type	\mathfrak{g}' type	$k + h_{\mathfrak{g}}^{\vee}$	$\ell + h_{\mathfrak{s}}^{\vee}$
\mathfrak{sl}_{mn}	$\mathfrak{sl}_n \otimes \mathfrak{sl}_m$	A	A	$\frac{h_{\mathfrak{g}}^{\vee}+1}{n}$ $\frac{h_{\mathfrak{g}}^{\vee}}{n+1}$ $\frac{h_{\mathfrak{g}}^{\vee}-1}{n}$	$h_{\mathfrak{s}}^{\vee} + 1$ $\frac{h_{\mathfrak{s}}^{\vee}}{n+1}$ $h_{\mathfrak{s}}^{\vee} - 1$
\mathfrak{so}_{mn}	$\mathfrak{so}_n \otimes \mathfrak{so}_m$	B	B	$\frac{h_{\mathfrak{g}}^{\vee}+2}{n+1}$ $\frac{h_{\mathfrak{g}}^{\vee}+1}{n}$ $\frac{h_{\mathfrak{g}}^{\vee}}{n}$	$\frac{h_{\mathfrak{s}}^{\vee}+2}{n+1}$ $h_{\mathfrak{s}}^{\vee} + 1$ $h_{\mathfrak{s}}^{\vee}$
\mathfrak{sp}_{2mn}	$\mathfrak{so}_n \otimes \mathfrak{sp}_{2m}$	C	B	$\frac{h_{\mathfrak{g}}^{\vee}}{n}$ $\frac{h_{\mathfrak{g}}^{\vee}-\frac{1}{2}}{n}$ $\frac{h_{\mathfrak{g}}^{\vee}-1}{n+1}$	$h_{\mathfrak{s}}^{\vee}$ $h_{\mathfrak{s}}^{\vee} - \frac{1}{2}$ $\frac{h_{\mathfrak{s}}^{\vee}-1}{n+1}$
	$\mathfrak{sp}_{2n} \otimes \mathfrak{so}_m$		C	$\frac{h_{\mathfrak{g}}^{\vee}}{2n+1}$ $\frac{h_{\mathfrak{g}}^{\vee}-\frac{1}{2}}{2n}$ $\frac{h_{\mathfrak{g}}^{\vee}-\frac{m}{2}}{2n-1}$	$\frac{h_{\mathfrak{s}}^{\vee}}{2n+1}$ $h_{\mathfrak{s}}^{\vee} + 1$ $\frac{2mn-h_{\mathfrak{s}}^{\vee}}{2n-1}$
\mathfrak{so}_{4mn}	$\mathfrak{sp}_{2n} \otimes \mathfrak{sp}_{2m}$	D	C	$\frac{h_{\mathfrak{g}}^{\vee}+1}{2n}$ $\frac{h_{\mathfrak{g}}^{\vee}}{2n+1}$ $\frac{h_{\mathfrak{g}}^{\vee}-2m}{2n-1}$	$h_{\mathfrak{s}}^{\vee} - \frac{1}{2}$ $\frac{h_{\mathfrak{s}}^{\vee}}{2n+1}$ $\frac{2mn-h_{\mathfrak{s}}^{\vee}}{2n-1}$

Table 1.1: Levels at which the central charges of the rectangular \mathcal{W} -algebra $\mathcal{W}_k(\mathfrak{g}, \iota(f))$ and the affine vertex algebra $\mathcal{V}_{\ell}(\mathfrak{s})$ coincide. Note that \mathfrak{s} is the maximal Lie algebra in \mathfrak{g}' . For the rectangular vertex algebra $\mathcal{W}_k(\mathfrak{g}, \iota(f))$, \mathfrak{sl}_2 is chosen to be principally embedded into \mathfrak{g}' .

Chapter 2

Background

2.1 Vertex algebras

This thesis discusses various aspects of the theory of vertex algebras. We commence by stating their definition and provide examples for later reference. Further background will be given only as far as is necessary for this work. The definition of a vertex algebra was first given in [Bor86]. For additional information on the basics of vertex algebras one may consult [FB04, FHL93, Kac98].

Let V be a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space and write $V = V_0 \oplus V_1$ where the subscripts denote the cosets of $\mathbb{Z}/2\mathbb{Z}$. Such a vector space will be referred to as a *superspace*. Define the function $p : V_i \rightarrow \mathbb{Z}/2\mathbb{Z}$ via $v \mapsto i$ for $i \in \mathbb{Z}/2\mathbb{Z}$. For an element $v \in V$ we implicitly assume v to be homogeneous, that is, and element of either V_0 or V_1 , whenever we write $p(v)$. The value $p(v)$ is referred to as *parity of v* . Elements of V_0 and V_1 are referred to as *even* and *odd*, respectively. Whenever $\dim(V) < \infty$ we define the *superdimension* $\text{sdim}(V) = \dim(V_0) - \dim(V_1)$.

A vertex algebra relies on the following data:

1. (*Space of states*) a superspace V ;
2. (*Vacuum vector*) an element $\mathbf{1} \in V_0$;
3. (*Translation operator*) an element $T \in \text{End}(V)$;

4. (*Vertex operators*) a linear map

$$Y(\cdot, z) : V \rightarrow \text{End}(V)[[z, z^{-1}]]$$

$$v \mapsto \sum_{n \in \mathbb{Z}} v_{(n)} z^{-n-1}.$$

Let $a \in V$. It is common convention to denote a vertex operator by an abbreviated expression $Y(a, z) = a(z)$. We will refer to a vertex operator as a *field* if $a_{(n)}b = 0$ for $n \gg 0$ and any $b \in V$. The definition of a vertex algebra now reads as follows.

Definition 2.1.1. A *vertex algebra* \mathcal{V} is a quadruple $(V, \mathbf{1}, T, Y)$ subject to the following conditions:

1. (*Vacuum axiom*) $Y(\mathbf{1}, z) = id_V$ and $Y(v, z)\mathbf{1}|_{z=0} = v$ for all $v \in V$;
2. (*Translation axiom*) $[T, Y(v, z)] = \partial_z Y(v, z)$ for all $v \in V$;
3. (*Locality axiom*) For any two elements $a, b \in V$ there exists an integer $N \gg 0$ such that

$$(z - w)^N Y(a, z) Y(b, w) = (-1)^{p(a)p(b)} (z - w)^N Y(b, z) Y(a, w).$$

As a first example, it is a commonly known fact that a vertex algebra is a commutative unital algebra with a derivation if and only if the locality axiom holds for $N = 0$ for any two elements $a, b \in V$.

As can be seen from their series expansion, a product of the form $a(z)b(z)$ is in general ill-defined. The *normally ordered product* between fields $a(z)$ and $b(w)$ is defined by

$$: a(z)b(w) := a(z)_+ b(w) + (-1)^{p(a)p(b)} b(w) a(z)_-$$

where we used

$$a(z)_- = \sum_{n \geq 0} a_{(n)} z^{-n-1} \quad \text{and} \quad a(z)_+ = \sum_{n < 0} a_{(n)} z^{-n-1}.$$

Acting on an element $v \in V$ shows¹ that the expression $: a(z)b(z) :$ is well defined. In case of multiple fields the normally ordered product is defined recursively

$$: c_0(z)c_1(z) \cdots c_n(z) :=: c_0(z) (: c_1(z) \cdots c_n(z)) :$$

Let $a, b, c_0, \dots, c_{N-1}$ be elements of the vertex algebra's underlying vector space. The following expression is known as the *operator product expansion* (OPE)

$$a(z)b(w) = \sum_{i=0}^{N-1} \frac{c_i(w)}{(z-w)^{i+1}} + : a(z)b(w) : .$$

It is common practice to abbreviate this expression by writing an equivalence up to terms which are regular in the limit $z \rightarrow w$, i.e. the above expression will be abbreviated by

$$a(z)b(w) \sim \sum_{i=0}^{N-1} \frac{c_i(w)}{(z-w)^{i+1}} .$$

A vertex algebra is \mathcal{V} *strongly generated* if there exist fields $c_0(z), c_1(z), \dots, c_n(z) \in \mathcal{V}$ such that all fields in \mathcal{V} can be written as a normally ordered product of the form

$$: \partial^{i_0} c_0(z) \partial^{i_1} c_1(z) \cdots \partial^{i_n} c_n(z) : .$$

Many of the vertex algebras encountered in this thesis contain an element $L(z) = \sum_n L_n z^{-n-2}$ the modes of which satisfy the relations of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0} .$$

A vertex algebra containing such an element will be referred to as a *vertex operator algebra* (VOA). For all vertex operator algebras and their modules considered here, the element L_0 acts semisimply and the VOA itself is given a grading by the L_0 -action. In case of a vertex (operator) algebra that is strongly generated by the fields $c_0(z), c_1(z), \dots, c_n(z)$ we say that the VOA is of type $\mathcal{W}(d_0, \dots, d_n)$ where d_i is the grading of the field $c_i(z)$. Note that the grading need not necessarily coincide with the grading under the L_0 -action.

¹This is not obvious from the exposition given here. For details we refer to the standard textbooks stated in the beginning of this section.

Further definitions such as vertex subalgebra, ideal, module, etc. are straight forward and will be omitted. Further details can be found in the references stated in the beginning of this section.

Let G be a group and let \mathcal{V} be a vertex algebra which is a G -module where the group action is given by automorphisms. The fixed points under this action, denoted by \mathcal{V}^G , form a sub vertex algebra $\mathcal{V}^G \subset \mathcal{V}$. This vertex algebra is commonly referred to as a G -orbifold.

Let \mathfrak{g} be a finite-dimensional Lie (super)algebra with a (super)symmetric, invariant bilinear form $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$. The *universal affine vertex (super)algebra at level k* $\mathcal{V}^k(\mathfrak{g}, B)$ is freely generated by the fields $a(z)$ for $a \in \mathfrak{g}$. Their operator product expansion is given by

$$a(z)b(w) \sim \frac{kB(a, b)}{(z-w)^2} + \frac{[a, b](w)}{z-w}.$$

In the case when B equals the (appropriately normalized) Killing form, the vertex algebra shall be denoted by $\mathcal{V}^k(\mathfrak{g})$. In case when \mathfrak{g} is simple, B is non-degenerate, and $k+h^\vee \neq 0$ the vertex algebra contains a conformal vector and has central charge $c = \frac{k \cdot \text{sdim}(\mathfrak{g})}{k+h^\vee}$.

Let L be a finitely generated free abelian group and assume it to be an integral lattice under the bilinear form $(\cdot|\cdot)_L$. Denote its group algebra by $\mathbb{C}[L]$ for which multiplication and unit is given by $e^a e^b = e^{a+b}$ and $e^0 = 1$ for $a, b \in L$. Let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ be the complexification of L . Extend the bilinear form to the complexified lattice by bilinearity. One may equip \mathfrak{h} with a product that vanishes for a choice of non-equal basis vectors, view it as a commutative Lie algebra and define its affinization as $\widehat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C}K$ with K being a central element. The affinization may be decomposed

$$\widehat{\mathfrak{h}} = \widehat{\mathfrak{h}}_{<} \oplus \widehat{\mathfrak{h}}_0 \oplus \widehat{\mathfrak{h}}_{>}$$

where

$$\widehat{\mathfrak{h}}_{<} = \sum_{i < 0} h \otimes t^i, \quad \widehat{\mathfrak{h}}_0 = \mathfrak{h} \oplus \mathbb{C}K, \quad \widehat{\mathfrak{h}}_{>} = \sum_{i > 0} \mathfrak{h} \otimes t^i.$$

We will from here onwards use the abbreviation $h_j \stackrel{\text{def}}{=} h \otimes t^j \in \mathfrak{h}[t, t^{-1}]$. Let S denote the symmetric space over $\widehat{\mathfrak{h}}_{<}$. It can be shown that the vector space

$V_L = S \otimes \mathbb{C}[L]$ can be given a vertex algebra structure where the vacuum vector equals $\mathbf{1} = 1 \otimes 1$ and the generators are

$$Y(h_{-1} \otimes 1, z) = h(z) \quad \text{for all } h \in \mathfrak{h}$$

and

$$V_a(z) = Y(1 \otimes e^a, z) = e^a z^{a(0)} e^{-\sum_{j<0} \frac{z^{-j}}{j} a^{(j)}} e^{-\sum_{j>0} \frac{z^{-j}}{j} a^{(j)}} c_a \quad \text{for all } a \in L$$

where the operators c_a satisfy the conditions

$$c_0 = 1 \tag{2.1}$$

$$c_a \mathbf{1} = 0 \tag{2.2}$$

$$[h_j, c_a] = 0 \tag{2.3}$$

$$e^a c_a e^b c_b = (-1)^{p(a)p(b)+(a|b)_L} e^b c_b e^a c_a \tag{2.4}$$

for $h_j \in \mathfrak{h}[t, t^{-1}]$ and $a, b \in L$. It can be shown that any solution to the above conditions yields a unique vertex algebra (see e.g. Proposition 5.4 in [Kac98]).

This vertex algebra is referred to as *lattice vertex algebra* \mathcal{V}_L .

Let $\epsilon : L \times L \rightarrow \mathbb{C}$. For a particular class of solutions which satisfy $c_a(s \otimes e^b) = \epsilon(a, b)s \otimes e^b$ for $s \in S$ and $a, b \in L$ the above conditions in (2.1)-(2.4) can be rewritten to

$$\epsilon(\alpha, 0) = \epsilon(0, \alpha) = 1 \tag{2.5}$$

$$\epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma) \tag{2.6}$$

$$\epsilon(\alpha, \beta) = (-1)^{p(\alpha)p(\beta)+(\alpha|\beta)} \epsilon(\beta, \alpha). \tag{2.7}$$

Introduce the twisted group algebra $\mathbb{C}_\epsilon[L]$ where the multiplication is given by $e^a e^b = \epsilon(a, b)e^{a+b}$. The first two conditions follow from the requirement of the group algebra being unital and associative. The last equation is a consequence of having a well defined vertex algebra structure. Considering the non-degenerate case, i.e.

$$\epsilon : L \times L \rightarrow \mathbb{C}^\times$$

conditions (2.5) and (2.6) above set ϵ to be a 2-cocycle. It can be checked that multiplying an element of the group algebra by an arbitrary constant changes

ϵ by a coboundary. As we are interested in a vertex algebra structure only up to isomorphism it follows that ϵ is an element of the second group cohomology with coefficients in \mathbb{C}^\times . Hence, isomorphic vertex algebra structures are determined by an element $\epsilon \in H^2(L, \mathbb{C}^\times)$ which in addition satisfies the last condition in (2.7). A vertex algebra structure over the vector space $S \otimes_{\mathbb{C}_\epsilon} [L]$ is unique up to isomorphism and in particular independent of the choice of ϵ by [Kac98, Theorem 5.5a]. Furthermore, one can construct such a vertex algebra structure choosing $\epsilon : L \times L \rightarrow \{\pm 1\}$ by [Kac98, Theorem 5.5b]. It can be seen from this that the above procedure leads to a central extension of the lattice L

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \tilde{L} \rightarrow L \rightarrow 0.$$

The lattice vertex algebra in this particular case will be referred to as the *lattice vertex superalgebra* \mathcal{V}_L .

We end this introduction by providing examples of vertex algebras which will be encountered throughout this work.

Example 2.1.2. The *Heisenberg vertex algebra* $\mathcal{H}(n)$ is freely generated by even fields $h^i(z)$ for $i = 1, \dots, n$ and their non-regular OPEs are given by

$$h^i(z)h^j(w) \sim \frac{\delta_{i,j}}{(z-w)^2}.$$

Example 2.1.3. The *free fermion vertex algebra* $\mathcal{F}(n)$ is freely generated by odd fields $\phi^i(z)$ for $i = 1, \dots, n$ and their non-regular OPEs are given by

$$\phi^i(z)\phi^j(w) \sim \frac{\delta_{i,j}}{(z-w)}.$$

Example 2.1.4. The *bc-system* $\mathcal{E}(n)$ is a vertex algebra that is isomorphic to $\mathcal{F}(2n)$ and freely generated by odd fields $b^i(z), c^i(z)$ for $i = 1, \dots, n$ and their non-regular OPEs are given by

$$b^i(z)c^j(w) \sim \frac{\delta_{i,j}}{(z-w)} \quad \text{and} \quad c^i(z)b^j(w) \sim \frac{\delta_{i,j}}{(z-w)}.$$

Example 2.1.5. The *$\beta\gamma$ -system* $\mathcal{S}(n)$ is a vertex algebra that is freely generated by even fields $\beta^i(z), \gamma^i(z)$ for $i = 1, \dots, n$ and their non-regular OPEs are given by

$$\beta^i(z)\gamma^j(w) \sim \frac{\delta_{i,j}}{(z-w)} \quad \text{and} \quad \gamma^i(z)\beta^j(w) \sim -\frac{\delta_{i,j}}{(z-w)}.$$

2.2 The chiral de Rham complex and invariant theory

It is hard to anticipate from their definition that vertex algebras exhibit connections to geometry. One of them is that any smooth manifold admits a vertex algebra valued sheaf. Its introduction will be the subject of this section. In doing so we will closely follow [MSV99] where the construction has first appeared and adopt their notation. Note that contrary to examples 2.1.4 and 2.1.5 the set of strong generators of the vertex algebras $\mathcal{E}(n)$ and $\mathcal{S}(n)$ are given by $\{\phi^i(z), \psi^i(z)\}_{i=1}^n$ and $\{a^i(z), b^i(z)\}_{i=1}^n$, respectively. Some additional background will be given only as far as is necessary.

Consider the Heisenberg algebra \mathcal{H} and the Clifford algebra \mathcal{C} of rank $2N$

$$[a_m^i, b_n^j] = \delta_{i,j} \delta_{m+n,0} \cdot C \quad \text{and} \quad [\phi_m^i, \psi_n^j]_+ = \delta_{i,j} \delta_{m+n,0} \cdot C$$

for $i, j = 1, \dots, N$. As seen in the previous section each of their associated vertex algebras - denote them V_N and Λ_N - allows for a conformal element with its central charge being equal to $2N$ and $-2N$, respectively. Straightforwardly, the central charge of the vertex algebra over the tensor product $\Omega_N = V_N \otimes \Lambda_N$ vanishes and the conformal element is simply the sum of the conformal elements of both sub vertex algebras. It is well known that the vertex algebra Ω_N contains a vertex algebra which is an extension of the Virasoro algebra. In particular, this vertex algebra is generated by 2 even and 2 odd fields. Apart from the Virasoro element, the vectors associated to these fields are

$$J = \sum_{i=1}^N \phi_0^i \psi_{-1}^i, \quad Q = \sum_{i=1}^N a_{-1}^i \phi_0^i, \quad G = \sum_{i=1}^N \psi_{-1}^i b_{-1}^i$$

with their OPEs being defined as follows

$$\begin{aligned}
T(z)T(w) &\sim \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} \\
T(z)J(w) &\sim -\frac{d}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{(z-w)} \\
T(z)Q(w) &\sim \frac{Q(w)}{(z-w)^2} + \frac{\partial_w Q(w)}{(z-w)} \\
T(z)G(w) &\sim \frac{2G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{(z-w)} \\
J(z)J(w) &\sim \frac{d}{(z-w)^2} \\
J(z)Q(w) &\sim \frac{Q(w)}{(z-w)} \\
J(z)G(w) &\sim -\frac{G(w)}{(z-w)} \\
Q(z)G(w) &\sim \frac{d}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{T(w)}{(z-w)}
\end{aligned} \tag{2.8}$$

There exist two important endomorphisms of Ω_N

$$F = J_0 = \sum_{i=1}^N \sum_{n=-\infty}^{\infty} : \phi_n^i \psi_{-n}^i : \quad \text{and} \quad d = -Q_0 = -\sum_{i=1}^N \sum_{n=-\infty}^{\infty} : a_n^i \phi_{-n}^i :$$

which are commonly referred to as the *fermionic charge operator* and the *chiral de Rham differential*. The relations

$$[F, \phi_n^i] = \phi_n^i, \quad [F, \psi_n^i] = -\psi_n^i, \quad [F, a_n^i] = 0, \quad [F, b_n^i] = 0$$

and $F1 = 0$ can be inferred from the vertex algebra structure. This allows for a vector space decomposition

$$\Omega_N = \bigoplus_{j=-\infty}^{\infty} \Omega_N^j \quad \text{for} \quad \Omega_N^j = \{\omega \in \Omega_N | F\omega = j\omega\}$$

from which one is quick to see that Ω_N^0 admits a vertex algebra structure. Observe that the OPE of the field $Q(z)$ with itself is regular which implies that $d^2 = 0$ and so indeed a differential. Moreover, it holds that this endomorphism increases the fermionic charge by 1

$$d : \Omega_N^i \hookrightarrow \Omega_N^{i+1}.$$

Now, let $\Omega(\mathbb{A}^N) = \bigoplus_{j=0}^N \Omega^j(\mathbb{A}^N)$ denote the algebraic de Rham complex of the affine space \mathbb{A}^N . In what follows, note that $\Lambda[\phi_0^1, \dots, \phi_0^N]$ denotes the exterior algebra on the symbols $\phi_0^1, \dots, \phi_0^N$ and is a sub-algebra of \mathcal{C} . There is an obvious isomorphism of dg-algebras

$$\Omega_N \cong \mathbb{C}[b_0^1, \dots, b_0^N] \otimes \Lambda[\phi_0^1, \dots, \phi_0^N]$$

where the left hand side can be identified with the right hand side. Under this identification note that b_0^1, \dots, b_0^N label the coordinate functions and $\phi_0^1, \dots, \phi_0^N$ their differentials. Furthermore, the de Rham differential can be written as

$$d_{\text{dR}} = \sum_{i=1}^N a_0^i \phi_0^i.$$

It can be shown (see Theorem 2.4 in [MSV99]) that the embedding of complexes

$$(\Omega(\mathbb{A}^N), d_{\text{dR}}) \hookrightarrow (\Omega_N, d) \tag{2.9}$$

is compatible with the differentials and a quasi-isomorphism.² The complex (Ω_N, d) is suggestively referred to as the *chiral de Rham complex*.

Let $\mathcal{B} = \mathbb{C}[b_0^1, \dots, b_0^N]$ and denote its completion under the l -adic topology by $\widehat{\mathcal{B}}$. It is immediate that $\mathcal{B} \subset \mathcal{H}$ is a subalgebra and \mathcal{H} a \mathcal{B} -module. One can associate a vertex algebra structure to the algebra

$$\widehat{\mathcal{H}} = \widehat{\mathcal{B}} \otimes_{\mathcal{B}} \mathcal{H}$$

as follows: By the obvious one-to-one mapping, any element of the vacuum module of $\widehat{\mathcal{H}}$ can be written as a finite sum of elements of the form $f(b_0)g(a, b)$ where $f(b_0) \in \widehat{\mathcal{B}}$ is a formal power series in the letters b_0^1, \dots, b_0^N and $g(a, b) \in \mathcal{H}$ is a monomial in the letters $a_{i_1}^1, \dots, a_{i_N}^N, b_{j_1}^1, \dots, b_{j_N}^N$ for $i_1, \dots, i_N < 0$ and $j_1, \dots, j_N \leq 0$. In particular, the definition of $Y(g(a, b), z)$ is the same as in the definition of the vertex algebra structure over \mathcal{H} . For the remaining cases it can be shown that the assignments

$$Y(f(b_0), z) = Y(f(b_0^1, \dots, b_0^N), z) = f(b_0^1(z), \dots, b_0^N(z))$$

²See Theorem 4.4 in [MSV99] for a stronger statement for sheaves in the algebraic, complex analytic, and C^∞ setting.

and

$$Y(f(b_0)g(a, b), z) =: Y(f(b_0), z)Y(g(a, b), z) :$$

yield a well defined map

$$Y : \widehat{\mathcal{H}} \rightarrow \text{End}(\widehat{\mathcal{H}})[[z, z^{-1}]].$$

It follows immediately that the vacuum element can be defined as the image of the vacuum element under the natural embedding $\mathcal{H} \rightarrow \widehat{\mathcal{H}}$. Furthermore, the existence of a Virasoro element follows by the same argument.

Given the now seemingly close connection to geometry, one may wonder how a vertex algebra structure can be given when considering localization

$$\mathcal{H}_f = \mathcal{B}_f \otimes_{\mathcal{B}} \mathcal{H}$$

for a non-zero polynomial $f \in \mathcal{B}$. Let $Y(f, z) = \sum_{n=-\infty}^{\infty} f_n z^{-n}$. Then

$$Y(f^{-1}, z) = \frac{1}{f_0 + \sum_{n \neq 0} f_n z^{-n}} = \frac{1}{f_0} \sum_{i=0}^{\infty} \left(f_0^{-1} \sum_{n \neq 0} f_n z^{-n} \right)^i$$

is a well defined vertex operator by the above construction provided that f_0 is invertible and it follows that a vertex algebra structure can be associated to \mathcal{H}_f . The above discussion culminates in the following: Let X equal the affine scheme $\text{Spec}(\mathcal{B})$ and consider the quasi-coherent sheaf \mathcal{O}^{ch} that corresponds to the \mathcal{B} -module \mathcal{H} . The above construction shows that a vertex algebra structure can be defined over the open set $U_f = \text{Spec}(\mathcal{B}_f)$ in the Zariski topology that is an associated vertex algebra to $\mathcal{H}_f = \Gamma(U_f, \mathcal{O}^{ch})$. This defines a vertex algebra valued presheaf since the restriction morphism $\mathcal{H}_g \rightarrow \mathcal{H}_f$ for an embedding $U_f \hookrightarrow U_g$ yields a morphism of vertex algebras. One can check that this construction defines a vertex algebra valued sheaf. In a similar way as above a vertex algebra structure can be associated to

$$\widehat{\Omega}_N = \widehat{\mathcal{B}} \otimes_{\mathcal{B}} \Omega_N.$$

The corresponding vertex algebra valued sheaf is denoted by Ω^{ch} . One can define a partial ordering on the monomials in the tensor product of the Heisenberg and Clifford algebra which induces a filtration on the vector spaces associated

to fields of fixed conformal weight. The associated graded pieces are direct sums of symmetric powers of the tangent bundle, exterior powers of the bundle of one-forms, and tensor products of said objects.

Remark 2.2.1. Regarding $\widehat{\Omega}_N$ one can replace $\widehat{\mathcal{B}}$ with any commutative \mathcal{B} -algebra that is a $Der(\mathcal{B})$ -module extending which restricts to the natural action on the subalgebra \mathcal{B} . Regarding applications, it follows that one may define a corresponding sheaf Ω^{ch} in the algebraic, complex analytic or smooth setting.

Let X be a smooth scheme of finite type over \mathbb{C} . It was shown in [MSV99] that the global sections of the sheaf Ω^{ch} canonically form a conformal vertex algebra in the complex analytic framework which can be lifted to a vertex algebra containing the fields $J(z), Q(z)$ and $G(z)$ as given above if the first Chern class vanishes. It was further shown that the fermionic charge operator F and the chiral de Rham differential d are well defined endomorphisms of Ω^{ch} which implies that the chiral de Rham complex (Ω^{ch}, d) is a complex of sheaves that are graded by the fermionic charge. In the C^∞ setting a similar picture holds. In case of a Riemannian manifold the sheaf Ω^{ch} contains a $\mathcal{N} = 1$ structure, in case of a Kähler metric and a Ricci-flat manifold it can be lifted to a $\mathcal{N} = 2$ structure, and if the manifold is hyper-Kähler it further lifts to a $\mathcal{N} = 4$ structure [BHS08] with central charge $c = 3dim_{\mathbb{C}}(X)$.

The question that is being addressed and answered in chapter 4 is fundamentally a problem within the framework of invariant theory. Consider the following: Let G be a linearly reductive group, V a finite dimensional G -module over a field k , $k[V]$ the ring of polynomial functions on V and $k[V]^G$ the subring of G -invariant polynomials. In 1893 Hilbert proved that $\mathbb{C}[V]^G$ is finitely generated. A fundamental problem is then to find generators and relations for the subring $k[V]^G$. The Basis Theorem, Nullstellensatz and Syzygy Theorem were introduced by Hilbert in connection to this problem. Now, consider the module $W = \bigoplus_{j \in \mathcal{S}} V_j$ where we assume that $V_j \cong V$ for all $j \in \mathcal{S}_0 \subseteq \mathcal{S}$ and $V_j \cong V^*$ for all $j \in \mathcal{S} \setminus \mathcal{S}_0$. Let $R = \mathbb{C}[W]^G$. For G a classical group and \mathcal{S} of finite cardinality, Weyl's first fundamental Theorem [Wey46] provides a

set of generators for R . Furthermore, his second fundamental Theorem (see *op. cit.*) yields generators for the ideal of relations on R . In this thesis we are mainly (but not exclusively) assuming G to be finite.

Example 2.2.2. Let $G = \mathbb{Z}/2\mathbb{Z}$ and V the one-dimensional non-trivial representation. Take $\mathcal{S} = \mathbb{N}$ and let $\{x_j\}$ be a basis for V_j for all $j \in \mathcal{S}$. Then $R = \mathbb{C}[W]^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{C}[x_1, x_2, x_3, \dots]^{\mathbb{Z}/2\mathbb{Z}}$ is the subalgebra of R of even degree. The first fundamental Theorem states that the generators of R can be given by $g_{i,j} = x_i x_j$ for $i \leq j$. The second fundamental Theorem states that the ideal of relations is generated by $g_{i,j} g_{k,l} - g_{i,k} g_{j,l}$.

Consider the Heisenberg vertex algebra $\mathcal{H} = \mathcal{H}(1)$ and denote its strong generator by $h(z)$. A basis of the underlying vector space of \mathcal{H} is given by the set $\{1\} \cup \{\partial^{j_1} h(z) \cdots \partial^{j_k} h(z) : |0 \leq j_1 \leq \cdots \leq j_k\}_{k=1}^{\infty}$ via the state-field correspondence. Let g denote the generator of the automorphism group $Aut(\mathcal{H}) \cong \mathbb{Z}/2\mathbb{Z}$ which acts via $g(h(z)) = -h(z)$. The space of states of \mathcal{H} is linearly isomorphic to the polynomial ring $\mathbb{C}[x_0, x_1, \dots]$ where $x_i \leftrightarrow \partial^i h(z)$. Comparing with example 2.2.2 we see that $R \cong \mathcal{H}^{\mathbb{Z}/2\mathbb{Z}}$, hence a set of strong generators corresponding to $g_{i,j}$ is given by the quadratics $\omega_{i,j} =: \partial^i h(z) \partial^j h(z) : \text{ for } i \leq j$. Note that there exist further relations among the generators of the polynomial ring due to the existence of a differential by the virtue of the vertex algebra. This manifests itself in the relation $\partial x_i = x_{i+1}$ which induces a relation on the strong generators of R and $\mathcal{H}^{\mathbb{Z}/2\mathbb{Z}}$. Thus, a minimal generating set on R as a differential algebra is given by $\{g_{0,2j} | j \geq 0\}$. On the contrary, $\mathcal{H}^{\mathbb{Z}/2\mathbb{Z}}$ is strongly generated by $\{\omega_{0,2j} | j \geq 0\}$, however, this strong generating set is not minimal. As shown in [DN99] the orbifold $\mathcal{H}^{\mathbb{Z}/2\mathbb{Z}}$ is of type $\mathcal{W}(2, 4)$ and a minimal strong generating set can be given by $\{\omega_{0,0}, \omega_{0,2}\}$. Why is this so? Note the following relation

$$\begin{aligned} \omega_{0,4} &= -\frac{4}{5} (: \omega_{0,0} \omega_{1,1} : - : \omega_{0,1} \omega_{0,1} :) + \frac{7}{5} \partial^2 \omega_{0,2} - \frac{7}{30} \partial^4 \omega_{0,0} \\ &= -\frac{2}{5} : \omega_{0,0} \partial^2 \omega_{0,0} : + \frac{4}{5} : \omega_{0,0} \omega_{0,2} : + \frac{1}{5} : \partial \omega_{0,0} \partial \omega_{0,0} : + \frac{7}{5} \partial^2 \omega_{0,2} - \frac{7}{30} \partial^4 \omega_{0,0} \end{aligned}$$

and observe its similarity to a generator of the ideal of relations in the G -invariant subring in example 2.2.2. The "classical" relation $g_{0,0} g_{1,1} - g_{0,1} g_{0,1} = 0$

does not vanish in the vertex algebraic framework : $\omega_{0,0}\omega_{1,1} : - : \omega_{0,1}\omega_{0,1} : \neq 0$. In a similar way one can find that the second fundamental Theorem induces relations $\omega_{0,2j+4} = P_j(\omega_{0,0}, \omega_{0,2})$ for $j \geq 0$. For further information on $\mathcal{H}(n)^{O(n)}$ we refer to chapter 5 in [Lin13].

In [LL07] Lian and Linshaw introduced the proper framework to deal with this type of problem by introducing a functor from a certain category of vertex algebras to the category of supercommutative rings with a differential which exhibits a reconstruction property such that the problem of finding a minimal strong generating set can be reduced to finding relations in certain rings. This will be made precise and used heavily in chapter 4.

2.3 The quantum Drinfeld-Sokolov reduction and \mathcal{W} -algebras

Constructing a vertex algebra can be done in many ways. Some of them are grounded in Lie theory, an example of which being the affine vertex algebra $\mathcal{V}^k(\mathfrak{g})$ which has been introduced at the beginning of this chapter. Another class of vertex algebras are the so-called \mathcal{W} -algebras $\mathcal{W}(\mathfrak{g})$ which can be seen as an affinization of the center $\mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. As the main objects of study in chapter 6 are \mathcal{W} -algebras let us briefly recall their definition. In doing so we follow the exposition given in [KRW03] and [KW04, KW05]. For another realization of \mathcal{W} -algebras see [Gen17] where it is shown that $\mathcal{W}^k(\mathfrak{g}, x, f)$ at generic level k is isomorphic to the intersection of kernels of certain operators, acting on the tensor vertex superalgebra of an affine vertex superalgebra and a neutral free superfermion vertex superalgebra.

Let \mathfrak{g} be a simple finite-dimensional Lie superalgebra equipped with an invariant non-degenerate supersymmetric even bilinear form $(\cdot|\cdot)$. Furthermore, let $x, f \in \mathfrak{g}$ such that (i) f is even with $[x, f] = -f$, (ii) ad_x is diagonalizable with eigenvalues in $\frac{1}{2}\mathbb{Z}$ such that the eigenvalues on the centralizer of f in \mathfrak{g} - from here onwards denoted by \mathfrak{g}^f - are non-positive, and (iii) the map $\text{ad}_f : \mathfrak{g}_j \rightarrow \mathfrak{g}_{j-1}$ is injective for $j \geq \frac{1}{2}$ and surjective for $j \leq \frac{1}{2}$. It follows that

there exists a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_- \quad \text{where} \quad \mathfrak{g}_j = \{g \in \mathfrak{g} \mid [x, g] = jg\}$$

with $\mathfrak{g}_+ = \bigoplus_{j>0} \mathfrak{g}_j$ and $\mathfrak{g}_- = \bigoplus_{j<0} \mathfrak{g}_j$. It is clear that $\mathfrak{h} \subset \mathfrak{g}_0$.

Next we introduce 3 vertex algebras associated to this datum; the first being the universal affine vertex algebra $V^k(\mathfrak{g})$ associated to the affinization of the Lie algebra \mathfrak{g} and the bilinear form $(\cdot|\cdot)$. The bilinear form $\langle \cdot | \cdot \rangle_f$ defined via

$$\langle a|b \rangle_f = (f|[a, b])$$

is even and skew-supersymmetric. It follows from (iii) and the non-degeneracy of $(\cdot|\cdot)$ that this bilinear form is also non-degenerate on $\mathfrak{g}_{\frac{1}{2}}$. Let $A_{\text{ne}} \cong \mathfrak{g}_{\frac{1}{2}}$ be a vector superspace with a bilinear form given by the tuple $\langle \cdot | \cdot \rangle_f$ and denote its Clifford affinization by \hat{A}_{ne} . $F(A_{\text{ne}})$ denotes the associated vertex algebra of neutral free fermions. Lastly, let $A \cong \pi(\mathfrak{g}_+)$ and $A^* \cong \pi(\mathfrak{g}_+^*)$ be two vector superspaces where π is a homomorphism of vector superspaces which restricts to isomorphisms of vector spaces but exchanges the parity of all elements (wherever defined) of a superspace. Let $A_{\text{ch}} = A \oplus A^*$ be a vector superspace together with a skew-supersymmetric bilinear form defined by

$$\langle A|A \rangle = \langle A^*|A^* \rangle = 0 \quad \text{and} \quad \langle a|b \rangle = b(a) \quad \text{for } a \in A, b \in A^*.$$

It is clear that this form is non-degenerate. $F(A_{\text{ch}})$ denotes the vertex algebra of charged free fermions associated to the Clifford affinization \hat{A}_{ch} .

The object of interest now is the vertex algebra

$$\mathcal{C}_k(\mathfrak{g}, x, f) = V^k(\mathfrak{g}) \otimes F(A_{\text{ch}}) \otimes F(A_{\text{ne}})$$

for which we assume that the level is non-critical, i.e. $k \neq -h^\vee$. This vertex algebra has an induced grading

$$\mathcal{C}_k^\bullet(\mathfrak{g}, x, f) = V^k(\mathfrak{g}) \otimes F^\bullet(A_{\text{ch}}) \otimes F(A_{\text{ne}})$$

given by the charge gradation. Note that $\mathcal{C}_k^0(\mathfrak{g}, x, f) \subset \mathcal{C}_k(\mathfrak{g}, x, f)$ is a vertex subalgebra. It can be shown that there exists a field $d(z) \in \mathcal{C}_k(\mathfrak{g}, x, f)$ depending on f for which

$$[d(z), d(w)] = 0 \quad \text{and} \quad [d_0, \mathcal{C}_k^m(\mathfrak{g}, x, f)] \subset \mathcal{C}_k^{m-1}(\mathfrak{g}, x, f)$$

where $d_0 = \text{Res}(d(z))$. The former statement implies that $d_0^2 = 0$, hence $(\mathcal{C}_k^\bullet(\mathfrak{g}, x, f), d_0)$ is a chain complex for which the cohomological grading is given by the charge. The cohomology, denoted here by $H_k^\bullet(\mathfrak{g}, x, f)$, is a vertex superalgebra with an appropriate \mathbb{Z} -grading (which differs to the grading by charge). Moreover, the cohomology is acyclic. i.e. $H_k^i(\mathfrak{g}, x, f) = 0$ for $i \neq 0$ [KW04, Theorem 4.1 (c)]. The resulting vertex algebra $H_k^0(\mathfrak{g}, x, f)$ is denoted by $\mathcal{W}^k(\mathfrak{g}, x, f)$ and referred to as the *quantum Drinfeld-Sokolov reduction* of the quadruple (\mathfrak{g}, x, f, k) . In addition, it contains a Virasoro field. Let $\{u_i\}_{i \in S}$ be a basis of \mathfrak{g}_+ that is compatible with its grading, i.e. $[x, u_i] = iu_i$. The central charge c is given by (see Theorem 2.2 (a) in [KRW03])

$$c(\mathfrak{g}, x, f, k) = \frac{k \, \text{sdim}(\mathfrak{g})}{k + h^\vee} - 12(x|x) - \sum_{i \in S} (-1)^{p(u_i)} (12i^2 - 12i + 2) + \frac{1}{2} \text{sdim}(\mathfrak{g}_{\frac{1}{2}}). \quad (2.10)$$

For a basis $\{g_i\}$ of \mathfrak{g}^f that is compatible with the grading $\mathfrak{g}^f = \bigoplus_j \mathfrak{g}_j^f$ induced from the decomposition of \mathfrak{g} as given above, the vertex algebra $\mathcal{W}^k(\mathfrak{g}, x, f)$ is strongly generated by fields $J_{g_i}(z)$ of conformal weight $1 + i$ for $g_i \in \mathfrak{g}_{-i}^f$ [KW04, Theorem 4.1 (b)].

Remark 2.3.1. Note that it follows from condition (i) that f is a nilpotent element. Hence, by the Jacobson-Morozov theorem, it may be embedded in a \mathfrak{sl}_2 -triple $\{e, f, h\}$ and all such triples are conjugate under an action of the centralizer \mathfrak{g}^f . Taking the elements $\{x, f\}$ to be elements of an \mathfrak{sl}_2 -triple has implications on the grading. We will not go into detail here and simply refer to [KRW03] where gradings with and without this property are discussed from section 2.4 onwards.

Remark 2.3.2. Recall that we have assumed non-criticality of the level k . \mathcal{W} -algebras at the critical level have been explored in the literature. See for example [Ara12] where it is shown that the center of $\mathcal{W}^k(\mathfrak{g}, f)$ for the critical level k coincides with the Feigin-Frenkel center of the affine Lie algebra associated with \mathfrak{g} .

This construction can be extended as follows: Let M be a restricted $\hat{\mathfrak{g}}$ -module. Any such module extends to a $V^k(\mathfrak{g})$ -module which trivially extends

to a $\mathcal{C}_k(\mathfrak{g}, x, f)$ -module $M \otimes F(A_{\text{ch}}) \otimes F(A_{\text{ne}})$. Following the previous steps, the module

$$\mathcal{C}^\bullet(M) = M \otimes F^\bullet(A_{\text{ch}}) \otimes F(A_{\text{ne}})$$

has a \mathbb{Z} -grading given by charge, $(\mathcal{C}^\bullet(M), d_0)$ is a chain complex of $\mathcal{C}_k(\mathfrak{g}, x, f)$ -modules and thus its cohomology is a direct sum of $\mathcal{W}^k(\mathfrak{g}, x, f)$ -modules $H(M) = \bigoplus_{j \in \mathbb{Z}} H^j(M)$. In particular, this construction yields a functor

$$\mathcal{H}_f : \mathbf{Rep}(\hat{\mathfrak{g}})^{\text{res}} \rightarrow \mathbf{Rep}(\mathcal{W}^k(\mathfrak{g}, x, f))$$

from the category of restricted $\hat{\mathfrak{g}}$ -modules to the category of $\mathcal{W}^k(\mathfrak{g}, x, f)$ -modules. This functor maps any integrable $\hat{\mathfrak{g}}$ -module to zero (cf. [Ara05, KRW03]). Moreover, it has been shown that this functor is exact and maps any irreducible module to zero or to an irreducible module [Ara04, Ara07].³

³See [Ara05] for an earlier result where it is proved that the functor \mathcal{H}_f preserves irreducibility in the case of $\mathcal{W}^k(\mathfrak{g}, f_\theta)$ -modules where f_θ is the root vector corresponding to the lowest root $-\theta$ of \mathfrak{g} .

Chapter 3

Self-dual vertex operator superalgebras and superconformal field theory

3.1 Introduction

The elliptic genus of a complex K3 surface X is a weak Jacobi form of weight zero and index one. It may be realized in the following three ways: via the chiral de Rham complex of X [BHS08, Hel09, Bor01, BL00], as the S^1 -equivariant χ_y -genus of the loop space of X [Hir78, Höh91, Kri90], and as a trace on the Ramond-Ramond sector of a sigma model on X [EOTY89, DY93, Wit94]. The small $N = 4$ superconformal algebra at central charge $c = 6$ appears in each of these three pictures. Firstly, it is the algebra of global sections of the chiral de Rham complex [Son16, Son15]. Secondly, the χ_y -genus can be viewed as a virtual module for a certain vertex operator superalgebra that contains this Lie superalgebra [CH14, TZZ99]. Finally, it appears as a supersymmetry algebra of the string theory sigma model [ET88b]. (We refer to [Wen15] for a recent detailed review of these topics.)

The K3 elliptic genus and the $N = 4$ superconformal algebra were absorbed into the orbit of moonshine when Eguchi–Ooguri–Tachikawa [EOT11] suggested a relationship between the largest Mathieu group, M_{24} , and character contributions of the $N = 4$ superconformal algebra to the K3 elliptic genus. This ignited a resurgence of interest in connections between string the-

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ory, modular forms and finite groups. Umbral moonshine [CDH14a, CDH14b, DGO15b], Thompson moonshine [HR16, GM16] and the recently announced O’Nan moonshine [DMO17] all belong to the quickly developing legacy of this *Mathieu moonshine* observation, although the connections to string theory are so far more obscure in the latter two cases. We refer to [DGO15a] for a fuller review, more references, and for comparison to the original monstrous moonshine [CN79, Tho79a, Tho79b] that appeared in the 1970s.

By now there are indications that Mathieu and monstrous moonshine are interrelated. An instance of this, and a primary motivation for the present work is [DM16], wherein the K3 elliptic genus apparently makes a fourth appearance: as a trace function on the moonshine module [Dun07, DM15] for Conway’s group, Co_0 [Con68, Con69]. On the one hand, this Conway moonshine module—a vertex operator superalgebra with $N = 1$ structure—is a direct supersymmetric analogue of the monstrous moonshine module of Frenkel–Lepowsky–Meurman [FLM84, FLM85, FLM89]. It manifests a genus zero property for Co_0 [DM15], just as the monstrous moonshine module does for the monster [Bor92]. On the other hand, M_{24} is a subgroup of Co_0 , and the Conway moonshine construction of the K3 elliptic genus may be twined by (most) elements of M_{24} . In many, but not all instances the resulting trace functions coincide with those that arise in Mathieu moonshine.

This tells us that the Conway moonshine module comes close to providing a vertex algebraic realization of the as yet elusive Mathieu moonshine module, whose structure as a representation of M_{24} was conjecturally determined in [Che10, GHV10b, GHV10a, EH11], and confirmed in [Gan16]. The Conway moonshine module has been used to realize analogues of the Mathieu moonshine module for other sporadic simple groups in [CDD⁺15, CHKW15]. See [TW15b, TW13, TW15a, GKH17, TW17] for the development of a promising geometric approach to the problem.

The Conway moonshine module is also connected to string theory on K3 surfaces. As is explained in [DM16], the Conway moonshine construction of the K3 elliptic genus may also be twined, in an explicitly computable way, by any automorphism of a K3 sigma model that preserves its supersymmetry. Such

automorphisms are classified in [GHV12]. It appears that the construction of [DM16] is (except for a small number of possible exceptions) in agreement [CHVZ18] with the twined K3 elliptic genera that one expects [CHVZ18] to arise from string theory. One reason this is surprising is that K3 sigma models, and in particular their automorphisms, are difficult to construct in general (cf. e.g. [NW01]). The Conway moonshine module seems to serve as a shortcut, to certain computations which might otherwise require the explicit construction of sigma models.

In view of these connections it is natural to ask how the Conway moonshine realization of the K3 elliptic genus is related to the three we began with above. Katz–Klemm–Vafa [KKV99] conjectured a method for computing the Gromov–Witten invariants of a K3 surface in terms of the χ_y -genera of its symmetric powers, and the generating function of these χ_y -genera can be realized in terms of a lift of the K3 elliptic genus. So the second mentioned realization of the K3 elliptic genus, as a generalization of Hirzebruch’s χ_y -genus, suggests a connection between Conway moonshine and enumerative geometry. This perspective is developed in [CDHK17], where equivariant counterparts to the conjecture of Katz–Klemm–Vafa are formulated, which explicitly describe equivariant versions of Gromov–Witten invariants of K3 surfaces. The conjecture of Katz–Klemm–Vafa was proved recently by Pandharipande–Thomas [PT16]. Katz–Klemm–Pandharipande [KKPT16] have extended the conjecture of Katz–Klemm–Vafa to refined Gopakumar–Vafa invariants. Conjectural descriptions of equivariant refined Gopakumar–Vafa invariants of K3 surfaces are also formulated in [CDHK17].

In this work we develop the relationship between Conway moonshine and the third mentioned realization, in terms of K3 sigma models. We do this by formalizing a new relationship between vertex algebra and conformal field theory, and by realizing the Conway moonshine module, and other vertex operator superalgebras, in examples.

Traditionally, vertex operator algebras satisfying suitable conditions are considered to define “chiral halves” of conformal field theories. More specifically, the bulk Hilbert space of a conformal field theory may be regarded as

(a completion of) a suitable sum of modules for a tensor product of vertex operator algebras (cf. [Hua92, Gab00, Wen15]). The alternative viewpoint we pursue here develops from the observation that a bulk Hilbert space of a conformal field theory, taken as a whole, resembles a self-dual vertex operator algebra. We explain this observation more fully in §3.4.1. It motivates our

Main Question: *Can a self-dual vertex operator algebra be identified with a bulk conformal field theory in some sense?*

We answer this question positively by formulating the notion of *potential (bulk) conformal field theory* (cf. Definition 3.4.1) and by identifying self-dual vertex operator algebras as examples (cf. Propositions 3.5.1 and 3.5.3). In fact, we formulate supersymmetric counterparts to potential conformal field theories as well (cf. Definitions 3.4.3 and 3.4.6), and find more examples amongst self-dual vertex operator superalgebras (cf. Theorems 3.5.6, 3.5.8 and 3.5.9). To support the analysis we also present a classification result (Theorem 3.3.1) for self-dual vertex operator superalgebras with central charge up to 12.

Equipped with the notion of potential bulk superconformal field theory we relate the Conway moonshine module to four superconformal field theories in §3.5. One of these is the superconformal field theory underlying the tetrahedral K3 sigma model (cf. §3.5.3), which was analyzed in detail by Gaberdiel–Taormina–Volpato–Wendland [GTVW14] (see also [TW17]). Another is the Gepner model (1)⁶ (cf. §3.5.4), and it is clear that there are further interesting examples waiting to be considered, that may shed more light on the role of Conway moonshine in K3 string theory.

An implication of our analysis is that there should be self-dual vertex operator superalgebras besides the Conway moonshine module that have analogous relationships to other string theory compactifications. In §3.5.2 we identify a self-dual vertex operator superalgebra—the $N = 1$ vertex operator superalgebra naturally attached to the E_8 lattice—which realizes the bulk superconformal field theory underlying a sigma model with a 4-torus as target (cf. Theorem 3.5.6). Volpato [Vol14] has shown that the supersymmetry preserving automorphism groups of 4-torus sigma models are subgroups of the Weyl

group of the E_8 lattice. In light of these results it seems likely that the E_8 vertex operator superalgebra that appears in §3.5.3 can serve as a counterpart to the Conway moonshine module for nonlinear sigma models on 4-dimensional tori.

It is interesting to compare the approach presented here to recent work [TW17] of Taormina–Wendland. In loc. cit. the relationship between superconformal field theory and vertex operator superalgebra is also reconsidered, but the starting point is a fully fledged superconformal field theory. A notion of *reflection* is introduced which, in special circumstances, produces a vertex operator superalgebra. The superconformal field theory underlying the tetrahedral K3 sigma model is considered in detail, and it is shown that the Conway moonshine module arises when reflection is applied in this case. In this way Taormina–Wendland independently obtain results equivalent to those we present in §3.5.3. Our notion of potential superconformal field theory serves to answer the question of what reflection produces from a superconformal field theory in general, except that a reflected superconformal field theory comes equipped with extra structure, on account of the richness of the superconformal field theory axioms. For example, the reflected tetrahedral K3 theory recovers the vertex operator superalgebra structure on the Conway moonshine module, but also furnishes an intertwining operator algebra structure on the direct sum of itself with its unique irreducible canonically twisted module (cf. §4 of [TW17]). Moving forward, we can expect that Taormina–Wendland reflection will play a key role in further elucidating the relationships between superconformal field theories, potential superconformal field theories and vertex operator superalgebras.

We now describe the structure of the article. We present background material in §3.2. We explain our conventions on vertex operator superalgebras in §3.2.1, and we review some modularity results for vertex operator superalgebras in §3.2.2. We recall the small $N = 4$ superconformal algebra in §3.2.3, and describe an explicit construction at $c = 6$ in §3.2.4. In §3.3 we establish our classification result for self-dual vertex operator superalgebras with central charge at most 12. Then in §3.4 we discuss the new relationship

between vertex algebra and conformal field theory that motivates this work, and explain our approach to answering the Main Question. We begin with conformal field theory in §3.4.1, discuss superconformal field theory in §3.4.2, and consider superconformal field theories with superconformal structure in §3.4.3. We present examples of bulk superconformal field theory interpretations of self-dual vertex operator superalgebras in §3.5. We begin with the diagonal conformal field theories associated to type D lattice vertex operator algebras in §3.5.1, and then discuss super analogues of these in §3.5.2. We discuss the superconformal field theory underlying the tetrahedral K3 sigma model in §3.5.3, and discuss the relationship between the Conway moonshine module and the Gepner model (1)⁶ in §3.5.4.

3.2 Background

3.2.1 Vertex Superalgebra

We assume some familiarity with the basics of vertex (operator) superalgebra theory. Good references for this include [FB04, FLM89, Kac98, LL04].

We adopt the convention, common in physical settings, of writing $(-1)^F$ for the canonical involution on a superspace $W = W^{\text{even}} \oplus W^{\text{odd}}$, so that $(-1)^F|_{W^{\text{even}}} = I$ and $(-1)^F|_{W^{\text{odd}}} = -I$. We write $Y(a, z) = \sum a_{(k)} z^{-k-1}$ for the vertex operator attached to an element a in a vertex superalgebra W . A vertex superalgebra W is called C_2 -cofinite if $W/C_2(W)$ is finite-dimensional, where $C_2(W) := \{a_{(-2)}b \mid a, b \in W\}$. Following [DLMM98] we say that a vertex operator superalgebra W is of *CFT type* if the L_0 -grading $W = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} W_n$ is bounded below by 0, and if W_0 is spanned by the vacuum vector. We assume that $W^{\text{even}} = \bigoplus_{n \in \mathbb{Z}} W_n$ and $W^{\text{odd}} = \bigoplus_{n \in \mathbb{Z} + \frac{1}{2}} W_n$. A vertex operator superalgebra that is C_2 -cofinite and of CFT type is *nice (schön)* in the sense of [Höh07].

Say that a vertex operator algebra is *rational* if all of its admissible modules are completely reducible. We refer to [DLM98] for the definition of admissible module. It is proven in loc. cit. that a rational vertex operator algebra has finitely many irreducible admissible modules up to equivalence. We say

that a vertex operator superalgebra is rational if its even sub vertex operator algebra is rational. We will apply results from [DZ05] in what follows, so we should note that our notion of rationality for a vertex operator superalgebra is stronger than that which appears there. A vertex operator superalgebra that is rational in our sense is both rational and $(-1)^F$ -rational in the sense of loc. cit. The equivalence of the two notions of rationality is proven in [HA15] under an assumption on fusion products of canonically twisted modules.

We say that a vertex operator superalgebra W is *self-dual* if W is rational (in our sense), irreducible as a W -module, and if W is the only irreducible admissible W -module up to isomorphism. Note that the term self-dual is sometimes used differently elsewhere in the literature, to refer to the situation in which W is isomorphic to its contragredient as a W -module.

According to Theorem 8.7 of [DZ05] a self-dual C_2 -cofinite vertex operator superalgebra W has a unique (up to isomorphism) irreducible $(-1)^F$ -stable canonically twisted module. We denote it W_{tw} . The $(-1)^F$ -stable condition on a canonically twisted module M for W is equivalent to the requirement of a superspace structure $M = M^{\text{even}} \oplus M^{\text{odd}}$ that is compatible with the superspace structure on W , so that elements of W^{even} and W^{odd} induce even and odd transformations of M , respectively. Modules that are not $(-1)^F$ -stable will not arise in this work so we henceforth assume the existence of a compatible superspace structure to be a part of the definition of untwisted or canonically twisted module for a vertex operator superalgebra. However, we will not require morphisms of modules to preserve a particular superspace structure. So for example, if W is a vertex operator superalgebra and Π is the parity change functor on superspaces then W and ΠW are not isomorphic as superspaces, but we do regard them as isomorphic W -modules.

Write V_L for the vertex superalgebra attached to an integral lattice L , which is naturally a vertex operator superalgebra if L is positive definite. Write $F(n)$ for the vertex operator superalgebra of n free fermions. According to the boson-fermion correspondence [Fre81, DM94] the vertex operator superalgebra attached to \mathbb{Z}^n is isomorphic to $F(2n)$. So the even sub vertex operator algebra $F(2n)^{\text{even}} < F(2n)$ is isomorphic to the lattice vertex operator algebra attached

to the *type D* lattice

$$D_n := \{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid x_1 + \dots + x_n = 0 \pmod{2}\}. \quad (3.2.1)$$

The discriminant group of D_n is $D_n^*/D_n \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and we label coset representatives as follows.

$$\begin{aligned} [0] &:= (0, \dots, 0, 0), & [1] &:= \frac{1}{2}(1, \dots, 1, 1), \\ [2] &:= (0, \dots, 0, 1), & [3] &:= \frac{1}{2}(1, \dots, 1, -1). \end{aligned} \quad (3.2.2)$$

Set $D_n^+ := D_n \cup D_n + [1]$. Then D_n^+ is a self-dual integral lattice—the rank n *spin lattice*—whenever $n = 0 \pmod{4}$. It is even if $n = 0 \pmod{8}$. We have $D_4^+ \cong \mathbb{Z}^4$ and $D_8^+ \cong E_8$, and D_{12}^+ is the unique self-dual integral lattice of rank 12 such that $\lambda \cdot \lambda \leq 1$ implies $\lambda = 0$. The lattice vertex operator superalgebras attached to D_n and D_{4n}^+ will play a prominent role later on.

Set $A_1 = \sqrt{2}\mathbb{Z}$. We will make use of the fact that D_{2n} admits A_1^{2n} as a sub lattice. Explicitly, denoting $e_1 := (1, 0, \dots, 0)$, $e_2 := (0, 1, \dots, 0)$, et cetera, we may take the first copy of A_1 to be generated by $e_1 + e_{n+1}$, the second copy to be generated by $e_1 - e_{n+1}$, the third copy to be generated by $e_2 + e_{n+2}$, et cetera. In the case that $n = 1$ this embedding is actually an isomorphism, $D_2 \cong A_1 \oplus A_1$. More generally, D_{2n}/A_1^{2n} embeds in the discriminant group of A_1^{2n} , which is naturally isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2n} \cong \mathbb{F}_2^{2n}$. As such, it is natural to use binary codewords of length $2n$ to label cosets of A_1^{2n} in its dual. Given such a codeword $C \in \mathbb{F}_2^{2n}$, define $\text{wt}(C)$ —the *weight* of C —to be the number of non-zero entries of C . Define a binary code $\mathcal{D}_{2n} < \mathbb{F}_2^{2n}$ by setting

$$\mathcal{D}_{2n} := \{C = (c_1, \dots, c_{2n}) \mid c_i = c_{n+i} \text{ for } 1 \leq i \leq n, \text{wt}(C) = 0 \pmod{4}\}. \quad (3.2.3)$$

We will abuse notation somewhat by also using $[i]$ to denote the following length $2n$ codewords,

$$[0] := (0^{2n}), \quad [1] := (1^n 0^n), \quad [2] := (0^{n-1} 1 0^{n-1} 1), \quad [3] := (1^{n-1} 0^n 1). \quad (3.2.4)$$

The next result may be checked directly, and smooths out any conflict between (3.2.2) and (3.2.4).

Lemma 3.2.1. *With the above conventions, the image of $(D_{2n} + [i])/A_1^{2n}$ in \mathbb{F}_3^{2n} is $\mathcal{D}_{2n} + [i]$ for $i \in \{0, 1, 2, 3\}$.*

The above discussion shows, in particular, that $A_1^{12} \cong \sqrt{2}\mathbb{Z}^{12}$ embeds in D_{12}^+ . In §3.5.4 we will make use of the fact that $\sqrt{3}\mathbb{Z}^{12}$ also embeds in D_{12}^+ . To see this recall that the (*extended*) *ternary Golay code* is a linear sub space $\mathcal{G} < \mathbb{F}_3^{12}$ of dimension 6 such that if

$$C \cdot D := \sum_i c_i d_i \quad (3.2.5)$$

for $C = (c_1, \dots, c_{12})$ and $D = (d_1, \dots, d_{12})$ then $C \cdot D = 0$ when $C, D \in \mathcal{G}$, and no non-zero codeword $C \in \mathcal{G}$ has less than six non-zero entries. These properties determine \mathcal{G} uniquely, up to permutations of coordinates, and multiplications of coordinates by ± 1 (cf. e.g. [CS88]).

We will denote the elements of \mathbb{F}_3 by $\{0, +, -\}$ when convenient. To obtain an embedding of $\sqrt{3}\mathbb{Z}^{12}$ in D_{12}^+ fix a copy \mathcal{G} of the ternary Golay code in \mathbb{F}_3^{12} . Multiplying some components by -1 if necessary we may assume that $(+^{12}) \in \mathcal{G}$. Then there are exactly 11 code words $C^i = (c_1^i, \dots, c_{12}^i) \in \mathcal{G}$ such that the first entry of C^i is $+1$, five further entries are $+1$, and the remaining six entries are -1 . Set $C^{12} = (+^{12})$ and define $\lambda^i := (\lambda_1^i, \dots, \lambda_{12}^i)$ for $1 \leq i \leq 12$ by setting $\lambda_j^i = \pm \frac{1}{2}$ when $c_j^i = \pm 1$. Then the λ^i all belong to D_{12}^+ , and satisfy $\lambda^i \cdot \lambda^j = 3\delta_{ij}$. So the λ^i constructed in this way generate a sub lattice of D_{12}^+ isomorphic to $\sqrt{3}\mathbb{Z}^{12}$.

The discriminant group of $\sqrt{3}\mathbb{Z}^{12}$ is \mathbb{F}_3^{12} , so it is natural to consider the image of $D_{12}^+/\sqrt{3}\mathbb{Z}^{12}$ in \mathbb{F}_3^{12} . Denote it \mathcal{G}_{12}^+ . Since D_{12}^+ is a self-dual lattice, \mathcal{G}_{12}^+ is a linear subspace such that $C \cdot D = 0$ for $C, D \in \mathcal{G}_{12}^+$. From the fact that $\lambda \in D_{12}^+$ can only satisfy $\lambda \cdot \lambda \leq 1$ if $\lambda = 0$ we obtain that \mathcal{G}_{12}^+ has no non-zero words with fewer than six non-zero entries. Applying the uniqueness of the ternary Golay code we obtain the following result.

Lemma 3.2.2. *The image of $D_{12}^+/\sqrt{3}\mathbb{Z}^{12}$ in \mathbb{F}_3^{12} is a copy of the ternary Golay code.*

3.2.2 Modularity

We now review some results on modularity for vertex operator superalgebras.

Zhu proved [Zhu96] that certain trace functions on irreducible modules for suitable vertex operator algebras span representations of the modular group $SL_2(\mathbb{Z})$. More general modularity results that incorporate twisted modules have been obtained by Dong–Li–Mason [DLM00], Dong–Zhao [DZ05, DZ10], and Van Ekeren [Van13]. We will use the extension of [Zhu96, DLM00] to vertex operator superalgebras established in [DZ05].

To describe the relevant results let $W = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} W_n$ be a rational C_2 -cofinite vertex operator superalgebra of CFT type. Let I_0 be an index set for the isomorphism classes of irreducible W -modules, let I_1 be an index set for the isomorphism classes of irreducible canonically twisted W -modules, and set $I := I_0 \cup I_1$. Write M_i for a representative (untwisted or canonically twisted) W -module corresponding to $i \in I$, and choose a compatible superspace structure $M_i = M_i^{\text{even}} \oplus M_i^{\text{odd}}$ for each $i \in I$. For M an untwisted or canonically twisted W -module define *vertex operators on the torus* $Y[a, z] : M \rightarrow M((z))$ for $a \in W$ by requiring that $Y[a, z] = Y(a, e^z - 1)e^{nz}$ when $a \in W_n$, and define $a_{[n]} \in \text{End}(M)$ by requiring $Y[a, z] = \sum_k a_{[n]} z^{-n-1}$. Also, write $W = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} W_{[n]}$ for the eigenspace decomposition of W with respect to the action of $\tilde{\omega}_{[1]}$, where $\tilde{\omega} := \omega - \frac{c}{24}\mathbf{v}$, and ω and \mathbf{v} are the Virasoro and vacuum elements of W , respectively.

Theorem 3.2.3 ([DZ05]). *Suppose that W is a rational C_2 -cofinite vertex operator superalgebra with central charge c . Then with I and $\{M_i\}_{i \in I}$ as above there are maps $\rho_{ij} : SL_2(\mathbb{Z}) \rightarrow \mathbb{C}$ for $i, j \in I$ such that if $v \in W_{[n]}$ for some $n \in \frac{1}{2}\mathbb{Z}$ then*

$$\text{tr}_{M_i} \left(o(v)(-1)^{\ell F} q^{L_0 - \frac{c}{24}} \right) \Big|_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_{j \in I_{\tilde{k}}} \rho_{ij} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{tr}_{M_j} \left(o(v)(-1)^{\tilde{\ell} F} q^{L_0 - \frac{c}{24}} \right) \quad (3.2.6)$$

for $\ell \in \mathbb{Z}/2\mathbb{Z}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, where $\tilde{k} = 1 + a(k+1) + c(\ell+1) \pmod{2}$ and $\tilde{\ell} = 1 + b(k+1) + d(\ell+1) \pmod{2}$ when $i \in I_k$.

In (3.2.6) we utilize the usual slash notation from modular forms, setting

$$(f|_n \begin{pmatrix} a & b \\ c & d \end{pmatrix})(\tau) := f\left(\frac{a\tau+b}{c\tau+d}\right) \frac{1}{(c\tau+d)^n} \quad (3.2.7)$$

for a holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. If W is a rational C_2 -cofinite vertex operator algebra then we can apply Theorem 3.2.3 to W by regarding it as a vertex operator superalgebra with trivial odd part. Then $I_0 = I_1$ and we recover a specialization of Zhu's results [Zhu96] for W by taking $k = \ell = 1$ in (3.2.6).

Resume the assumption that W is a rational C_2 -cofinite vertex operator superalgebra of CFT type. Define the *characters* of an untwisted or canonically twisted W -module M by setting

$$\text{ch}^\pm[M](\tau) := \text{tr}_M \left((\pm 1)^F q^{L_0 - \frac{c}{24}} \right). \quad (3.2.8)$$

Then taking v to be the vacuum in Theorem 3.2.3 we obtain

$$\text{ch}^\epsilon[M_i] \left(\frac{a\tau + b}{c\tau + d} \right) = \sum_{j \in I_{\bar{k}}} \rho_{ij}(\gamma) \text{ch}^{\tilde{\epsilon}}[M_j](\tau), \quad (3.2.9)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, where $\epsilon = (-1)^\ell$ and $\tilde{\epsilon} = (-1)^{\tilde{\ell}}$.

In this work we will be especially interested in the situation in which a superspace M is a module for a tensor product $V' \otimes V''$ of vertex operator superalgebras, and is thus equipped with two commuting actions of the Virasoro algebra. We write L'_n and L''_n for the Virasoro operators corresponding to V' and V'' , respectively, and write c' and c'' for the corresponding central charges. Then it is natural to consider the *refined characters*

$$\widehat{\text{ch}}^\pm[M](\tau', \tau'') := \text{tr}_M \left((\pm 1)^F q'^{L'_0 - \frac{c'}{24}} q''^{L''_0 - \frac{c''}{24}} \right) \quad (3.2.10)$$

where $q' := e^{2\pi i\tau'}$ and $q'' := e^{2\pi i\tau''}$. Krauel–Miyamoto [KM15] have shown how Zhu's theory [Zhu96] extends so as to yield a modularity result for such refined characters in the vertex operator algebra case. By replacing Zhu's results with the appropriate vertex operator superalgebra counterparts from [DZ05] in the proof of Theorem 1 in [KM15] we readily obtain a direct analogue for vertex operator superalgebras. Here we require the following special case of this.

Theorem 3.2.4 ([DZ05, KM15]). *Let W be a rational C_2 -cofinite vertex operator algebra and suppose that $\omega = \omega' + \omega''$ where ω is the conformal vector of W , and ω' and ω'' generate commuting representations of the Virasoro algebra. Then with I , $\{M_i\}_{i \in I}$ and $\rho_{ij}(\gamma)$ as in (3.2.6) we have*

$$\widehat{\text{ch}}^\epsilon[M_i] \left(\frac{a\tau' + b}{c\tau' + d}, \frac{a\tau'' + b}{c\tau'' + d} \right) = \sum_{j \in I_{\tilde{k}}} \rho_{ij}(\gamma) \widehat{\text{ch}}^{\tilde{\epsilon}}[M_j](\tau', \tau'') \quad (3.2.11)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Here \tilde{k} and $\tilde{\ell}$ are as in (3.2.6), for $\epsilon = (-1)^\ell$ and $\tilde{\epsilon} = (-1)^{\tilde{\ell}}$.

3.2.3 Superconformal Algebras

In this section we recall some properties of the small $N = 4$ superconformal algebra from [Ali99, ET87, ET88a, ET88b]. It is strongly generated by a Virasoro field T of dimension 2, four odd fields G^a ($a = 0, 1, 2, 3$) of dimension $\frac{3}{2}$, and three even fields J^i ($i = 1, 2, 3$) of dimension 1. Define

$$\alpha_{a,b}^i := \frac{1}{2} (\delta_{a,i} \delta_{b,0} - \delta_{b,i} \delta_{a,0}) + \frac{1}{2} \epsilon_{iab} \quad (3.2.12)$$

where ϵ_{ijk} is totally antisymmetric for $i, j, k \in \{1, 2, 3\}$, normalized so that $\epsilon_{123} = 1$, and defined to be zero if one of the indices is zero. Also let k be a positive integer and set $c = 6k$. The operator product algebra in a representation with central charge c is then

$$T(z)T(w) \sim \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}, \quad (3.2.13)$$

$$T(z)G^a(w) \sim \frac{\frac{3}{2}G^a(w)}{(z-w)^2} + \frac{\partial_w G^a(w)}{(z-w)}, \quad (3.2.14)$$

$$T(z)J^i(w) \sim \frac{J^i(w)}{(z-w)^2} + \frac{\partial_w J^i(w)}{(z-w)}, \quad (3.2.15)$$

$$G^a(z)G^b(w) \sim \frac{\frac{2}{3}c\delta_{a,b}}{(z-w)^3} - \frac{\sum_{i=1}^3 8\alpha_{a,b}^i J^i(w)}{(z-w)^2} + \frac{2\delta_{a,b}T(w) - \sum_{i=1}^3 4\alpha_{a,b}^i \partial_w J^i(w)}{(z-w)}, \quad (3.2.16)$$

$$J^i(z)G^a(w) \sim \frac{\sum_{b=0}^3 4\alpha_{a,b}^i G^b(w)}{(z-w)}, \quad (3.2.17)$$

$$J^i(z)J^j(w) \sim \frac{-\frac{1}{2}k\delta_{i,j}}{(z-w)^2} + \frac{\sum_{k=1}^3 \epsilon_{ijk}J^k(w)}{(z-w)}. \quad (3.2.18)$$

Equivalently, we have the following commutation relations of modes, where $m, n \in \mathbb{Z}$ and $r, s \in \mathbb{Z} + \frac{1}{2}$.

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} = \frac{c}{12} (M^3 - m) \delta_{m+n,0}, \\ [L_m, G_r^a] &= \left(\frac{m}{2} - r\right) G_{m+r}^a, \\ [L_m, J_n^i] &= -nJ_{m+n}^i, \\ \{G_r^a, G_s^b\} &= 2\delta_{a,b}L_{r+s} - \sum_{i=1}^3 4(r-s)\alpha_{a,b}^i J_{r+s}^i + \frac{c}{3} \left(r^2 - \frac{1}{4}\right) \delta_{a,b} \delta_{r+s,0}, \quad (3.2.19) \\ [J_m^i, G_r^a] &= \sum_{b=0}^3 \alpha_{a,b}^i G_{m+r}^b, \\ [J_m^i, J_n^j] &= \sum_{k=1}^3 \epsilon_{ijk} J_{m+n}^k - m \frac{k}{2} \delta_{i,j} \delta_{m+n,0}. \end{aligned}$$

Note that the currents $J^i(z)$ represent the affine Lie algebra of \mathfrak{sl}_2 at level k , and k is a positive integer in any unitary representation. Often one uses another basis for these; namely

$$J(z) := -2iJ^1(z), \quad J^+(z) := J^2(z) - iJ^3(z), \quad J^-(z) := -J^2(z) - iJ^3(z). \quad (3.2.20)$$

Also for the odd currents there is another useful basis:

$$G^{-,1}(z) := G^0(z) - iG^1(z), \quad G^{+,1}(z) := -G^2(z) + iG^3(z), \quad (3.2.21)$$

$$G^{-,2}(z) := G^2(z) + iG^3(z), \quad G^{+,2}(z) := -G^0(z) + iG^1(z), \quad (3.2.22)$$

which yields

$$[J_m^\pm, G_r^{\mp,x}] = G_{m+r}^{\pm,x}, \quad [J_m, G_r^{\pm,x}] = \pm G_{m+r}^{\pm,x} \quad (3.2.23)$$

for $x \in \{1, 2\}$.

The mode algebra of the affine Lie algebra of \mathfrak{sl}_2 at level k has a family of automorphisms, called spectral flow. They are induced from affine Weyl translations so they are parameterized by the translation lattice which is isomorphic

to \mathbb{Z} . For $\ell \in \mathbb{Z}$ the corresponding action is denoted σ^ℓ , and satisfies

$$\begin{aligned}\sigma^\ell(J_n^\pm) &= J_{n \mp \ell}^\pm, \\ \sigma^\ell(J_n) &= J_n - \delta_{n,0} \ell k, \\ \sigma^\ell(L_n) &= L_n + \delta_{n,0} \left(\frac{\ell}{2} J_0 + \frac{\ell^2 k}{4} \right).\end{aligned}\tag{3.2.24}$$

We may consider twisted modules as in (2.10) of [CR13]. That is, given a module M for the affine Lie algebra of \mathfrak{sl}_2 at level k let σ_ℓ^* be the unique invertible linear transformation of M such that the action of the mode algebra satisfies

$$X \sigma_\ell^*(v) = \sigma_\ell^*(\sigma^{-\ell}(X)v).\tag{3.2.25}$$

This σ^ℓ -twisting of M is isomorphic to M as a module for the mode algebra, but since the grading is changed they are in general not isomorphic as modules for the vertex operator algebra $L_k(\mathfrak{sl}_2)$ attached to \mathfrak{sl}_2 at level k . Identification of the twisted modules can often be achieved via characters.

So we define a character of M by setting

$$\text{ch}[M](y, u, \tau) := \text{tr}_M \left(y^k z^{J_0} q^{L_0 - \frac{c}{24}} \right)\tag{3.2.26}$$

where $z = e^{2\pi i u}$ and $q = e^{2\pi i \tau}$. Then from the action of σ^ℓ we see that

$$\text{ch}[\sigma_\ell^*(M)](y, z, q) = \text{ch}[M](yz^\ell q^{\frac{1}{4}\ell^2}, zq^{\frac{1}{2}\ell}, q).\tag{3.2.27}$$

If k is a positive integer and M is an irreducible integrable highest-weight module of level k , then the character of M is a component of a vector-valued Jacobi form and the twisted module can be seen to be isomorphic to the original one if ℓ is even. If ℓ is odd it maps the irreducible integrable highest-weight module with highest weight j (corresponding to the $j + 1$ dimensional representation of \mathfrak{sl}_2) to the one with highest weight $k - j$. Thus it corresponds to fusion with the order two simple current module.

We now consider a vertex operator superalgebra V of central charge $c = 6k$ that contains a commuting pair of sub vertex operator algebras $L_k(\mathfrak{sl}_2)$ and U , where the Virasoro elements of V and $L_k(\mathfrak{sl}_2) \otimes U$ coincide. We require U

to be rational, and suppose that V is of the form

$$V = \bigoplus_i L_k(\lambda_i) \otimes U_i \quad (3.2.28)$$

where $L_k(\lambda_i)$ is the irreducible $L_k(\mathfrak{sl}_2)$ -module with highest weight λ_i , and U_i is an irreducible U -module. Further we require that V contain strong generators for the small $N = 4$ superconformal algebra at $c = 6k$. In this situation the parity of elements in V is given by $(-1)^{J_0}$ where J_0 is as in (3.2.20), (3.2.23). The spectral flow automorphism $\sigma := \sigma^1$ extends naturally to V in such a way that

$$\sigma^*(V) = \bigoplus_i (L_k(\lambda_i) \boxtimes_{L_k(\mathfrak{sl}_2)} L_k(\lambda_*)) \otimes U_i \quad (3.2.29)$$

as a $L_k(\mathfrak{sl}_2) \otimes U$ -module, where $L_k(\lambda_*)$ denotes the order two simple current of $L_k(\mathfrak{sl}_2)$, and \boxtimes_V is the fusion product for modules over V .

Under spectral flow we have $\sigma^{-1}(L_0) = L_0 + \frac{1}{2}J_0 + \frac{c}{24}$, so the conformal weight of the modes $G_r^{\pm, x}$ becomes $r \pm \frac{1}{2}$ as endomorphisms on twisted modules. Especially, the $G_{\mp \frac{1}{2}}^{\pm, x}$ act with conformal dimension zero, and thus induce grading-preserving maps from the even part to the odd part of the twisted module. We compute

$$\begin{aligned} \left(G_{-\frac{1}{2}}^{+,1} + G_{\frac{1}{2}}^{-,2} \right)^2 &= \frac{1}{2} \left\{ G_{-\frac{1}{2}}^{+,1} + G_{\frac{1}{2}}^{-,2}, G_{-\frac{1}{2}}^{+,1} + G_{\frac{1}{2}}^{-,2} \right\} \\ &= -2L_0 + J_0 \\ &= 2\sigma^{-1} \left(L_0 - \frac{c}{24} \right). \end{aligned} \quad (3.2.30)$$

This shows that $G_{-\frac{1}{2}}^{+,1} + G_{\frac{1}{2}}^{-,2}$ is an invertible map from the even conformal grade n subspace of $\sigma^*(M)$ to the odd one as long as $n - \frac{c}{24}$ does not vanish.

We summarize this as follows.

Lemma 3.2.5. *If M is a V -module and $\sigma^*(M) = \bigoplus_n \sigma^*(M)_n$ is the grading of the corresponding twisted module then $\text{sdim } \sigma^*(M)_n = 0$ unless $n - \frac{c}{24} = 0$.*

3.2.4 Free Fields

We now describe a free field realization of the small $N = 4$ superconformal algebra at $c = 6$. Consider the tensor product of the vertex operator superalgebra of four free fermions with the rank four Heisenberg vertex operator

algebra. An action of the compact Lie group $SU(2)$ is given by organizing both the fermions and bosons in the standard and conjugate representations of $SU(2)$. By Lemma 3.4 of [CH14] the sub vertex operator superalgebra of fixed points for this $SU(2)$ action contains the small $N = 4$ superconformal algebra at $c = 6$. So by using bosonization we obtain a free field realization of the small $N = 4$ superconformal algebra at $c = 6$ in terms of twelve free fermions. The precise expression for the latter can be found in §2 of [GTVW14]. In terms of lattice vertex operator superalgebras this free field realization may be described as follows.

The vertex operator superalgebra of four free fermions may be identified with the lattice vertex operator superalgebra associated to \mathbb{Z}^2 . We have $\mathbb{Z}^2 = D_2 \cup (D_2 + [2])$ in the notation of §3.2.1, and the exceptional isomorphism $D_2 \cong A_1 \oplus A_1$. So $V_{D_2} \cong V_{A_1} \otimes V_{A_1}$. Four free fermions are thus the simple current extension of $V_{A_1} \otimes V_{A_1}$ by the unique simple current with dimension $\frac{1}{2}$. The free field realization in terms of twelve free fermions can be inspected to be a sub vertex operator superalgebra of

$$V_{D_2}^{\otimes 3} \oplus V_{D_2+[2]}^{\otimes 3}, \quad (3.2.31)$$

and we have just seen that this vertex operator superalgebra is isomorphic to

$$V_{A_1}^{\otimes 6} \oplus V_{A_1+(1)}^{\otimes 6} \quad (3.2.32)$$

where $A_1 + (1)$ denotes the unique non-trivial coset of A_1 in its dual. For completeness we describe a precise realization. Consider odd fields b_1, \dots, b_6 and c_1, \dots, c_6 with operator products

$$b_i(z)c_j(w) \sim \frac{\delta_{i,j}}{(z-w)}, \quad b_i(z)b_j(w) \sim c_i(z)c_j(w) \sim 0, \quad (3.2.33)$$

generating a copy $F(12) \cong V_{\mathbb{Z}^6}$ of the vertex operator superalgebra of 12 free fermions. From the above we have $(V_{D_2} \oplus V_{D_2+[2]})^{\otimes 3} \cong V_{\mathbb{Z}^6}$. Then b_1, b_2, c_1, c_2 span the first copy of $V_{D_2} \oplus V_{D_2+[2]}$, the b_3, b_4, c_3, c_4 span the second copy, and b_5, b_6, c_5, c_6 span the last one. The three fields

$$h = :b_1c_1: + :b_2c_2:, \quad e = :b_1b_2:, \quad f = :c_1c_2:, \quad (3.2.34)$$

are all in the even sub vertex operator algebra V_{D_2} of the first copy of $V_{D_2} \oplus V_{D_2+[2]}$. These three fields strongly generate a vertex operator algebra isomorphic to $L_1(\mathfrak{sl}_2)$. The four fields

$$G^{+,1} = :b_1 b_3 b_5 :, \quad G^{-,1} = :c_2 b_3 b_5 :, \quad G^{-,2} = :c_1 c_3 c_5 :, \quad G^{+,2} = :b_2 c_3 c_5 :, \quad (3.2.35)$$

are all fields in $V_{D_2+[2]}^{\otimes 3}$. These seven fields together with the Virasoro field strongly generate a vertex operator superalgebra isomorphic to the small $N = 4$ superconformal algebra at $c = 6$.

3.3 Self-Dual Vertex Operator Superalgebras

In this section we prove our first main result, which is a classification of self-dual C_2 -cofinite vertex operator superalgebras of CFT type with central charge less than or equal to 12.

Theorem 3.3.1. *If W is a self-dual C_2 -cofinite vertex operator superalgebra of CFT type with central charge $c \leq 12$ then either $W \cong F(n)$ for some $0 \leq n \leq 24$, or $W \cong V_{E_8} \otimes F(n)$ for $0 \leq n \leq 8$, or $W \cong V_{D_{12}^+}$.*

Proof. We first prove the claimed result for the special case that $c = 12$. For this we require to show that a self-dual C_2 -cofinite vertex operator superalgebra of CFT type with $c = 12$ is isomorphic to one of $V_{D_{12}^+}$, $V_{E_8} \otimes F(8)$ or $F(24)$.

So let W be a self-dual C_2 -cofinite vertex operator superalgebra of CFT type with central charge 12. We will constrain the possibilities for W by extending the methods used in §5.1 of [Dun07]. There, $V_{D_{12}^+}$ is characterized as the unique such vertex operator superalgebra with an $N = 1$ superconformal structure and vanishing weight $\frac{1}{2}$ subspace. We will also employ the arguments of §4.2 of [DM15], in which the hypothesis of superconformal structure is removed.

We begin by applying Theorem 3.2.3 to W . In this situation I_0 and I_1 are singletons. Let us set $I_k = \{k\}$, so that $M_0 = W$ and $M_1 = W_{\text{tw}}$. Then taking v to be the vacuum in (3.2.6) we obtain that $Z_{\text{NS-NS}}^+(\tau) := \text{tr}_W q^{L_0 - \frac{1}{2}}$ is a weakly holomorphic modular form with character ρ_{00} of weight 0 for $\Gamma_\theta := \langle S, T^2 \rangle$,

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Further, ρ_{00} is, a priori, trivial on T^2 , and ± 1 on S . This fact is the basis of the proof of Proposition 5.7 in [Dun07], which shows that if $d := \dim W_{\frac{1}{2}}$ then

$$\begin{aligned} Z_{\text{NS-NS}}^+(\tau) &= \frac{\eta(\tau)^{48}}{\eta(2\tau)^{24}\eta(\frac{\tau}{2})^{24}} + d - 24 \\ &= q^{-\frac{1}{2}} + d + 276q^{\frac{1}{2}} + O(q) \end{aligned} \tag{3.3.1}$$

so that in particular, ρ_{00} is trivial on Γ_θ . Also, similar to Proposition 5.9 of [Dun07], we find from taking $\gamma = TS$ in (3.2.6) that $(W_{\text{tw}})_n$ vanishes unless $n \geq \frac{1}{2}$. For another application let $u, u' \in W_1$. Then taking $v = u_{[-1]}u'$ in (3.2.6), setting $X(\tau) := \text{tr}_W o(u_{[-1]}u')q^{L_0 - \frac{1}{2}}$ and using the triviality of ρ_{00} on Γ_θ we find that $X(\tau)$ is a weakly holomorphic modular form of weight 2 for Γ_θ satisfying $X(\tau) = Cq^{-\frac{1}{2}} + O(1)$ as $\tau \rightarrow i\infty$ for some $C \in \mathbb{C}$. Note that Γ_θ has two cusps, represented by $i\infty$ and 1. From (3.2.6) and the fact that W_{tw} has L_0 -grading bounded below by $\frac{1}{2}$ we see that $X(\tau) = O(1)$ as $\tau \rightarrow 1$. Since the space of modular forms of weight 2 for Γ_θ is one-dimensional, spanned by $\theta_{D_4}(\frac{\tau+1}{2}) = 1 - 24q^{\frac{1}{2}} + O(q)$ (cf. Theorem 7.1.6 in [Ran77]), it follows that

$$\begin{aligned} X(\tau) &= -2Cq \frac{d}{dq} Z_{\text{NS-NS}}^+(\tau) + D\theta_{D_4}(\frac{\tau+1}{2}) \\ &= Cq^{-\frac{1}{2}} + D + (-276C - 24D)q^{\frac{1}{2}} + O(q) \end{aligned} \tag{3.3.2}$$

for some $C, D \in \mathbb{C}$ (with C as above), which we can expect will depend on u and u' .

We now endeavour to connect C and D in (3.3.2) to the Lie algebra structure on W_1 . For this note that the first paragraph of the proof of Theorem 4.5 in [DM15] applies to W , showing that W admits a unique (up to scale) non-degenerate invariant bilinear form, which we henceforth denote $\langle \cdot, \cdot \rangle$. We also have that W_1 is contained in the kernel of L_1 , so by Theorem 1.1 of [DM02] the Lie algebra structure on W_1 is reductive. Applying the argument of the proof of Theorem 5.12 in [Dun07] to W —this uses the identity (3.3.1)—we see that the Lie rank of W_1 is bounded above by 12. We can identify a simple component of W_1 just by considering $d = \dim W_{\frac{1}{2}}$. To do this note that the invariant bilinear form $\langle \cdot, \cdot \rangle$ on W is non-degenerate when restricted to $W_{\frac{1}{2}}$. If we let U denote the sub vertex operator superalgebra of W generated by

$W_{\frac{1}{2}}$ then, arguing as in [GS88] (cf. also §3 of [Tam99]), we see that U is isomorphic to the Clifford algebra vertex operator superalgebra defined by the orthogonal space structure on $W_{\frac{1}{2}}$, and if V is the commutant of U in W then W is naturally isomorphic to $U \otimes V$. So W_1 contains U_1 , which is the Lie algebra naturally associated to the orthogonal structure on $W_{\frac{1}{2}}$. Since W_1 has Lie rank bounded above by 12 we must have $d \leq 24$. Indeed, the case that $d = 24$ is realized by $W = U = F(24)$.

Table 3.1: Dual Coxeter numbers of simple complex Lie algebras

\mathfrak{g}	h^\vee
A_n	$n + 1$
B_n	$2n - 1$
C_n	$n + 1$
D_n	$2n - 2$
E_6	12
E_7	18
E_8	30
F_4	9
G_2	4

So suppose henceforth that $d < 24$. Then $\dim U_1 < \dim W_1 = 276$, and $V_1 \neq \{0\}$. To further constrain W_1 we consider (3.3.2) with $u, u' \in V_1$. Note that since $V_{\frac{1}{2}} = \{0\}$ by construction we have $\text{tr}_W o(u)o(u')q^{L_0 - \frac{1}{2}} = \kappa(u, u')q^{\frac{1}{2}} + O(q)$ where κ is the Killing form on W_1 . We have $u_{[1]}u' = \langle u, u' \rangle \mathbf{v}$ according to Lemma 5.1 of [Dun07], so by an application of Sublemma 6.9 of [DZ05] we find that

$$\begin{aligned} X(\tau) &= \text{tr}_W o(u)o(u')q^{L_0 - \frac{1}{2}} - \frac{1}{12} \langle u, u' \rangle E_2(\tau) Z_{\text{NS-NS}}^+(\tau) \\ &= -\frac{1}{12} \langle u, u' \rangle q^{-\frac{1}{2}} - \frac{1}{12} \langle u, u' \rangle d + (\kappa(u, u') - 21 \langle u, u' \rangle) q^{\frac{1}{2}} + O(q) \end{aligned} \quad (3.3.3)$$

where $E_2(\tau) = 1 - 24 \sum_{n>0} \frac{nq^n}{1-q^n}$ is the quasi-modular Eisenstein series of weight 2, and $Z_{\text{NS-NS}}^+(\tau)$ is as in (3.3.1). Comparing (3.3.2) with (3.3.3) we find that $C = -\frac{1}{12} \langle u, u' \rangle$, $D = -\frac{1}{12} \langle u, u' \rangle d$, and

$$\kappa(u, u') = (44 + 2d) \langle u, u' \rangle. \quad (3.3.4)$$

In particular, the Killing form is non-degenerate on V_1 , so the Lie algebra structure on V_1 is semisimple. So let \mathfrak{g} be a simple component of V_1 . From

the main theorem of [DM06] we know that the vertex operators attached to V_1 represent the affine Lie algebra associated to \mathfrak{g} with integral level; call it k . Then for α a long root of \mathfrak{g} we have $\kappa(\alpha, \alpha) = 4h$ where h is the dual Coxeter number of \mathfrak{g} , and $\langle \alpha, \alpha \rangle = 2k$. So (3.3.4) with $u = u' = \alpha$ yields $h = (22 + d)k$. We are reduced to finding the pairs (d, \mathfrak{g}) where d is an integer $0 \leq d < 24$, and \mathfrak{g} is a simple Lie algebra with Lie rank bounded above by $12 - \frac{1}{2}d$ such that the dual Coxeter number h^\vee of \mathfrak{g} is an integer multiple of $22 + d$. Inspecting Table 3.1 we see that either $d = 0$ and \mathfrak{g} is of type D_{12} , or $d = 8$ and \mathfrak{g} is of type E_8 . The first of these is realized by $W = V_{D_{12}^+}$. The second is realized by $W = V_{E_8} \otimes F(8)$. Thus we have dealt with the special case that $c = 12$.

To complete the proof let W be a self-dual C_2 -cofinite vertex operator superalgebra of CFT type with $c < 12$. By applying Theorem 11.3 of [DLM00] to W^{even} we may conclude that c is a rational number. Then the argument of Satz 2.2.2 of [Höh07] applies to W , and shows that $c \in \frac{1}{2}\mathbb{Z}$. So $n = 24 - 2c$ is a positive integer and $W' := W \otimes F(n)$ is a self-dual C_2 -cofinite vertex operator superalgebra of CFT type with central charge $c' = 12$. Since n is positive W' is one of $V_{E_8} \otimes F(8)$ or $F(24)$, by what we have already proved about self-dual vertex operator superalgebras with central charge 12. The desired conclusion follows. \square

3.4 Superconformal Field Theory

3.4.1 Potential Bulk Conformal Field Theory

The basic structure underlying a *bulk conformal field theory* is a module \mathcal{H} for a tensor product $V' \otimes V''$ of vertex operator algebras V' and V'' . It is required that

$$\mathcal{H} = \bigoplus_i N'_i \otimes N''_i \tag{3.4.1}$$

where the N'_i and N''_i are irreducible modules for V' and V'' , respectively. Also, $V' \otimes V''$ should appear exactly once as a summand of \mathcal{H} . A standard but very special case is that $V' = V''$ is rational and C_2 -cofinite, and the $N'_i = N''_i$

are all the irreducible V' -modules. We call this the *diagonal* conformal field theory for $V = V' = V''$.

Various further properties are required of \mathcal{H} , including *closure under fusion* and *modular invariance*. Define

$$\begin{aligned} \widehat{\text{ch}}[\mathcal{H}](\tau', \tau'') &:= \text{tr}_{\mathcal{H}} \left(q'^{L'_0 - \frac{c'}{24}} q''^{L''_0 - \frac{c''}{24}} \right) \\ &= \sum_i \text{ch}[N'_i](\tau') \text{ch}[N''_i](\tau'') \end{aligned} \tag{3.4.2}$$

(cf. (3.2.10)). Modular invariance is the requirement that the *partition function*

$$\begin{aligned} Z_{\mathcal{H}}(\tau) &:= \widehat{\text{ch}}[\mathcal{H}](\tau, -\bar{\tau}) \\ &= \sum_i \text{ch}[N'_i](\tau) \text{ch}[N''_i](-\bar{\tau}) \end{aligned} \tag{3.4.3}$$

be invariant for the natural action of $SL_2(\mathbb{Z})$. Closure under fusion is the requirement that operator products close on \mathcal{H} . So superficially at least, fusion and modularity force a bulk conformal field theory to resemble a self-dual vertex operator algebra. This is the motivation for the Main Question we formulated in §3.1.

We will answer the Main Question positively in what follows, using a certain convenient substitute for the notion of bulk conformal field theory. Also, we will find that there are more examples if we allow the vertex operator algebra in the question to be a vertex operator superalgebra.

To motivate our approach note that modular invariance is a strong constraint that is used in practice to classify possible examples of conformal field theories. There are additional properties of correlation functions that are harder to verify (cf. [TW17, Wen15]), but from a representation category point of view these correlation requirements are satisfied by symmetric special Frobenius algebra objects in the modular tensor category of a suitably chosen vertex operator algebra according to [FRS02]. In this work lattice vertex operator algebras underly all examples, so all simple objects are simple currents, and if a symmetric special Frobenius algebra object \mathcal{A} is a direct sum of inequivalent simple currents then there is a rather explicit prescription

for the decomposition (3.4.1); namely (5.85) of [FRS02]. For example, if we assume $V' \cong V''$ and choose $\mathcal{A} = V'$ then the partition function of the bulk is the charge conjugation invariant. In the cases we consider every irreducible V' -module will be invariant under charge conjugation, so the charge conjugation invariant will coincide with the ordinary diagonal modular invariant. Motivated by this we introduce the following.

Definition 3.4.1. A *potential bulk conformal field theory* is a $V' \otimes V''$ -module \mathcal{H} as in (3.4.1) such that the partition function (3.4.3) is modular invariant.

Now to formulate an answer to the Main Question, suppose that W is a self-dual C_2 -cofinite vertex operator superalgebra such that

$$W \cong \bigoplus_i N'_i \otimes N''_i \quad (3.4.4)$$

as a $V' \otimes V''$ -module, for V' and V'' a commuting pair of rational C_2 -cofinite sub vertex operator algebras, where the N'_i and N''_i are irreducible modules for V' and V'' , respectively. Define

$$Z_W(\tau) := \widehat{\text{ch}}^+[W](\tau, -\bar{\tau}) \quad (3.4.5)$$

where $\widehat{\text{ch}}^\pm[\cdot]$ is as in (3.2.10).

Proposition 3.4.2. *With W as in (3.4.4), if the S -matrix of V'' is real and the eigenvalues of the action of $L'_0 - L''_0$ on W belong to $\mathbb{Z} + \frac{1}{24}(c' - c'')$ then Z_W is modular invariant.*

Proof. Since W is self-dual Theorem 3.2.3 implies that $\text{ch}^+[W]$ (cf. (3.2.8)) is invariant under S . It follows from Theorem 3.2.4 then that $\widehat{\text{ch}}^+[W](\tau', \tau'')$ is S -invariant as well. In particular we have

$$\begin{aligned} \widehat{\text{ch}}^+[W] \left(-\frac{1}{\tau'}, -\frac{1}{\tau''} \right) &= \sum_{i,j,\ell} S'_{ij} \text{ch}^+[N'_j](\tau') S''_{i\ell} \text{ch}^+[N''_\ell](\tau'') \\ &= \sum_i \text{ch}^+[N'_i](\tau') \text{ch}^+[N''_i](\tau'') \end{aligned} \quad (3.4.6)$$

for suitable matrices S' and S'' . Consider the action of S on $Z_W(\tau)$. Using (3.4.6) and the hypothesis that the entries of S'' are real we compute

$$\begin{aligned}
Z_W\left(-\frac{1}{\tau}\right) &= \sum_i \text{ch}^+[N'_i]\left(-\frac{1}{\tau}\right) \overline{\text{ch}^+[N''_i]\left(-\frac{1}{\tau}\right)} \\
&= \sum_{i,j,\ell} S'_{i,j} \text{ch}^+[N'_j](\tau) \overline{S''_{i,\ell} \text{ch}^+[N''_\ell](\tau)} \\
&= \sum_{i,j,\ell} S'_{i,j} \text{ch}^+[N'_j](\tau) S''_{i,\ell} \text{ch}^+[N''_\ell](-\bar{\tau}) \\
&= \sum_i \text{ch}^+[N'_i](\tau) \text{ch}^+[N''_i](-\bar{\tau}) \\
&= Z_W(\tau).
\end{aligned} \tag{3.4.7}$$

So Z_W is S -invariant. Invariance under T follows from

$$\begin{aligned}
Z_W(\tau + 1) &= \text{tr}_W \left(e^{2\pi i(\tau+1)(L'_0 - \frac{c'}{24})} e^{-2\pi i(\bar{\tau}+1)(L''_0 - \frac{c''}{24})} \right) \\
&= e^{\frac{2\pi i}{24}(c'' - c')} \text{tr}_W \left(e^{2\pi i(L'_0 - L''_0)} q^{L'_0 - \frac{c'}{24}} \bar{q}^{L''_0 - \frac{c''}{24}} \right)
\end{aligned} \tag{3.4.8}$$

and our hypothesis on the eigenvalues of L'_0 and L''_0 . This completes the proof. \square

Proposition 3.4.2 gives a path to answering the Main Question positively, in the sense that if its hypotheses are satisfied then it identifies a self-dual vertex operator superalgebra W with a potential bulk conformal field theory \mathcal{H} as a module for the underlying vertex operator algebra $V' \otimes V''$. The two conditions of Proposition 3.4.2 are strong, but are satisfied in interesting cases as we will see in §3.5.

3.4.2 Potential Bulk Superconformal Field Theory

We are also interested in relating self-dual vertex operator superalgebras to superconformal field theories in this work. To define a supersymmetric counterpart to the notion of potential bulk conformal field theory we consider a $V' \otimes V''$ -module

$$\mathcal{H} = \bigoplus_i N'_i \otimes N''_i \tag{3.4.9}$$

as in (3.4.1), but allow V' and V'' to be vertex operator superalgebras, and allow the N'_i in (3.4.1) to be irreducible untwisted or canonically twisted modules for V' , and similarly for the N''_i . (Strictly speaking, \mathcal{H} is a module for the even sub vertex operator algebra of $V' \otimes V''$.) Following tradition we use subscripts NS and R to indicate restrictions to untwisted and canonically twisted modules for V' and V'' . So,

$$\mathcal{H}_{\text{NS-NS}} := \bigoplus_i \begin{matrix} N'_i \otimes N''_i \\ N'_i \text{ untwisted} \\ N''_i \text{ untwisted} \end{matrix}, \quad (3.4.10)$$

$$\mathcal{H}_{\text{NS-R}} := \bigoplus_i \begin{matrix} N'_i \otimes N''_i \\ N'_i \text{ untwisted} \\ N''_i \text{ twisted} \end{matrix}, \quad (3.4.11)$$

and so on. We call $\mathcal{H}_{\text{NS-NS}}$ the NS-NS sector, et cetera. We also assume that \mathcal{H} is equipped with a compatible superspace structure $\mathcal{H} = \mathcal{H}^{\text{even}} \oplus \mathcal{H}^{\text{odd}}$, so that \mathcal{H} is graded by $(\mathbb{Z}/2\mathbb{Z})^3$,

$$\mathcal{H} = \mathcal{H}_{\text{NS-NS}}^{\text{even}} \oplus \mathcal{H}_{\text{NS-NS}}^{\text{odd}} \oplus \mathcal{H}_{\text{NS-R}}^{\text{even}} \oplus \cdots \oplus \mathcal{H}_{\text{R-R}}^{\text{even}} \oplus \mathcal{H}_{\text{R-R}}^{\text{odd}}. \quad (3.4.12)$$

We may regard a bulk conformal field theory as a bulk superconformal field theory in which only the even part of the NS-NS sector is non-zero. From this point of view it is natural to expect examples in which the NS-NS sector of a superconformal field theory is identified with a self-dual vertex operator superalgebra, and the R-R sector is identified with its canonically twisted module. As such, the prescription (3.4.3) does not usually define a modular invariant function when the superspace structure is non-trivial. Rather, Theorem 3.2.3 indicates that we should consider the vector-valued function

$$Z_{\mathcal{H}}(\tau) := \begin{pmatrix} Z_{\text{NS-NS}}^+(\tau) \\ Z_{\text{NS-NS}}^-(\tau) \\ Z_{\text{R-R}}^+(\tau) \\ Z_{\text{R-R}}^-(\tau) \end{pmatrix} \quad (3.4.13)$$

where Z_{X-Y}^{\pm} is defined by setting $Z_{X-Y}^{\pm}(\tau) := \widehat{\text{ch}}^{\pm}[\mathcal{H}_{X-Y}](\tau, -\bar{\tau})$ (cf. (3.2.10)) for $X, Y \in \{\text{NS}, \text{R}\}$. Then modularity for $Z_{\mathcal{H}}$ is the requirement that

$$\begin{aligned} \mathbf{S} \cdot Z_{\mathcal{H}}\left(-\frac{1}{\tau}\right) &= Z_{\mathcal{H}}(\tau), \\ \mathbf{T} \cdot Z_{\mathcal{H}}(\tau + 1) &= Z_{\mathcal{H}}(\tau), \end{aligned} \quad (3.4.14)$$

where

$$\mathbf{S} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{T} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.4.15)$$

Motivated by this we formulate the following super-analogue of Definition 3.4.1.

Definition 3.4.3. A *potential bulk superconformal field theory* is a $V' \otimes V''$ -module \mathcal{H} as in (3.4.9) whose partition function (3.4.13) satisfies (3.4.14).

Remark 3.4.4. For bulk superconformal field theories we expect that correlation function requirements are encoded by a suitably formulated notion of symmetric special Frobenius *superalgebra* object. It would be interesting to generalize [FRS02] to the super setting, and determine whether the examples we describe in this work are compatible or not. Superalgebra objects are introduced in the context of vertex algebra tensor categories in [CKL19, CKM17].

To formulate a supersymmetric counterpart to Proposition 3.4.2 consider a self-dual C_2 -cofinite vertex operator superalgebra W such that

$$W \cong \bigoplus_i N'_i \otimes N''_i \quad (3.4.16)$$

as a $V' \otimes V''$ -module, for V' and V'' a commuting pair of rational C_2 -cofinite sub vertex operator superalgebras, where the N'_i and N''_i are irreducible (untwisted) modules for V' and V'' , respectively. Write W_{tw} for the unique irreducible canonically twisted W -module (cf. §3.2.1) and chose a superspace structure $W_{\text{tw}} = W_{\text{tw}}^{\text{even}} \oplus W_{\text{tw}}^{\text{odd}}$ that is compatible with the superspace structure on W . We then define $Z_W(\tau)$ in analogy with (3.4.13), so that

$$Z_W(\tau) := \begin{pmatrix} Z^+(\tau) \\ Z^-(\tau) \\ Z_{\text{tw}}^+(\tau) \\ Z_{\text{tw}}^-(\tau) \end{pmatrix} \quad (3.4.17)$$

where $Z^\pm(\tau) := \widehat{\text{ch}}^\pm[W](\tau, -\bar{\tau})$ and $Z_{\text{tw}}^\pm(\tau) := \widehat{\text{ch}}^\pm[W_{\text{tw}}^{\text{even}}](\tau, -\bar{\tau})$. The proof of the next result follows in a directly similar way to that of Proposition 3.4.2.

Proposition 3.4.5. *With W as in (3.4.16), if the S -matrix of V'' is real, the eigenvalues of $L'_0 - L''_0$ on W^{even} lie in $\mathbb{Z} + \frac{1}{24}(c' - c'')$, and the eigenvalues of $L'_0 - L''_0$ on W^{odd} lie in $\mathbb{Z} + \frac{1}{2} + \frac{1}{24}(c' - c'')$ then Z_W satisfies the modularity condition (3.4.14).*

Thus a vertex operator superalgebra W satisfying the hypotheses of Proposition 3.4.5 also answers the Main Question positively, in the sense that it is identified with the NS-NS sector of a potential bulk superconformal field theory as a module for the underlying vertex operator superalgebra $V' \otimes V''$, and similarly for W_{tw} and the R-R sector. Note that reality of the modular S -matrix holds for $L_k(\mathfrak{sl}_2)$ when k is positive and integral, and holds also for minimal models of the Virasoro algebra. Inspecting Weil's description of the modular group action on theta functions for cosets of an even lattice L in its dual L^* , we see that if the inner products on L^* are contained in $\frac{1}{2}\mathbb{Z}$ then the S -matrix for the lattice vertex operator algebra V_L is also real. This holds in particular for $L = D_{4n}$, which will play a prominent role in what follows.

3.4.3 Superconformal structure

Superconformal field theories are usually assumed to come equipped with supersymmetry. With \mathcal{H} as in (3.4.9) we define an $N = (2, 2)$ *superconformal structure* to be an identification of sub vertex operator superalgebras of V' and V'' with vacuum modules for the $N = 2$ superconformal algebra. In this situation it is natural to consider

$$Z_{\mathcal{H}}(u, \tau) := \begin{pmatrix} Z_{\text{NS-NS}}^+(u, \tau) \\ Z_{\text{NS-NS}}^-(u, \tau) \\ Z_{\text{R-R}}^+(u, \tau) \\ Z_{\text{R-R}}^-(u, \tau) \end{pmatrix} \quad (3.4.18)$$

where $Z_{X-Y}^{\pm}(u, \tau) := \widehat{\text{ch}}^{\pm}[\mathcal{H}_{X-Y}](u, \tau, -\bar{u}, -\bar{\tau})$ and

$$\widehat{\text{ch}}^{\pm}[M](u', \tau', u'', \tau'') := \text{tr}_M \left((\pm 1)^F z'^{J'_0} q'^{L'_0 - \frac{c'}{24}} z''^{J''_0} q''^{L''_0 - \frac{c''}{24}} \right). \quad (3.4.19)$$

Here $z' = e^{2\pi i u'}$ and $q' = e^{2\pi i \tau'}$, et cetera. We also have the *elliptic genus* of \mathcal{H} , defined by setting

$$E_{\mathcal{H}}(u, \tau) := \widehat{\text{ch}}^{-}[\mathcal{H}_{\text{R-R}}](u, \tau, 0, -\bar{\tau}). \quad (3.4.20)$$

Now the modularity conditions on \mathcal{H} are richer. The natural counterpart to (3.4.14) is

$$\begin{aligned} e^{-\pi i \left(\frac{c'}{6} \frac{u^2}{\tau} - \frac{c''}{6} \frac{\bar{u}^2}{\bar{\tau}} \right)} \mathbf{S} \cdot Z_{\mathcal{H}} \left(\frac{u}{\tau}, -\frac{1}{\tau} \right) &= Z_{\mathcal{H}}(u, \tau), \\ \mathbf{T} \cdot Z_{\mathcal{H}}(u, \tau + 1) &= Z_{\mathcal{H}}(u, \tau), \end{aligned} \quad (3.4.21)$$

where \mathbf{S} and \mathbf{T} are as in (3.4.15). Call this modularity for $Z_{\mathcal{H}}$. If \mathcal{H} is a superconformal field theory underlying a sigma model with Calabi–Yau target space X then $E_{\mathcal{H}}$ is also modular, in the sense that we have

$$\begin{aligned} e^{-\pi i d \frac{u^2}{\tau}} E_{\mathcal{H}} \left(\frac{u}{\tau}, -\frac{1}{\tau} \right) &= E_{\mathcal{H}}(u, \tau), \\ E_{\mathcal{H}}(u, \tau + 1) &= E_{\mathcal{H}}(u, \tau), \end{aligned} \quad (3.4.22)$$

where $d = \dim_{\mathbb{C}} X$, and $\tau \mapsto E_{\mathcal{H}}(u, \tau)$ remains bounded as $\tau \rightarrow i\infty$, for any fixed $u \in \mathbb{C}$. That is, $E_{\mathcal{H}}$ is a *weak Jacobi form* of weight 0 and index $\frac{1}{2} \dim_{\mathbb{C}} X$. In this situation $E_{\mathcal{H}}(0, \tau)$ is the Euler characteristic of X . (In particular, $E_{\mathcal{H}}(0, \tau)$ is a constant function of τ .) Let us say that \mathcal{H} is *elliptic* if $E_{\mathcal{H}}$ satisfies (3.4.22). Say that \mathcal{H} satisfies *spectral flow symmetry* if

$$Z_{\text{R-R}}^{\pm}(u', \tau', u'', \tau'') = (z')^{\frac{c'}{6}} (z'')^{\frac{c''}{6}} (q')^{\frac{c'}{24}} (q'')^{\frac{c''}{24}} Z_{\text{NS-NS}}^{\pm}(u' + \frac{1}{2}\tau', \tau', u'' + \frac{1}{2}\tau'', \tau''). \quad (3.4.23)$$

At this point it is natural to define a *potential bulk $N = (2, 2)$ superconformal field theory* to be a potential bulk superconformal field theory \mathcal{H} (cf. Definition 3.4.3) with $N = (2, 2)$ superconformal structure such that (3.4.21), (3.4.22) and (3.4.23) are satisfied. These are strict requirements. Interestingly, we will see in examples that the extra superconformal structure allows us to weaken these requirements in what seems to be a useful way. In anticipation of this we offer the following.

Definition 3.4.6. Say that \mathcal{H} as in (3.4.9) is a *quasi potential bulk $N = (2, 2)$ superconformal field theory* if \mathcal{H} is elliptic (3.4.22), satisfies spectral flow symmetry (3.4.23), and if $Z_{\mathcal{H}}$ is modular (3.4.21) for some finite index subgroup of the modular group.

So for the notion of quasi potential bulk $N = (2, 2)$ superconformal field theory we relax condition (3.4.21), which is the invariance of $Z_{\mathcal{H}}$ under the

action of $SL_2(\mathbb{Z})$ defined by the left hand sides of (3.4.21), and require invariance only for some subgroup $\Gamma < SL_2(\mathbb{Z})$ such that the coset space $\Gamma \backslash SL_2(\mathbb{Z})$ is finite. However, we retain the condition of full modular invariance (3.4.22) for the elliptic genus $E_{\mathcal{H}}$ (3.4.20). So it is perhaps surprising that there are examples of quasi potential bulk superconformal field theories that are not potential bulk superconformal field theories. We discuss one such example in detail, and further motivate the notion, in §3.5.4.

We may also be interested in the case that both V' and V'' contain the small $N = 4$ superconformal algebra at some central charge c . In this situation we call \mathcal{H} a *potential bulk $N = (4, 4)$ superconformal field theory* if (3.4.21), (3.4.22) and (3.4.23) hold, and if in addition $c = 6k$ for k a positive integer, and spectral flow for each of V' and V'' is realized by fusion with the order two simple current of $L_k(\mathfrak{sl}_2)$ (as discussed in §3.2.3). For the notion of *quasi potential bulk $N = (4, 4)$ superconformal field theory* we relax the requirement of modularity (3.4.21) of $Z_{\mathcal{H}}$ from the modular group to some finite index subgroup.

In order to present a counterpart to Proposition 3.4.5 we now consider a self-dual C_2 -cofinite vertex operator superalgebra W with a decomposition as in (3.4.16) such that V' and V'' contain copies of the vacuum module for the $N = 2$ superconformal algebra at some central charge $c = c' = c''$. We then define $Z_W(u, \tau)$ in analogy with (3.4.18), setting

$$Z_W(u, \tau) := \begin{pmatrix} Z^+(u, \tau) \\ Z^-(u, \tau) \\ Z_{\text{tw}}^+(u, \tau) \\ Z_{\text{tw}}^-(u, \tau) \end{pmatrix} \quad (3.4.24)$$

where $Z^{\pm}(u, \tau) := \widehat{\text{ch}}^{\pm}[W](u, \tau, -\bar{u}, -\bar{\tau})$ and $Z_{\text{tw}}^{\pm}(u, \tau) := \widehat{\text{ch}}^{\pm}[W_{\text{tw}}^{\text{even}}](u, \tau, -\bar{u}, -\bar{\tau})$, and $\widehat{\text{ch}}^{\pm}[\cdot]$ is as in (3.4.19). The proof of the next result is similar to the proofs of Propositions 3.4.2 and 3.4.5, but note the restriction that $c' = c''$.

Proposition 3.4.7. *Suppose that W is as in (3.4.16), and V' and V'' contain the vacuum module for the $N = 2$ superconformal algebra at some central charge $c = c' = c''$. If $L'_0 - L''_0$ acts on W^{even} and W_{tw} with eigenvalues in \mathbb{Z} , and on W^{odd} with eigenvalues in $\mathbb{Z} + \frac{1}{2}$, and if the S -matrices of V' and*

V'' are both real, then Z_W satisfies the modularity and spectral flow symmetry conditions, (3.4.21) and (3.4.23).

For simplicity we restrict to the case that $c' = c''$ in Proposition 3.4.7, but interesting examples with $c' \neq c''$ may also exist.

Define the *elliptic genus* of W by setting

$$E_W(u, \tau) := \widehat{\text{ch}}^- [W_{\text{tw}}](u, \tau, 0, -\bar{\tau}). \quad (3.4.25)$$

We will presently see examples W for which Proposition 3.4.7 fails but spectral flow symmetry holds and E_W is a weak Jacobi form of weight 0 and some index.

3.5 Examples

3.5.1 Type D Conformal Field Theory

Let n be a positive integer. The bulk of the diagonal conformal field theory with $V' = V'' = V_{D_{2n}}$ is

$$\mathcal{H} = \bigoplus_{i=0}^3 V_{D_{2n}+[i]} \otimes V_{D_{2n}+[i]}. \quad (3.5.1)$$

Observe that we may embed $D_{2n} \oplus D_{2n}$ in D_{4n} by taking the first copy of D_{2n} to be the vectors in D_{4n} supported on the first $2n$ components, and letting the second copy be its orthogonal complement. Then for the cosets of D_{4n} in its dual we have

$$\begin{aligned} D_{4n} + [0] &= (D_{2n} + [0], D_{2n} + [0]) \cup (D_{2n} + [2], D_{2n} + [2]), \\ D_{4n} + [1] &= (D_{2n} + [1], D_{2n} + [1]) \cup (D_{2n} + [3], D_{2n} + [3]), \\ D_{4n} + [2] &= (D_{2n} + [0], D_{2n} + [2]) \cup (D_{2n} + [2], D_{2n} + [0]), \\ D_{4n} + [3] &= (D_{2n} + [1], D_{2n} + [3]) \cup (D_{2n} + [3], D_{2n} + [1]). \end{aligned} \quad (3.5.2)$$

From this we immediately obtain the following result.

Proposition 3.5.1. *For n a positive integer, the bulk of the diagonal $V_{D_{2n}}$ conformal field theory is isomorphic to $V_{D_{4n}^+}$ as a $V_{D_{2n}} \otimes V_{D_{2n}}$ -module.*

Note that $D_4^+ \simeq \mathbb{Z}^4$ and $D_8^+ \simeq E_8$. So Proposition 3.5.1 furnishes bulk conformal field theory interpretations for the self-dual vertex operator superalgebras $F(8)$, V_{E_8} and $V_{D_{12}^+}$ appearing in Theorem 3.3.1.

An embedding of A_1^{2n} in D_{2n} is discussed in §3.2.1. We may use this to formulate a counterpart to Proposition 3.5.1 for tensor powers of $L_1(\mathfrak{sl}_2)$. For example, if \mathcal{H} denotes the bulk of the diagonal conformal field theory of $L_1(\mathfrak{sl}_2)^{\otimes 2n}$ then by virtue of the isomorphism $V_{A_1} \cong L_1(\mathfrak{sl}_2)$ we have

$$\begin{aligned} \mathcal{H} &= \bigoplus_{C \in \mathbb{F}_2^{2n}} V_{A_1^{2n}+C} \otimes V_{A_1^{2n}+C} \\ &\cong \bigoplus_{C \in \mathcal{Z}_{4n}} V_{A_1^{4n}+C} \end{aligned} \quad (3.5.3)$$

where

$$\mathcal{Z}_{2m} := \{C = (c_1, \dots, c_{2m}) \in \mathbb{F}_2^{2m} \mid c_i = c_{m+i} \text{ for } 1 \leq i \leq m\}. \quad (3.5.4)$$

Observe that $\mathcal{Z}_{2m} = \mathcal{D}_{2m} \cup \mathcal{D}_{2m} + [2]$ in the notation of (3.2.4). By applying Lemma 3.2.1 and noting that $\mathcal{D}_{2m} \cup \mathcal{D}_{2m} + [2] \cong \mathbb{Z}^{2m}$ we obtain the following result.

Proposition 3.5.2. *For n a positive integer, the bulk of the diagonal $L_1(\mathfrak{sl}_2)^{\otimes 2n}$ conformal field theory is isomorphic to $F(8n)$ as a $L_1(\mathfrak{sl}_2)^{\otimes 2n} \otimes L_1(\mathfrak{sl}_2)^{\otimes 2n}$ -module.*

Proposition 3.5.2 furnishes bulk conformal field theory interpretations for the vertex operator superalgebras $F(8)$, $F(16)$ and $F(24)$ in Theorem 3.3.1.

It is instructive to consider the analogue of Proposition 3.5.2 where $L_1(\mathfrak{sl}_2)^{\otimes 2n}$ is replaced by a simple current extension. Write (1^{2n}) as a shorthand for the “all ones” vector $(1, 1, \dots, 1) \in \mathbb{F}_2^{2n}$. We will consider $V' \cong V'' \cong V_L$ where $L = A_1^{2n} \cup A_1^{2n} + (1^{2n})$. Observe that the irreducible V_L -modules are the $V_{L+C} = V_{A_1^{2n}+C} \oplus V_{A_1^{2n}+(1^{2n})+C}$ for $C \in \mathbb{F}_2^{2n}$ with $\text{wt}(C) = 0 \pmod{2}$. For simplicity assume that n is even, so that L is an even lattice and V_L is a vertex operator algebra. Then for the bulk of the diagonal conformal field theory for V_L we have

$$\begin{aligned} \mathcal{H} &= \bigoplus_{\substack{C \in \mathbb{F}_2^{2n} \\ \text{wt}(C) = 0 \pmod{2} \\ c_{2n} = 0}} V_{L+C} \otimes V_{L+C} \\ &= \bigoplus_{\substack{C \in \mathbb{F}_2^{2n} \\ \text{wt}(C) = 0 \pmod{2}}} \left(V_{A_1^{2n}+C} \otimes V_{A_1^{2n}+C} \oplus V_{A_1^{2n}+(1^{2n})+C} \otimes V_{A_1^{2n}+C} \right). \end{aligned} \quad (3.5.5)$$

Comparing with (3.2.4) we see that

$$\mathcal{H} \cong \bigoplus_{C \in \mathcal{D}_{4n}^+} V_{A_1^{4n} + C} \quad (3.5.6)$$

where $\mathcal{D}_{4n}^+ := \mathcal{D}_{4n} \cup \mathcal{D}_{4n} + [1]$. Since $D_{4n}^+ = D_{4n} \cup D_{4n} + [1]$ by definition, an application of Lemma 3.2.1 yields the following alternative interpretation for $V_{D_{4n}^+}$ as a potential bulk conformal field theory, at least for n even.

Proposition 3.5.3. *Let n be an even positive integer and let $L = A_1^{2n} \cup A_1^{2n} + (1^{2n})$. Then the bulk diagonal conformal field theory associated to V_L is isomorphic to $V_{D_{4n}^+}$ as a $V_L \otimes V_L$ -module.*

Proposition 3.5.3 offers a bulk conformal field theory interpretation for V_{E_8} distinct from that of Proposition 3.5.1.

3.5.2 Type D Superconformal Field Theory

We now consider the diagonal superconformal field theory associated to $2n$ free fermions. By the boson-fermion correspondence we have $F(2n) \cong V_{D_n} \oplus V_{D_n+[2]}$, and $F(2n)_{\text{tw}} \cong V_{D_n+[1]} \oplus V_{D_n+[3]}$. So we have

$$\mathcal{H}_{\text{NS-NS}} = F(2n) \otimes F(2n), \quad \mathcal{H}_{\text{R-R}} = F(2n)_{\text{tw}} \otimes F(2n)_{\text{tw}}, \quad (3.5.7)$$

in this case. This gives us a bulk superconformal field theory interpretation for $F(4n)$ for n a positive integer (cf. Proposition 3.5.2).

Proposition 3.5.4. *Let n be a positive integer. Then the NS-NS sector of the bulk diagonal superconformal field theory associated to $2n$ free fermions is isomorphic to $F(4n)$ as a $F(2n) \otimes F(2n)$ -module, and the R-R sector is isomorphic to $F(4n)_{\text{tw}}$ as a canonically twisted $F(2n) \otimes F(2n)$ -module.*

Next we consider the diagonal superconformal field theory associated to the D_{2n} torus. In this case $V' = V'' = V_{D_{2n}} \otimes F(2n)$, so

$$\mathcal{H}_{\text{NS-NS}} = \bigoplus_i (V_{D_{2n}+[i]} \otimes F(2n)) \otimes (V_{D_{2n}+[i]} \otimes F(2n)) \quad (3.5.8)$$

as modules for $V' \otimes V''$, and

$$\mathcal{H}_{\text{R-R}} = \bigoplus_i (V_{D_{2n}+[i]} \otimes F(2n)_{\text{tw}}) \otimes (V_{D_{2n}+[i]} \otimes F(2n)_{\text{tw}}) \quad (3.5.9)$$

as canonically twisted modules for $V' \otimes V''$. Comparing this description with the decomposition of $D_{4n}^+ = D_{4n} \cup D_{4n} + [1]$ into cosets for $D_{2n} \oplus D_{2n}$ (cf. (3.5.2)) we obtain the following identification.

Proposition 3.5.5. *Let n be a positive integer and set $V = V_{D_{2n}} \otimes F(2n)$. Then the NS-NS sector of the bulk diagonal superconformal field theory associated to the D_{2n} torus is isomorphic to $V_{D_{4n}^+} \otimes F(4n)$ as a $V \otimes V$ -module, and the R-R sector is isomorphic to $V_{D_{4n}^+} \otimes F(4n)_{\text{tw}}$ as a canonically twisted $V \otimes V$ -module.*

We also obtain our first example of a potential bulk superconformal field theory with superconformal structure.

Theorem 3.5.6. *Let n be a positive integer. Then the vertex operator superalgebra $W = V_{D_{4n}^+} \otimes F(4n)$ is a potential bulk $N = (2, 2)$ superconformal field theory in the sense of §3.4, for $V' \cong V'' \cong V_{D_{2n}} \otimes F(2n)$, and the elliptic genus defined by this structure vanishes.*

Proof. The modularity (3.4.21) and spectral flow symmetry (3.4.23) follow from Proposition 3.4.7. We may compute directly that the elliptic genus (3.4.25) vanishes, so it trivially satisfies (3.4.22). \square

Taking $n = 2$ in Theorem 3.5.6 we obtain an interpretation for $V_{E_8} \otimes F(8)$ as the bulk superconformal field theory of the sigma model with D_4 torus as target. Volpato has observed [Vol14] that supersymmetry preserving symmetry groups of sigma models with arbitrary 4-dimensional torus as target space embed in the Weyl group of E_8 . The Weyl group of E_8 acts naturally on $V_{E_8} \otimes F(8)$. It appears that $V_{E_8} \otimes F(8)$ can play the analogous role for sigma models on 4-dimensional tori that $V_{D_{12}^+}$ has been shown [DM16] to play for sigma models on K3 surfaces.

3.5.3 Type A Superconformal Field Theory

In principle we may consider superconformal field theories with $V' \cong V'' \cong V_L$ for $L = A_1^{2n} \cup A_1^{2n} + (1^{2n})$ also for n odd (cf. Proposition 3.5.3). However, as we have explained in §3.2.4, the case $n = 3$ is distinguished by the presence of $N = 4$ superconformal structure. This fact underpins a significant part of the analysis of [GTVW14], in which it is shown that the diagonal superconformal field theory with $V' \cong V'' \cong V_L$ underlies a Kummer type K3 sigma model arising from the canonical $\mathbb{Z}/2\mathbb{Z}$ -orbifold of the D_4 torus. This is the *tetrahedral* K3 sigma model in [TW13]. We discuss this sigma model from the point of view of A_1^6 now. Although the motivation of [GTVW14] is somewhat different, their detailed analysis precedes, and may serve to flesh out the discussion we offer here.

So let $L = A_1^6 \cup A_1^6 + (1^6)$ in this section. Directly applying the observations preceding Proposition 3.5.3 with $n = 3$ we see that the NS-NS sector of the diagonal superconformal field theory associated to V_L satisfies $\mathcal{H}_{\text{NS-NS}} \cong V_{D_{12}} \oplus V_{D_{12}+[1]} = V_{D_{12}^+}$ as a $V_L \otimes V_L$ -module. Noting that the irreducible canonically twisted V_L -modules are the V_{L+C} with $C \in \mathbb{F}_2^6$ and $\text{wt}(C) = 1 \pmod{2}$ we find that $\mathcal{H}_{\text{R-R}} \cong V_{D_{12}+[2]} \oplus V_{D_{12}+[3]}$ as a canonically twisted $V_L \otimes V_L$ -module. The diagonal superconformal field theory for V_L underlies the tetrahedral K3 sigma model according to [GTVW14], so we have the following super-analogue of Proposition 3.5.3 for $n = 3$.

Proposition 3.5.7. *For $L = A_1^6 \cup A_1^6 + (1^6)$, the NS-NS sector of the tetrahedral K3 sigma model is isomorphic to $V_{D_{12}^+}$ as a $V_L \otimes V_L$ -module, and the R-R sector of the tetrahedral K3 sigma model is isomorphic to $V_{D_{12}+[2]} \oplus V_{D_{12}+[3]}$ as a canonically-twisted $V_L \otimes V_L$ -module.*

Now let us consider superconformal structure. As explained in §3.2.4 the vertex operator superalgebra $V_L \cong V_{A_1^6} \oplus V_{A_1^6+(1^6)}$ contains a copy of the vacuum module for the small $N = 4$ superconformal algebra at $c = 6$. As in §3.2.4 we choose the first copy of $L_1(\mathfrak{sl}_2)$ in $L_1(\mathfrak{sl}_2)^{\otimes 6} < V_L$ to generate the affine \mathfrak{sl}_2 subalgebra. Then spectral flow corresponds to fusion with $V_{L+C} \otimes V_{L+C}$ where $C = (10^5)$. In terms of $V_{D_{12}^+}$ this is the same as fusion with $V_{D_{12}+[2]}$ by force

of (3.2.4). Thus spectral flow interchanges the NS-NS and R-R sectors. An explicit calculation, such as is carried out in §D.3 of [GTVW14], verifies that $E_W(u, \tau)$ is a weak Jacobi form of weight 0 and index 1 such that $E_W(0, \tau) = 24$. Thus we have the following.

Theorem 3.5.8. *The vertex operator superalgebra $V_{D_{12}^+}$ is a potential bulk $N = (4, 4)$ superconformal field theory in the sense of §3.4, for $V' \cong V'' \cong V_L$, and the elliptic genus defined by this structure is the K3 elliptic genus.*

Note that $V_{D_{12}^+}$ serves as the moonshine module for Conway’s group [Dun07, DM15], and is precisely the vertex operator superalgebra that is used to attach weak Jacobi forms with level to supersymmetry preserving symmetries of K3 sigma models in [DM16]. Results equivalent to Proposition 3.5.7 and Theorem 3.5.8 have been obtained independently in [TW17].

3.5.4 Gepner Type Superconformal Field Theory

We now present a superconformal field theory interpretation for $V_{D_{12}^+}$ of a different nature. As mentioned in §3.2.1, the lattice vertex operator superalgebra $V_{\sqrt{3}\mathbb{Z}}$ realizes the vacuum module of the $N = 2$ superconformal algebra at $c = 1$. In this section we set $K = \sqrt{3}\mathbb{Z}^6$. Thus V_K contains the vacuum module of the $N = 2$ superconformal algebra at $c = 6$. We will show that $W = V_{D_{12}^+}$ is a quasi potential bulk $N = (2, 2)$ superconformal field theory (cf. Definition 3.4.6)—but not a potential bulk $N = (2, 2)$ superconformal field theory—for $V' \cong V'' \cong V_K$. It will develop that W is closely related to the Gepner model of type $(1)^6$, which is a superconformal field theory that also has $V' \cong V'' \cong V_K$.

Recall from §3.2.1 that the lattice $\sqrt{3}\mathbb{Z}^{12} \cong K \oplus K$ embeds in D_{12}^+ . Using such an embedding we may fix commuting sub vertex operator superalgebras $V', V'' < W$ such that $V' \cong V'' \cong V_K$. Since the discriminant group of $K \oplus K$ is \mathbb{F}_3^{12} it is natural to use ternary codewords of length 12 to describe the irreducible $V_K \otimes V_K$ -modules. According to Lemma 3.2.2 the $V_K \otimes V_K$ -modules that appear in W are indexed by the codewords in a copy \mathcal{G}_{12}^+ of the ternary Golay code. To make this more concrete let us assume that $(+^6 -^6)$ is a word

in \mathcal{G}_{12}^+ (if not then permute the coordinates so as to make this true), and define $C', C'' \in \mathbb{F}_3^6$ for $C \in \mathcal{G}_{12}^+$ by setting $C' = (c_1, \dots, c_6)$ and $C'' = (c_7, \dots, c_{12})$ when $C = (c_1, \dots, c_{12})$. Then we have

$$W = \bigoplus_{C \in \mathcal{G}_{12}^+} V_{K+C'} \otimes V_{K+C''} \quad (3.5.10)$$

as modules for $V_K \otimes V_K$. Our main result in this section is the following.

Theorem 3.5.9. *The vertex operator superalgebra $V_{D_{12}^+}$ is a quasi potential bulk $N = (2, 2)$ superconformal field theory in the sense of §3.4, for $V' \cong V'' \cong V_K$, and the elliptic genus defined by this structure is the K3 elliptic genus.*

Proof. Let w be the marked complete weight enumerator of \mathcal{G}_{12}^+ for the marking $C \mapsto (C', C'')$. That is, define w to be the 6-variate polynomial

$$w(X', Y', Z', X'', Y'', Z'') := \sum_{C \in \mathcal{G}_{12}^+} w_{C'}(X', Y', Z') w_{C''}(X'', Y'', Z'') \quad (3.5.11)$$

where $w_C(X, Y, Z)$, for $C \in \mathbb{F}_3^n$, is defined by $w_C(X, Y, Z) := X^{a_0} Y^{a_+} Z^{a_-}$ in case C has a_0 entries equal to 0, and a_{\pm} entries equal to ± 1 . Then the functions $\widehat{\text{ch}}^{\pm}[W](u', \tau', u'', \tau'')$ and $\widehat{\text{ch}}^{\pm}[W_{\text{tw}}](u', \tau', u'', \tau'')$ (cf. (3.4.19)) are obtained by substituting characters of suitable irreducible modules for the $N = 2$ superconformal algebra at $c = 1$. These characters can be expressed in terms of classical theta functions and the Dedekind eta function (cf. [RY87]) so they are invariant for the action of some finite index subgroup of $SL_2(\mathbb{Z})$. So Z_W (cf. (3.4.24)) is invariant for some finite index subgroup of $SL_2(\mathbb{Z})$. Also, the $N = 2$ characters satisfy spectral flow symmetry, so the same is true for W .

It remains to examine the elliptic genus E_W (cf. (3.4.25)) of W . For this define

$$f_s(u, \tau) := \eta(\tau)^{-1} \sum_{k \in \mathbb{Z}} (e^{\pi i z})^{k + \frac{s}{6}} q^{\frac{3}{2}(k + \frac{s}{6})^2}, \quad (3.5.12)$$

which is a character for the $N = 2$ superconformal algebra at $c = 1$ when $s \in \mathbb{Z}$. Note that f_s depends only on $s \bmod 6$, we have $f_s(0, \tau) = e^{\pm \frac{\pi i}{6}}$ when

$s = \pm 1 \pmod 6$, and $f_s(0, \tau)$ vanishes identically when $s = 3 \pmod 6$. Because of this we have

$$E_W(u, \tau) = w(f_3(u, \tau), f_1(u, \tau), f_{-1}(u, \tau), 0, e^{\frac{\pi i}{6}}, e^{-\frac{\pi i}{6}}). \quad (3.5.13)$$

Under our hypotheses on \mathcal{G}_{12}^+ the marked complete weight enumerator is given by

$$\begin{aligned} w(X', Y', Z', X'', Y'', Z'') &= (X'^6 + Y'^6 + Z'^6)(X''^6 + Y''^6 + Z''^6) \\ &\quad + 90(X'^4 Y' Z' + X' Y'^4 Z' + X' Y' Z'^4) X''^2 Y''^2 Z''^2 \\ &\quad + 20(X'^3 Y'^3 + X'^3 Z'^3 + Y'^3 Z'^3)(X''^3 Y''^3 + X''^3 Z''^3 + Y''^3 Z''^3) \\ &\quad + 90 X'^2 Y'^2 Z'^2 (X''^4 Y'' Z'' + X'' Y''^4 Z'' + X'' Y'' Z''^4). \end{aligned} \quad (3.5.14)$$

So we have

$$E_W = -2(f_3^6 + f_1^6 + f_{-1}^6) + 20(f_1^3 f_{-1}^3 + f_3^3 f_1^3 + f_{-1}^3 f_3^3). \quad (3.5.15)$$

We may check directly using (3.5.12) that this expression for E_W is a weak Jacobi form of weight 0 and index 1. Substituting $u = 0$ in (3.5.15) we obtain $E_W(0, \tau) = -2((-1) + (-1)) + 20(1) = 24$. So E_W is indeed the K3 elliptic genus. This completes the proof. \square

One can check directly using (3.5.14) that Z_W is not invariant for the full modular group. So the decomposition (3.5.10) does not make $W = V_{D_{12}^+}$ a potential bulk superconformal field theory. Nonetheless, it is closely related to such an object, and such relationships motivate the notion. To explain this we consider the superconformal field theory underlying the Gepner model of type (1)⁶, which realizes a K3 sigma model and also has $V' \cong V'' \cong V_K$. To describe the bulk define a ternary code \mathcal{U} of length 6 by setting

$$\mathcal{U} := \left\{ C = (c_1, \dots, c_6) \in \mathbb{F}_3^6 \mid \sum_i c_i = 0 \right\}. \quad (3.5.16)$$

Then, according to [GHV12] for example, the NS-NS sector is $\mathcal{H}_{\text{NS-NS}} = \bigoplus_{a \in \mathbb{F}_3} \mathcal{H}_{\text{NS-NS}}^a$ where

$$\mathcal{H}_{\text{NS-NS}}^a := \bigoplus_{C \in \mathcal{U}} V_{K+C+(a^6)} \otimes V_{K+C-(a^6)} \quad (3.5.17)$$

as a module for $V_K \otimes V_K$. Since there are $V_K \otimes V_K$ -modules in (3.5.17) whose corresponding codewords in \mathbb{F}_3^{12} have fewer than 6 non-zero entries, the $V_K \otimes V_K$ -module structure on W in Theorem 3.5.9 does not identify it with the superconformal field theory underlying the Gepner model (1)⁶.

However, W and the (1)⁶ model have closely related symmetry. To see this let $G^{N=2}$ be the group of symmetries of the bulk superconformal field theory of the (1)⁶ model that fix the states of the left and right moving $N = 2$ superconformal algebras at $c = 6$. According to [GHV12] this group is a split extension of S_6 by $(\mathbb{Z}/3\mathbb{Z})^6$. The subgroup of the automorphism group of \mathcal{G}_{12}^+ that stabilizes the splitting $C \mapsto (C', C'')$ (cf. (3.5.10)) is another copy of S_6 . From this it follows that $G^{N=2}$ also acts on W , preserving the states of the two commuting $N = 2$ superconformal algebras at $c = 6$ in $V_K \otimes V_K$.

Now let $G^{N=4}$ be the subgroup of $G^{N=2}$ that preserves the left and right moving copies of the small $N = 4$ superconformal algebra. Given $g \in G^{N=4}$ define the corresponding *equivariant elliptic genus* of the (1)⁶ model by setting

$$E_{\mathcal{H},g}(u, \tau) := \text{tr}_{\mathcal{H}_{\mathbb{R}-\mathbb{R}}} \left((-1)^F g z^{J'_0} q^{L'_0 - \frac{c'}{24}} \bar{q}^{L''_0 - \frac{c''}{24}} \right), \quad (3.5.18)$$

and make the analogous definition for W . Then, according to the results of [DM16], the actions of $G^{N=4}$ on W and the bulk of the (1)⁶ model give the same equivariant elliptic genera, $E_{W,g} = E_{\mathcal{H},g}$ for $g \in G^{N=4}$.

This indicates that quasi potential bulk superconformal field theories have a role to play in further elucidating the main question of this article, on the extent to which self-dual vertex operator superalgebras model bulk superconformal field theories. More specifically, we see an example whereby a self-dual vertex operator superalgebra is not identified with the bulk of a superconformal field theory, but does retain important information about it, such as its symmetry group, and the corresponding equivariant elliptic genera.

It is natural to ask what other invariants of (1)⁶ can be computed using W , and what other examples there are, of bulk superconformal field theories whose invariants are computed by quasi potential bulk superconformal field theories that are identified with self-dual vertex operator superalgebras. There are a number of $c = 6$ combinations of minimal $N = 2$ models V such that

$V \otimes V$ embeds in $V_{D_{12}^+}$. Do these embeddings produce further examples of quasi potential superconformal field theories with close relationships to corresponding Gepner models? We offer the further investigation of these questions as problems for future work.

Chapter 4

Invariant subalgebras of the small $\mathcal{N} = 4$ superconformal algebra

4.1 Introduction

The small $\mathcal{N} = 4$ superconformal algebra $V^k(\mathfrak{n}_4)$ at level k is a highly interesting family of vertex operator superalgebras. It is defined as the minimal \mathcal{W} -superalgebra of the universal affine vertex superalgebra $V^{-k-1}(\mathfrak{psl}_{2|2})$ of $\mathfrak{psl}_{2|2}$ at level $-k-1$ and its affine subalgebra is the universal affine vertex algebra $V^k(\mathfrak{sl}_2)$ of \mathfrak{sl}_2 at level k . Moreover it has four dimension $3/2$ odd fields, hence the name $\mathcal{N} = 4$ superconformal algebra. This algebra is a key ingredient in various problems of physics, as string theory on $K3$ surfaces [ET88b] and hence Mathieu moonshine [EOT11], the *AdS/CFT* correspondence [Mal99] and as chiral algebras of certain four-dimensional super Yang-Mills theories [BMR19]. In mathematics, it appears at level one as the global sections of the chiral de Rham complex of $K3$ surfaces [Son16] and more generally at level a positive integer n it is believed to be the algebra of global sections of $2n$ complex dimensional hyper Kähler manifolds, see e.g. [Hel09]. $V^k(\mathfrak{n}_4)$ is exceptional in the sense that its group of outer automorphism is $SU(2)$, i.e. not a finite group. It is surely important to gain a better understanding of vertex algebras related to $\mathcal{N} = 4$ superconformal algebras and so in this work we study invariant subalgebras of $V^k(\mathfrak{n}_4)$, while in [CFL19] it is studied how small and also large $\mathcal{N} = 4$ superconformal algebras can themselves be realized

as cosets.

One of our main motivations for this project is the vertex algebra of global sections on Enriques surfaces:

4.1.1 Global sections on Enriques surfaces

It was shown in [MSV99] that there exists a natural way to attach a vertex superalgebra valued sheaf Ω^{CdR} , named the *chiral de Rham* sheaf, to any smooth scheme X of finite type over \mathbb{C} . It comes equipped with a differential d that squares to zero such that there exists a canonical embedding of the de Rham complex which is moreover a quasi-isomorphism $(\Omega^{dR}, d^{dR}) \xrightarrow{\sim} (\Omega^{CdR}, d)$. The global sections contain a Virasoro field and if the first Chern class vanishes this extends to a $\mathcal{N} = 2$ structure. Moreover, if X is hyper-Kähler the global sections contain a $\mathcal{N} = 4$ structure with central charge $c = 3\dim_{\mathbb{C}}(X)$ [Hel09]. Further structure on $H^*(X, \Omega^{CdR})$ such as a chiral version of Poincaré duality are known to exist [MS99b]. It has been shown that this sheaf has connections to elliptic genera [BL00, MS03], mirror symmetry [Bor01], and physics [Kap05].

Unfortunately, specific examples of $H^*(X, \Omega^{CdR})$ are still lacking. Restricting to the vertex subalgebra of global sections, the only example in the literature so far has been given when X is a $K3$ surface and was constructed in [Son16] where it was shown that $H^0(X, \Omega^{CdR})$ is isomorphic to the simple $\mathcal{N} = 4$ vertex algebra $V_1(\mathfrak{n}_4)$, which has central charge $c = 6$.¹ (See also [MS99a] where $H^0(\mathbb{P}^N, \Omega^{CdR})$ was computed as a $\widehat{\mathfrak{sl}}_{N+1}$ -module.) Recently some progress was made in [Son18] where for X a compact Ricci-flat Kähler manifold $H^0(X, \Omega^{CdR})$ was shown to be isomorphic to a subspace of a $\beta\gamma - bc$ -system that is invariant under the action of a certain Lie algebra. A first motivation for this article is to provide a further example to this list by concretely constructing the vertex algebra of global sections of Ω^{CdR} on any complex Enriques surface. The following is shown in Corollary 4.6.5 and Remark 4.7.7

¹See the preceding work [Son15] for the construction restricting to Kummer surfaces.

Corollary 4.1.1. *The vertex algebra of global sections of the chiral de Rham complex on a complex Enriques surface is of type $\mathcal{W}(1, \frac{3^2}{2}, 2, \frac{7^2}{2}, 4^4)$. Its strong generators are explicitly constructed in the main text, and it can be regarded as an extension of $\mathcal{H} \otimes N_{-4}(\mathfrak{sl}_2)$. Here \mathcal{H} denotes the Heisenberg vertex algebra, and $N_{-4}(\mathfrak{sl}_2)$ denotes the parafermion algebra of \mathfrak{sl}_2 at level -4 .*

Any complex Enriques surface X can be constructed as the quotient of a K3 surface by an involution that is free of fixed points. Let ι be such an involution on a K3 surface Y . The action of the involution lifts to an action on the sheaf Ω^{CdR} and to its cohomology via automorphisms on the vertex algebra. A general construction of the chiral de Rham complex on orbifolds was given in [FS07]. For K3 surfaces the automorphism on the vertex algebra induced by ι was already stated in [Son16]. The vertex algebra of global sections on Enriques surfaces is given by the fixed point set under this involution $H^0(X, \Omega^{CdR}) = H^0(Y, \Omega^{CdR})^\iota$ (see Theorem 6.6 in *op. cit.*).

4.1.2 Invariant subalgebras of the small $\mathcal{N} = 4$ superconformal algebra

It is useful to place the problem of describing the $\mathbb{Z}/2\mathbb{Z}$ -orbifold of $V_1(\mathfrak{n}_4)$ in the larger context of orbifolds of $V^k(\mathfrak{n}_4)$ under general reductive automorphisms groups, and cosets of $V^k(\mathfrak{n}_4)$ by general subalgebras. These problems have in fact been considered in [ACKL17] for a general minimal \mathcal{W} -algebra. By Theorem 4.10 of [ACKL17], for any simple \mathfrak{g} and any reductive automorphism group G , the coset $\mathcal{W}^k(\mathfrak{g}, f_{\min})^G$ is strongly finitely generated for generic values of k . Additionally, the coset of $\mathcal{W}^k(\mathfrak{g}, f_{\min})$ by its affine subalgebra is also strongly finitely generated for generic k ; see Theorem 4.12 of [ACKL17]. However, these results are nonconstructive, and it is useful to give explicit minimal strong generating sets in specific cases. In this paper, we will give minimal strong generating sets for $V^k(\mathfrak{n}_4)^{\mathbb{Z}/2\mathbb{Z}}$ and $V^k(\mathfrak{n}_4)^{U(1)}$ for generic values of k , and also determine the set of nongeneric values where our description fails. We also correct the description of $\mathcal{C}^k = \text{Com}(V^k(\mathfrak{sl}_2), V^k(\mathfrak{n}_4))$ that appeared in [ACKL17]; it is in fact of type $\mathcal{W}(2, 3^3, 4, 5^3, 6)$. Furthermore, we show

that \mathcal{C}^k has an additional action of $U(1)$ coming from the outer automorphism group of $V^k(\mathfrak{n}_4)$, and that $(\mathcal{C}^k)^{U(1)}$ is of type $\mathcal{W}(2, 3, 4, 5, 6, 7, 8)$. It arises as a quotient of the universal two-parameter \mathcal{W}_∞ -algebra constructed in [Lin17], and we identify it as a one-parameter VOA with another, seemingly unrelated coset, namely,

$$\text{Com}(V^\ell(\mathfrak{sl}_2), V^{\ell+1}(\mathfrak{sl}_2) \otimes \mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}})),$$

where k and ℓ are related by $k = -\frac{\ell+1}{\ell+2}$. Finally, using this identification, we classify all isomorphisms between the simple quotient $(\mathcal{C}_k)^{U(1)}$ and various other structures such as type A principal \mathcal{W} -algebras, generalized parafermion algebras, and (conjecturally) cosets of type A subregular \mathcal{W} -algebras.

4.1.3 A new level-rank duality

Coincidences between $(\mathcal{C}^k)^{U(1)}$ and principal \mathcal{W} -algebras of type A appear at negative half integer k . On the other hand these principal \mathcal{W} -algebras of type A also appear as cosets by [ACL19] and these cosets at positive integer level enjoy a nice level-rank duality with Grassmannian cosets [OS14]. We found an extension of this picture to negative integral levels and Grassmannian supercosets. Let $L_r(\mathfrak{sl}_m)$ denote the simple quotient of $V^r(\mathfrak{sl}_m)$ and $\mathcal{H}(1)$ the rank one Heisenberg vertex algebra. Our Theorem 4.8.1 is

Theorem 4.1.2. *Let r, n, m be positive integers, then there exist vertex algebra extensions $A^{-n}(\mathfrak{sl}_m)$ and $A^m(\mathfrak{sl}_{r|n})$ of homomorphic images $\tilde{V}^{-n}(\mathfrak{sl}_m)$ and $\tilde{V}^m(\mathfrak{sl}_{r|n})$ of $V^{-n}(\mathfrak{sl}_m)$ and $V^m(\mathfrak{sl}_{r|n})$ such that the level-rank duality*

$$\begin{aligned} \text{Com}\left(V^{-n+r}(\mathfrak{sl}_m), A^{-n}(\mathfrak{sl}_m) \otimes L_r(\mathfrak{sl}_m)\right) &\cong \\ \text{Com}\left(V^{-m}(\mathfrak{sl}_n) \otimes L_m(\mathfrak{sl}_r) \otimes \mathcal{H}(1), A^m(\mathfrak{sl}_{r|n})\right) & \end{aligned}$$

holds.

It is natural to ask if the statement of the Theorem can be improved, i.e. one could ask for a level-rank duality of the form

$$\text{Com}\left(V^{-n+r}(\mathfrak{sl}_m), \tilde{V}^{-n}(\mathfrak{sl}_m) \otimes L_r(\mathfrak{sl}_m)\right) \stackrel{??}{\cong}$$

$$\text{Com}\left(V^{-m}(\mathfrak{sl}_n) \otimes L_m(\mathfrak{sl}_r) \otimes \mathcal{H}(1), \tilde{V}^m(\mathfrak{sl}_{r|n})\right).$$

Answering this question amounts to a better understanding of embeddings of the involved vertex superalgebras in the rank rm bc -system times the rank nm $\beta\gamma$ -system, i.e. improving the results of [LSS15].

4.1.4 Organization

The structure of the paper is as follows, In section 6.1 some necessary background of the theory of vertex algebras is recalled and some notation is fixed. Then we quickly prove in section 4.3 that the group of outer automorphisms of $V^k(\mathfrak{n}_4)$ is $SL(2)$. Sections 6.2 and 4.5 contain the construction of a few vertex algebras, the most important ones being $V^k(\mathfrak{n}_4)^{U(1)}$ and $V^k(\mathfrak{n}_4)^{\mathbb{Z}/2\mathbb{Z}}$. In section 4.6 some structure of the two central orbifolds from the previous sections are discussed and their simple quotients at all but finitely many levels k are determined. in section 4.7 we determine cosets of $V^k(\mathfrak{n}_4)$ by is affine subalgebra at generic and specific levels. Especially we identify its $U(1)$ orbifold with $\text{Com}(V^\ell(\mathfrak{sl}_2), V^{\ell+1}(\mathfrak{sl}_2) \otimes \mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}}))$ and in addition at special levels with Grassmanian cosets and principal \mathcal{W} -algebras of type A at degenerate admissible levels. The last section then discusses the new level-rank duality.

Notation: In this article we denote the $\mathcal{N} = 2$ and the small $\mathcal{N} = 4$ superconformal algebra by \mathfrak{n}_2 and \mathfrak{n}_4 , respectively. The universal affine vertex superalgebras at level ℓ are then denoted by $V^\ell(\mathfrak{n}_2)$ and $V^\ell(\mathfrak{n}_4)$ and their simple quotients by a subscript. Explicit dependence of fields on the formal variable is often dropped for easier readability and the benefit of the reader. The symbol \propto is used to indicate equality up to multiplication by a non-zero constant over the base field which is assumed to be \mathbb{C} throughout the article.

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4.2 Background

4.2.1 Vertex Algebras

We assume that the reader is familiar with the basics of the theory of vertex algebras which has been discussed from multiple angles in the literature, some of which can be found in [Bor86, FB04, Kac98]. A few of the basics will be recalled here in order to fix notation. In doing so we will adopt the viewpoint developed in [LZ95].

Let $V = V^0 \oplus V^1$ be a super vector space over \mathbb{C} . Furthermore, let z, w be formal variables and denote by $QO(V)$ the space of linear maps $V \rightarrow V((z))$. A representation of an element $a \in QO(V)$ may be given by a formal power series

$$a = a(z) = \sum_n a_n z^{-n-1} \in \text{End}(V)[[z, z^{-1}]].$$

We write $a = a_0 + a_1$ for $a_i \in V^i$ where $a_i : V^j \rightarrow V_{i+j}((z))$ for $i = 0, 1$. Moreover, let $|\cdot| : V^i \rightarrow \mathbb{Z}/2\mathbb{Z}$ given by $|a_i| = i$ denote the parity. Depending on the parity elements will be referred to as either even or odd. We call an element $a \in V$ homogeneous if a is even or odd. The space $QO(V)$ possesses a family of non-associative bilinear operations, indexed by $n \in \mathbb{Z}$, that is defined on homogeneous elements $a, b \in V$ as follows

$$a(w) \circ_n b(w) = \text{Res}_z (a(z)b(w) \iota_{|z|>|w|}(z-w)^n) - (-1)^{|a||b|} \text{Res}_z (b(w)a(z) \iota_{|w|>|z|}(z-w)^n).$$

Here, for a rational function $f \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]]$, the notation $\iota_{|z|>|w|}f$ and $\iota_{|w|>|z|}f$ indicates the power series expansion of f in the region $|z| > |w|$, respectively $|w| > |z|$. Throughout the text the expression $(z-w)^n$ is understood to mean its formal power series expansion in the region $|z| > |w|$; the maps $\iota_{|z|>|w|}$ and $\iota_{|w|>|z|}$ will hence be suppressed and thus expressions like $(z-w)^{-2}$ and $(-w+z)^{-2}$ will be understood to mean different formal power series. For elements $a, b \in QO(V)$ the *operator product expansion* (OPE) is an identity of formal power series and reads

$$a(z)b(w) = \sum_{n=0}^{\infty} a(w) \circ_n b(w) (z-w)^{-n-1} + : a(z)b(w) : .$$

Here the abbreviation $: a(z)b(w) := a(z)_-b(w) + (-1)^{|a||b|}b(w)a(z)_+$ is used where $a(z)_- = \sum_{n<0} a_n z^{-n-1}$ and $a(z)_+ = \sum_{n\geq 0} a_n z^{-n-1}$. The expression $: a(z)b(w) :$ is commonly known as the *Wick product* or *normally ordered product*. Observe that it is regular for $z = w$ and that in this case it coincides with $a(w) \circ_{-1} b(w) =: a(w)b(w) :$. It is a convention in notation to suppress this expression in the OPE and indicate equality modulo a normally ordered product by the symbol \sim as in

$$a(z)b(w) \sim \sum_{n=0}^{\infty} a(w) \circ_n b(w) (z-w)^{-n-1}.$$

The n -fold normally ordered product for elements $a_1, \dots, a_n \in QO(V)$ can be defined iteratively by

$$: a_1(z) \cdots a_n(z) :=: a_1(z)b(z) : \quad \text{where} \quad b(z) =: a_2(z) \cdots a_n(z) :.$$

A subspace $\mathcal{S} \subset QO(V)$ that is closed under all operations \circ_n and contains a unit is called a *quantum operator algebra* (QOA). Let $[\cdot, \cdot] : QO(V) \times QO(V) \rightarrow QO(V)$ denote the usual superbracket. Two elements $a, b \in QO(V)$ are said to be *local* if there exists an $N \in \mathbb{N}_0$ such that

$$(z-w)^n [a(z), b(w)] = 0$$

for all $n \geq N$. This is equivalent to the condition $a(w) \circ_n b(w) = 0$ for $n \geq N$ which ensures the OPE $a(z)b(w)$ to be a finite sum. Finally, a *vertex algebra* is a QOA such that all elements are pairwise local. We will call an element of a vertex algebra $a(z) = \sum_n a_n z^{-n-1}$ a *field* and refer to the coefficients a_n as *modes*.

Let \mathcal{V} be a vertex algebra. A set of fields \mathcal{S} is said to *strongly generate* \mathcal{V} if it generates \mathcal{V} under normally ordered product. This means every field of \mathcal{V} can be written as a normally ordered polynomial of the fields in \mathcal{S} and their iterated derivatives. If \mathcal{S} is minimal with this property and the fields \mathcal{W} of \mathcal{S} have conformal weight h_W then one says that \mathcal{V} is of type $W(\{h_W \mid W \in \mathcal{S}\})$.

We will conclude this subsection with a few basic examples of conformal vertex algebras which will make an appearance in the body of this work.

Let $\mathcal{H}(n)$ be the *Heisenberg* vertex algebra of rank n . It is generated by n even fields $h^i(z)$ with $i = 1, \dots, n$ for which the only OPEs with a non-regular part are

$$h^i(z)h^j(w) \sim \frac{\delta_{i,j}}{(z-w)^2}.$$

The Virasoro element is given by $L(z) = \frac{1}{2} \sum_{i=1}^n : h^i(z)h^i(z) :$ with the central charge equal to n and for which all generating fields are primary of weight 1. The automorphism group is isomorphic to the orthogonal group $Aut(\mathcal{H}(n)) \cong O(n)$.

Let $\mathcal{A}(n)$ be the *symplectic fermion* vertex algebra of rank n . It is generated by $2n$ odd fields $e^i(z)$ and $f^i(z)$ with $i = 1, \dots, n$ for which the only OPEs with a non-regular part are

$$e^i(z)f^j(w) \sim \frac{\delta_{i,j}}{(z-w)^2} \quad , \quad f^i(z)e^j(w) \sim -\frac{\delta_{i,j}}{(z-w)^2}.$$

The Virasoro field is given by $L(z) = -\sum_{i=1}^n : e^i(z)f^i(z) :$ with the central charge equal to $-2n$ and for which all generating fields are primary of weight 1. The automorphism group is isomorphic to the symplectic group $Aut(\mathcal{A}(n)) \cong Sp(2n)$.

Let $\mathcal{S}(n)$ denote the vertex algebra commonly referred to as $\beta\gamma$ -*system* of rank n . It is generated by $2n$ even fields $\beta^i(z)$ and $\gamma^i(z)$ with $i = 1, \dots, n$ for which the only OPEs with a non-regular part are

$$\beta^i(z)\gamma^j(w) \sim \frac{\delta_{i,j}}{(z-w)} \quad , \quad \gamma^i(z)\beta^j(w) \sim -\frac{\delta_{i,j}}{(z-w)}.$$

The Virasoro field is given by $L(z) = \frac{1}{2} \sum_{i=1}^n : \beta^i(z)\partial\gamma^i(z) : - : \partial\beta^i(z)\gamma^i(z) :$ with the central charge equal to $-n$ and for which all generating fields are primary of weight $\frac{1}{2}$. The automorphism group is isomorphic to the symplectic group $Aut(\mathcal{S}(n)) \cong Sp(2n)$.

4.2.2 Filtrations

Let \mathcal{V} be a vertex algebra. Suppose that \mathcal{V} has a filtration

$$\mathcal{V}_{(0)} \subset \mathcal{V}_{(1)} \subset \dots \quad \text{where} \quad \mathcal{V} = \bigcup_{n=0}^{\infty} \mathcal{V}_{(n)}$$

such that for any two elements $a \in \mathcal{V}_{(r)}$ and $b \in \mathcal{V}_{(s)}$

$$a \circ_n b \in \mathcal{V}_{(r+s)} \quad \text{for } n \in \mathbb{Z}.$$

We refer to such a filtration as a *weak increasing filtration*. Setting $\mathcal{V}_{(-1)} = \{0\}$, then the associated graded vector space $gr(\mathcal{V}) = \bigoplus_{n=0}^{\infty} \mathcal{V}_{(n)}/\mathcal{V}_{(n-1)}$ can be given a vertex algebra structure that is induced from \mathcal{V} . Moreover, a strongly generating set on \mathcal{V} can be inferred from a strongly generating set on $gr(\mathcal{V})$ (see Lemma 4.4.1 and the comment following it).

The filtration defined above is a generalization of the slightly more restrictive *good increasing filtration* which was introduced in [Li04] and requires for any two elements $a \in \mathcal{V}_{(r)}$ and $b \in \mathcal{V}_{(s)}$ that

$$a \circ_n b \in \begin{cases} \mathcal{V}_{(r+s)} & \text{for } n < 0 \\ \mathcal{V}_{(r+s-1)} & \text{for } n \geq 0. \end{cases}$$

This property ensures that the vertex algebra on the associated graded vector space is abelian. Hence, $gr(\mathcal{V})$ is a graded, associative, (super-)commutative unital ring with a derivation where the multiplication is induced from the Wick product. We refer to such a ring as a ∂ -ring. A ∂ -ring \mathcal{A} is said to be generated by a set $\{a_i \in \mathcal{A} | i \in I\}$ if the fields in the set $\{\partial^n a_i | i \in I, n \geq 0\}$ generate \mathcal{A} as a ring.

4.2.3 Associative G -modules and orbifolds

We will make use of an isomorphism that can be found in [KR96]. For similar work see also [DLM96]. We recall here the result that will be used later on.

Let A be an associative algebra over \mathbb{C} . Furthermore, let G be a group and let $\phi : G \rightarrow Aut(A)$ be a group homomorphism. If a G -module V is simultaneously an A -module that is G -equivariant, i.e. if

$$g(av) = (\phi(g)a)(gv) \quad \text{for } g \in G, a \in A, v \in V,$$

then we call V a (G, A) -module. Let $A_0 \subset A$ be the subalgebra that is invariant under the G -action. Assume that V is a direct sum of at most countably many finite dimensional and irreducible G -modules. Let $M \subset V$ be such a module

and set $V^M = \bigoplus_i M_i$ to be the sum of all G -submodules of V which are isomorphic to M . The action of A_0 and G commute which implies that V^M is an A_0 -module. In particular, the action of A_0 viewed as a map from M given by $m \mapsto am$ for $a \in A_0$ shows that this map is a G -homomorphism. Choosing a 1-dimensional subspace $f \subset M$ fixes unique 1-dimensional subspaces f_i in all other G -modules $M_i \subset V^M$ by Schur's Lemma. Schur's Lemma further implies that acting by any $a \in A_0$ on f then either yields one of these subspaces $f_i \subset M_i$ or zero. Hence, $\bigoplus_i f_i = V^M \subset V^M$ is an A_0 -module. This leads to an isomorphism $V^M \cong M \otimes V^M$ of (G, A_0) -modules thereby showing that

$$V \cong \bigoplus_{M \in \mathcal{S}} M \otimes V^M$$

is a (G, A_0) -module isomorphism where \mathcal{S} is the set of equivalence classes of simple G -modules.

Later on we will make use of this isomorphism in the context when A is the algebra of modes which span the vector space of a vertex algebra and view it as a module over itself.

4.2.4 Associated vertex algebra to \mathfrak{n}_4

In this article we denote the small $\mathcal{N} = 4$ superconformal algebra by \mathfrak{n}_4 and its associated vertex superalgebra at level k by $V^k(\mathfrak{n}_4)$. Note that here the level refers to the level of the affine subalgebra. $V^k(\mathfrak{n}_4)$ is actually the minimal \mathcal{W} -superalgebra of $\mathfrak{psl}(2|2)$ at level $-k - 1$. The operator product expansions

of its associated vertex algebra $V^k(\mathfrak{n}_4)$ are as follows.

$$\begin{aligned}
T(z)T(w) &\sim \frac{\frac{1}{2}c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} \\
T(z)G^{\pm,x}(w) &\sim \frac{\frac{3}{2}G^{\pm,x}(w)}{(z-w)^2} + \frac{\partial_w G^{\pm,x}(w)}{(z-w)} \\
T(z)X(w) &\sim \frac{X(w)}{(z-w)^2} + \frac{\partial_w X(w)}{(z-w)} \quad \text{for } X \in \{J, J^\pm\} \\
J(z)G^{\pm,x}(w) &\sim \pm \frac{G^{\pm,x}(w)}{(z-w)} \\
J^\pm(z)G^{\mp,x}(w) &\sim (-1)^x \frac{G^{\pm,x}(w)}{(z-w)} \\
J(z)J(w) &\sim \frac{2k}{(z-w)^2} \\
J(z)J^\pm(w) &\sim \pm \frac{2J^\pm(w)}{(z-w)} \\
J^\pm(z)J^\mp(w) &\sim \frac{k}{(z-w)^2} \pm \frac{J(w)}{(z-w)} \\
G^{\pm,2}(z)G^{\pm,1}(w) &\sim \frac{2J^\pm(w)}{(z-w)^2} + \frac{\partial_w J^\pm(w)}{(z-w)} \\
G^{\mp,2}(z)G^{\pm,1}(w) &\sim \frac{\frac{1}{3}c}{(z-w)^3} \mp \frac{J(w)}{(z-w)^2} + \frac{T(w) \mp \frac{1}{2}\partial_w J(w)}{(z-w)}
\end{aligned} \tag{4.2.1}$$

From here onwards we adopt the following conventions:

$$\begin{aligned}
Q^+(z) &= G^{-,2}(z) & G^+(z) &= G^{+,2}(z) \\
Q^-(z) &= G^{+,1}(z) & G^-(z) &= G^{-,1}(z)
\end{aligned}$$

4.3 Automorphisms of $V^k(\mathfrak{n}_4)$

For later use, we determine the group of automorphisms G of $V^k(\mathfrak{n}_4)$. First, there is the group of inner automorphisms G_{Inn} obtained by exponentiating the zero modes of the weight 1 fields; this is a copy of SL_2 . Since the affine subalgebra $V^k(\mathfrak{sl}_2)$ has no outer automorphisms, the outer automorphism group G_{Out} of $V^k(\mathfrak{n}_4)$ is just the subgroup of G consisting of automorphisms that fix $V^k(\mathfrak{sl}_2)$ pointwise.

Lemma 4.3.1. *G_{Out} is a normal subgroup of G , and G is the semidirect product $G_{\text{Out}} \rtimes G_{\text{Inn}}$.*

Proof. Clearly any inner automorphism which fixes $V^k(\mathfrak{sl}_2)$ is trivial, so $G_{\text{Inn}} \cap G_{\text{Out}}$ is trivial. Let $\omega \in G$. The restriction of ω to $V^k(\mathfrak{sl}_2)$ is an automorphism of $V^k(\mathfrak{sl}_2)$, which has only inner automorphisms, so there exists $\alpha \in G_{\text{Inn}}$ such that $\omega = \alpha$ on $V^k(\mathfrak{sl}_2)$. Letting $\beta = \alpha^{-1}\omega$ and $\gamma = \omega\alpha^{-1}$, it is easy to see that $\beta, \gamma \in G_{\text{Out}}$ and are the unique elements of G_{Out} such that $\alpha\beta = \omega = \gamma\alpha$. The normality of G_{Out} is obvious from the definition, so the claim follows. \square

By weight considerations, G_{Out} must act linearly on the weight $\frac{3}{2}$ subspace, which is the span of $\{G^\pm, Q^\pm\}$, and since it preserves OPEs between the weight 1 fields and weight $\frac{3}{2}$ fields, it must preserve the two-dimensional spaces $\{G^+, Q^-\}$ and $\{Q^+, G^-\}$. Using the fact that G_{Out} preserves OPEs between the weight $\frac{3}{2}$ fields, it is not difficult to check that $\omega \in G_{\text{Out}}$ must have the form

$$\begin{aligned} \omega(G^+) &= a_0 G^+ + a_1 Q^-, & \omega(Q^-) &= b_0 G^+ + b_1 Q^-, \\ \omega(Q^+) &= a_0 Q^+ - a_1 G^-, & \omega(G^-) &= -b_0 Q^+ + b_1 G^-, \end{aligned} \tag{4.3.1}$$

for constants $a_0, a_1, b_0, b_1 \in \mathbb{C}$, where $a_0 b_1 - a_1 b_0 = 1$. One can then identify G_{Out} with SL_2 via $\omega \mapsto \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \end{pmatrix}$. Moreover, it is easy to verify that G_{Out} commutes with G_{Inn} . We obtain

Theorem 4.3.2. *The automorphism group G of $V^k(\mathfrak{n}_4)$ is isomorphic to $SL_2 \times SL_2$.*

4.4 Construction of the vertex algebras

In this section we construct the orbifolds of $V^k(\mathfrak{n}_4)$ and for this we use ideas from [AL17, ACL17, CL18]

4.4.1 Construction of $V^k(\mathfrak{n}_4)^{U(1)}$.

In order to construct cyclic orbifolds of $V^k(\mathfrak{n}_4)$ we start by determining the $U(1)$ -orbifold of the vertex subalgebra $\mathcal{V} = V^k(\mathfrak{sl}_2)$ that is strongly generated by the fields $\{J^-(z), J(z), J^+(z)\}$. Define a weak increasing filtration

$$\mathcal{V}_{(0)} \subset \mathcal{V}_{(1)} \subset \mathcal{V}_{(2)} \subset \cdots$$

where $\mathcal{V}_{(n)}$ is spanned by all iterated Wick products of the minimal strong generators and their derivatives which contain at most n elements of the set $\{\partial^k J^-(z), \partial^k J^+(z)\}_{k=0}^\infty$. All relevant operator product expansions in (4.2.1) remain unaffected in the associated graded vertex algebra \mathcal{W} except that the image of $J^\pm(z)J^\mp(w)$ is regular for $z = w$. Introducing a good increasing filtration

$$\mathcal{W}_{(0)} \subset \mathcal{W}_{(1)} \subset \mathcal{W}_{(2)} \subset \dots$$

where $\mathcal{W}_{(n)}$ is spanned by all iterated Wick products of the minimal strong generators and their derivatives of length at most n yields an associated graded vertex algebra that is abelian. Moreover, the associated graded vertex algebra is isomorphic to a polynomial ring with an induced derivation. We abuse notation and denote the image of the fields by the same symbol. Suppressing the z -dependence it reads

$$gr(\mathcal{W}) \cong \mathbb{C}[J, \partial J, \partial^2 J, \dots, J^+, \partial J^+, \partial^2 J^+, \dots, J^-, \partial J^-, \partial^2 J^-, \dots].$$

For any non-trivial equivariant $U(1)$ -action the vertex algebra $\mathcal{V}^{U(1)}$ is spanned by elements of the form

$$: \partial^{i_0} J \dots \partial^{i_r} J \partial^{j_0} J^+ \dots \partial^{j_s} J^+ \partial^{k_0} J^- \dots \partial^{k_s} J^- :$$

such that $i_0 \geq \dots \geq i_r$, $j_0 \geq \dots \geq j_s$ and $k_0 \geq \dots \geq k_s$ with $r, s \geq 0$. Moreover, the set of these monomials is a basis because \mathcal{V} is freely generated. In what follows it will be convenient to define the fields $U_{i,j}(z) =: \partial^i J^+(z) \partial^j J^-(z) :$.

In [LL07] a functor from a certain category of vertex algebras \mathfrak{R} to the category of supercommutative rings was constructed. Assuming that \mathcal{V} possesses a good increasing filtration its key property is captured by the following Lemma.

Lemma 4.4.1. [LL07] *Let \mathcal{V} be a vertex algebra in \mathfrak{R} . Suppose that $gr(\mathcal{V})$ is generated as a ∂ -ring by a collection $\{a_i | i \in I\}$, where a_i is homogeneous of degree d_i . Choose vertex operators $a_i(z) \in \mathcal{V}_{(d)}$ such that $\phi_{d_i}(a_i(z)) = a_i$. Then \mathcal{V} is strongly generated by the collection $\{a_i(z) | i \in I\}$.*

This reconstruction property was shown to hold more generally for weak increasing filtrations (see Lemma 4.1 in [ACL17]).

Lemma 4.4.2. $\mathcal{V}^{U(1)}$ is strongly generated by the fields $\{J(z), U_{n,0}(z)\}_{n=0}^{\infty}$.

Proof. The weak increasing filtration restricts to the fixed point set

$$\mathcal{V}_{(0)}^{U(1)} \subset \mathcal{V}_{(1)}^{U(1)} \subset \mathcal{V}_{(2)}^{U(1)} \subset \dots$$

where $\mathcal{V}_{(i)}^{U(1)} = \mathcal{V}^{U(1)} \cap \mathcal{V}_{(i)}$. The group action descends to the associated graded object and we have an isomorphism $gr(\mathcal{V}^{U(1)}) \cong \mathcal{W}^{U(1)}$. Likewise, we have $gr(\mathcal{W}^{U(1)}) \cong gr(\mathcal{W})^{U(1)}$. For the latter the generating set of fields as a ∂ -ring is given by $\{J, u_{a,b}\}_{a,b=0}^{\infty}$ where $u_{a,b} = \partial^a J^+ \partial^b J^-$. The vertex algebra $\mathcal{W}^{U(1)}$ is an object in \mathfrak{A} . By the previous Lemma it is strongly generated by $\{J(z), U_{a,b}(z)\}_{a,b=0}^{\infty}$ where $U_{a,b}(z) =: \partial^a J^+(z) \partial^b J^-(z) \therefore$. Since the Lemma also holds for weak increasing filtrations and the map gr is equivariant with respect to the group action up to isomorphism we can repeat this argument to obtain a strongly generating set of $\mathcal{V}^{U(1)}$. Therefore, and by making use of the Leibniz rule, $\mathcal{V}^{U(1)}$ is strongly generated by $\{J(z), U_{a,0}(z)\}_{a=0}^{\infty}$ \square

Lemma 4.4.2 can be improved upon. The proof made use of the vertex algebra $gr(\mathcal{W})^{U(1)}$ which is strongly generated by $\{J, u_{a,b}\}_{a,b=0}^{\infty}$. Note that due to commutativity there exist identities between these generators

$$u_{a,b}u_{c,d} = u_{a,d}u_{c,b}.$$

We will show that a preimage of $u_{a,b}u_{c,d} - u_{a,d}u_{c,b}$ in $\mathcal{V}^{U(1)}$ does not vanish, thereby showing the existence of relations between generators. From here onwards any z -dependence of the fields will be suppressed for easier readability.

Considering conformal weight and Lemma 4.4.2 it follows that

$$: U_{a,b}U_{c,d} - U_{a,d}U_{c,b} := c_N U_{N,0} + f(J, U_{0,0}, \dots, U_{N-1,0})$$

for some scalar c_N where $a + b + c + d + 2 = N$ and f is a sum of normally ordered products in the strong generators of conformal weight less than $N + 2$

and their derivatives. Invoking the Leibniz rule yields the identity $U_{a,b} = \sum_{i=0}^b (-1)^i \binom{b}{i} \partial^{b-i} U_{a+i,0}$ which implies that the last equality can be rewritten

$$c_N U_{N,0} = g(J, U_{0,0}, \dots, U_{N-1,0})$$

where g is again a normally ordered polynomial in the strong generators of conformal weight less than $N+2$ and their derivatives. This shows that the field $U_{N,0}$ can be written as a normally ordered polynomial in strong generators of lower conformal weight, provided that c_N does not vanish. Following [ACL17] we will refer to such an expression as a *decoupling relation*. What follows are two technical Lemmas which will serve as a preparation for the proof of Proposition 4.4.6.

For some $n \in \mathbb{N}$ we may write a given element $\omega \in \mathcal{V}_{(2)}^{U(1)}$ of conformal weight $n+2$ as a sum of normally ordered products $g_\omega(J, U_{0,0}, \dots, U_{n,0})$ in the strong generators of conformal weight at most $n+2$ and their derivatives. Such a normally ordered product is not unique due to the existence of decoupling relations and different conventions for normal ordering. Let the coefficient of $:\partial^{n-i} J^+ \partial^i J^-:$ in g_ω be denoted by $c_{n,i}(\omega)$ and define

$$c_n(\omega) = \sum_{i=0}^n (-1)^i c_{n,i}(\omega).$$

Lemma 4.4.3. *For any $\omega \in \mathcal{V}_{(2)}^{U(1)}$ of weight $n+5$ for $n \in \mathbb{N}$ the coefficient of $U_{n+3,0}$ appearing in g_ω is independent of all choices of normal orderings and is equal to $c_n(\omega)$.*

Proof. The proof is analogous to the proof of Lemma 5.2 in [ACL17]. \square

Remark 4.4.4. Note that we use strong generators of the form $U_{n,0}$ (*cf. loc. cit.*). We may simply rewrite these by using the Leibniz rule as above with their difference being a total derivative. The factor $(-1)^n$ does not appear in our case because of the different choice in strong generators.

Lemma 4.4.5. *Let $n \in \mathbb{N}_0$ and let P_m denote a sum of normally ordered products of strong generators of $\mathcal{V}^{U(1)}$ of weight less than m and their derivatives.*

$$:U_{0,0}U_{1,n} := \left(\frac{2}{n+2} + \frac{k}{n+2} \right) U_{1,n+2} - \frac{1}{n+1} U_{2,n+1} +$$

$$\begin{aligned}
& + \left(1 + \frac{k}{3}\right) U_{3,n} + \frac{2}{(n+1)(n+2)(n+3)} U_{n+3,0} + P_{n+5} \\
: U_{0,n} U_{1,0} := & \left(\frac{2}{n+2} + \frac{k}{2}\right) U_{1,n+2} - U_{2,n+1} + \\
& + \left\{ \frac{2}{(n+1)(n+2)(n+3)} + (-1)^n \left(\frac{1}{n+1} + \frac{k}{n+3}\right) \right\} U_{n+3,0} + P_{n+5}
\end{aligned}$$

Proof. The proof is a straightforward computation using only the definition of normal ordering and the commutation relations of $\widehat{\mathfrak{sl}}_2$ at level k . \square

Proposition 4.4.6. *For any non-vanishing level k the vertex algebra $V^k(\mathfrak{sl}_2)^{U(1)}$ is of type $\mathcal{W}(1, 2, 3, 4, 5)$. A set of minimal strong generators is $\{J, U_{n,0}\}_{n=0}^3$.*

Proof. Since all strong generators of $\mathcal{V}^{U(1)}$ have a different conformal weight by Lemma 4.4.2 and the smallest non-trivial identity between the strong generators of $gr(\mathcal{W})^{U(1)}$ is $u_{0,0}u_{1,1} = u_{0,1}u_{1,0}$ we will focus on a one-parameter family of decoupling relations that involve the expression

$$\omega_n \stackrel{\text{def}}{=} U_{0,0}U_{1,n} - U_{0,n}U_{1,0} :$$

for $n \in \mathbb{N}$. Note that the non-regular part of the OPE of $J^\pm(z)J^\mp(w)$ does not contain the field $J^+(z)$ or $J^-(z)$ and therefore $\omega_n \in \mathcal{V}_{(2)}^{U(1)}$. The element ω_n can be written as a sum of normally ordered products in the strong generators of $\mathcal{V}^{U(1)}$ and their derivatives. The coefficient of $U_{n+3,0}$ is canonical in the sense of Lemma 4.4.3. Due to Lemma 4.4.5 it can be easily computed and equals

$$c_{n+3}(\omega_n) = (-1)^{n+1} k \frac{n(n+5)}{6(n+2)(n+3)}.$$

This shows existence of a decoupling relation for $U_{n+4,0}$ for all $n \in \mathbb{N}_0$ at any non-vanishing level k . Hence, $V^k(\mathfrak{sl}_2)^{U(1)}$ is strongly generated by $\{J, U_{n,0}\}_{n=0}^3$ for $k \neq 0$. That there exists no further decoupling relation between the remaining strong generators can be checked directly and implies minimality of the set. \square

Remark 4.4.7. It is easy to find a maximal subset of minimal strong generators which have no non-regular OPEs with the Heisenberg field. By the identity $V^k(\mathfrak{sl}_2)^{U(1)} = \mathcal{H} \otimes Com(\mathcal{H}, V^k(\mathfrak{sl}_2))$ this is another proof that $Com(\mathcal{H}, V^k(\mathfrak{sl}_2))$ is of type $\mathcal{W}(2, 3, 4, 5)$ which was initially proven in [DW10].

The last Proposition can be used to prove a similar statement about $V^k(\mathfrak{n}_2)^{U(1)}$. It will be convenient to introduce the following fields

$$V_{a,b} \stackrel{\text{def}}{=} \partial^a G^+ \partial^b G^- : .$$

Proposition 4.4.8. *For any level $k \neq 0, -2$ the vertex algebra $V^k(\mathfrak{n}_2)^{U(1)}$ is of type $\mathcal{W}(1, 2, 3, 4, 5)$. A set of minimal strong generators is $\{J, T, V_{n,0}\}_{n=0}^2$.*

Proof. Denote the standard strong generators of $V^k(\mathfrak{sl}_2)$ by $\{H, X^\pm\}$ and let \mathcal{E} be the bc -system of rank 1. Considering the tensor product $V^k(\mathfrak{sl}_2) \otimes \mathcal{E}$ we will abuse notation and denote the strong generators by the same symbols. Let $K = \frac{1}{2}H - :bc:$. The zero mode K_0 integrates to a $U(1)$ -action on $V^k(\mathfrak{sl}_2) \otimes \mathcal{E}$. Lemma 8.6 in [CL19] shows that if $k \neq -2$ then

$$V^\ell(\mathfrak{n}_2) \cong \text{Com}(\mathcal{H}_1, V^k(\mathfrak{sl}_2) \otimes \mathcal{E}) \quad (4.4.1)$$

with central charge $\frac{3k}{k+2}$ where \mathcal{H}_1 is the Heisenberg vertex algebra generated by K . The zero mode H_0 integrates to a $U(1)$ -action as well. Denote the group associated to the zero mode of K (H) by G_1 (G_2) and let \mathcal{H}_2 be the Heisenberg vertex algebra generated by H .

$$\begin{aligned} \mathcal{H}_1 \otimes V^\ell(\mathfrak{n}_2)^{G_2} &\cong ((V^k(\mathfrak{sl}_2) \otimes \mathcal{E})^{G_2})^{G_1} \\ &\cong (\mathcal{H}_2 \otimes \text{Com}(\mathcal{H}_2, V^k(\mathfrak{sl}_2) \otimes \mathcal{E}))^{G_1} \\ &\cong \mathcal{H}_2 \otimes \text{Com}(\mathcal{H}_2, V^k(\mathfrak{sl}_2)) \otimes \mathcal{E}^{G_1} \end{aligned}$$

The commutativity of the two group actions was used in the first equality. All G_1 -invariant fields in \mathcal{E} can be strongly generated by the fields $: \partial^i b \partial^j c :$ for $i, j \geq 0$. By the action of the derivation a set of strong generators is given by $\{ : \partial^i bc : \}_{i=0}^\infty$. It is easy to show that the equality

$$: (\partial^n bc) : (: bc :) := \frac{n+2}{n+1} : \partial^{n+1} bc : + \partial \omega$$

holds for $n \geq 0$ where ω is a linear combination of the fields $\partial^{n-i} : \partial^i bc :$ for $i = 0, \dots, n$. This implies that $: bc :$ strongly generates \mathcal{E}^{G_1} . From the above isomorphism one can deduce

$$V^\ell(\mathfrak{n}_2)^{G_2} \cong \text{Com}(\mathcal{H}_1, \mathcal{H}_2 \otimes \text{Com}(\mathcal{H}_2, V^k(\mathfrak{sl}_2)) \otimes \mathcal{E}^{G_1}).$$

It now follows from Proposition 4.4.6 that the only strong generator of weight 1 is an element of the commutant $Com(\mathcal{H}_1, \mathcal{H}_2 \otimes \mathcal{E}^{G_1})$ which is isomorphic to a Heisenberg vertex algebra. Hence, $V^k(\mathfrak{n}_2)^{G_2}$ is of type $\mathcal{W}(1, 2, 3, 4, 5)$. The Heisenberg and the Virasoro field of $V^k(\mathfrak{n}_2)$ are elements of the kernel of H_0 . The isomorphism in (4.4.1) for the weight $\frac{3}{2}$ fields is given by

$$: X^\pm b : \mapsto \delta^\pm G^\pm \quad \text{where} \quad \delta^+ \delta^- = 2 + k.$$

From the action of the zero mode H_0 it is immediate that a set of strong generators of $V^k(\mathfrak{n}_2)^{G_2}$ of weight 3, 4 and 5 can be given by $\{V_{n,0}\}_{n=0}^2$. \square

We now shift our focus towards $V^k(\mathfrak{n}_4)^{U(1)}$. Let J^+ and G^+ be in identical $U(1)$ -representations ρ_i for $i = 1, 2$. The group action on $V^k(\mathfrak{n}_4)$ will be chosen to be such that it restricts to (and is compatible with) ρ_1 and ρ_2 . All strong generators on which the group action is left undetermined by this requirement are chosen to be in the trivial representation. Note that restricting to $\mathbb{Z}/2\mathbb{Z}$ at level $k = 1$ yields the automorphism on $H^0(X, \Omega^{CdR})$ on a $K3$ surface X that is induced from a fixed point free involution as mentioned in the introduction.

The vertex algebras from Proposition 4.4.6 and 4.4.8 are vertex subalgebras of $V^k(\mathfrak{n}_4)^{U(1)}$. Their minimal strong generators are part of the set of strong generators of $V^k(\mathfrak{n}_4)^{U(1)}$. In contrast to before there exists a decoupling relation for $U_{3,0}$ for all non-vanishing levels k (see (A.1.1)). All other strong generators of the two vertex subalgebras are part of the minimal set of strong generators of $V^k(\mathfrak{n}_4)^{U(1)}$ which follows from the OPEs in (4.2.1). The proof of the following Lemma is a straightforward computation and will be omitted.

Lemma 4.4.9. *Let P_m be a sum of normally ordered products of weight m such that each summand includes the field $\partial^i T$ or $\partial^j J$, and at most one field $\partial^k V_{l,0}$ for some $i, j, k, l, m \in \mathbb{N}_0$. Let $n \in \mathbb{N}_0$.*

$$\frac{k(6+n)}{3(3+n)} V_{n+3,0} = : V_{0,0} V_{n,0} : + k \partial V_{n+2,0} - \left(\frac{1}{2+2n} + k \right) \partial^2 V_{n+1,0} + \frac{k}{3} \partial^3 V_{n,0} + P_{n+6}$$

The last Lemma now implies that the union of the sets of minimal strong generators of the subalgebras $V(\mathfrak{sl}_2)^{U(1)}$ and $V(\mathfrak{n}_2)^{U(1)}$ yields a set of even minimal strong generators of $V(\mathfrak{n}_4)^{U(1)}$. All possible homogeneous minimal odd

generators (up to sums of normally ordered products of fields of lower weight) necessarily are in the linear span of the fields in the set $\{Q^+, Q^-, A_{n,0}, B_{n,0}\}_{n=0}^\infty$ where we define

$$A_{n,0} \stackrel{\text{def}}{=} \partial^n J^+ G^- : \quad \text{and} \quad B_{n,0} \stackrel{\text{def}}{=} \partial^n J^- G^+ :$$

for $n \in \mathbb{N}_0$. The proof of the following Lemma is again a straightforward computation and will be omitted.

Lemma 4.4.10. *Let $n \in \mathbb{N}_0$ and let P_m be a sum of normally ordered products of strong generators of $V^k(\mathfrak{n}_4)^{U(1)}$ of weight less than m and their derivatives.*

$$\begin{aligned} \frac{nk}{2(n+2)} A_{n+2,0} + P_{n+\frac{9}{2}} &= : U_{n,0} A_{0,0} : - : U_{0,0} A_{n,0} : + : (U_{n+1,0} - \partial U_{n,0}) Q^- : \\ &\quad - \frac{1}{n+1} \sum_{i=0}^{n+1} (-1)^i \binom{n}{i} : \partial^i U_{n+1-i,0} Q^- : \\ \frac{nk}{2(n+2)} B_{n+2,0} + P_{n+\frac{9}{2}} &= \sum_{i=0}^n (-1)^i \binom{n}{i} \{ : \partial^i U_{n-i,0} B_{0,0} : + : \partial^i U_{n+1-i,0} Q^+ : \} \\ &\quad - \frac{1}{n+1} : U_{n+1,0} Q^+ : + (-1)^{n+1} : U_0 B_{n,0} : \end{aligned}$$

The last Lemma shows the existence of decoupling relations for the fields in the set $\{A_{i,0}, B_{i,0}\}_{i=3}^\infty$. As before it can be checked directly that there are no decoupling relations for the remaining strong generators. Hence, this yields the following.

Theorem 4.4.11. *For any level $k \neq -2, 0$ the vertex algebra $V^k(\mathfrak{n}_4)^{U(1)}$ is of type $\mathcal{W}(1, \frac{3}{2}, 2^2, \frac{5}{2}, 3^2, \frac{7}{2}, 4^2, \frac{9}{2}, 5)$. A set of minimal strong generators is given by $\{J, Q^\pm, T, U_{i,0}, A_{i,0}, B_{i,0}, V_{i,0}\}_{i=0}^2$.*

4.5 Construction of the cyclic orbifold

Let $\mathcal{U}(\mathfrak{n}_4)^G$ be the universal enveloping algebra of \mathfrak{n}_4 that is invariant under the group G . From here onwards we will restrict the group to be cyclic $G \cong \mathbb{Z}/N\mathbb{Z}$. By section 4.2.3 there are isomorphisms

$$\bigoplus_{i=0}^{N-1} \mathbb{C}_i \otimes W_i \cong V^k(\mathfrak{n}_4) \cong \bigoplus_{i=-\infty}^{\infty} \mathbb{C}_i \otimes V_i$$

as a $(\mathbb{Z}/N\mathbb{Z}, \mathcal{U}(\mathfrak{n}_4)^{\mathbb{Z}/N\mathbb{Z}})$ -module and as a $(U(1), \mathcal{U}(\mathfrak{n}_4)^{U(1)})$ -module, respectively. By restriction this leads to an isomorphism

$$W_a \cong \bigoplus_{k=-\infty}^{\infty} V_{kN+a}$$

for $a = 0, \dots, N-1$ as a $\mathcal{U}(\mathfrak{n}_4)^{\mathbb{Z}/N\mathbb{Z}}$ -module.

It will be convenient to define the following fields

$$Y(\sigma_{a_1, \dots, a_{jN}}^{(i)\pm}, z) = \Sigma_{a_1, \dots, a_{jN}}^{(i)\pm} \stackrel{\text{def}}{=} \partial^{a_1} J^\pm \dots \partial^{a_{jN-i}} J^\pm \partial^{a_{jN-i+1}} G^\pm \dots \partial^{a_{jN}} G^\pm :$$

for $i = 0, \dots, jN$ and $a_i \in \mathbb{N}_0$.

Lemma 4.5.1. *The vertex algebra $V^k(\mathfrak{n}_4)^{\mathbb{Z}/N\mathbb{Z}}$ is strongly generated by the strong generators of $V^k(\mathfrak{n}_4)^{U(1)}$ and the fields $\Sigma_{a_1, \dots, a_N}^{(i)\pm}$ for $i = 0, \dots, N$ and $a_1, \dots, a_N \geq 0$.*

Proof. Let $V_\pm = \bigoplus_{j \in \mathbb{N}} V_{\pm jN}$ and let $i = 0, \dots, N$. Each vector space V_{jN} for $j \in \mathbb{Z} \setminus \{0\}$ is a $\mathcal{U}(\mathfrak{n}_4)^{U(1)}$ -module and generated by the vectors in the set $\{\sigma_{a_1, \dots, a_{jN}}^{(i)+}\}_{a_1, \dots, a_{jN}=0}^\infty$. The vertex algebra on the vector space $V^0 \oplus V_+$ is induced from $V^k(\mathfrak{n}_4)$. Due to commutativity of the fields in the set $\{\Sigma_{a_1, \dots, a_N}^{(i)+}\}_{i=0}^N$ the vertex algebra is strongly generated by these fields and the strong generators of V^0 . By the same argument the vertex algebra over the vector space $V^0 \oplus V_-$ is strongly generated by the fields in the set $\{\Sigma_{a_1, \dots, a_N}^{(i)-}\}_{i=0}^N$ and the strong generators of V^0 . Observe that the fields that appear in the OPEs between the strong generators in V_N and V_{-N} are necessarily elements of V^0 . This proves the proposition. \square

Some fields will be superfluous as strong generators due to the action of the derivation d . The generating function of the fields $\Sigma_{a_1, \dots, a_N}^{(l)\pm}$ for fixed l with respect to weight will show which of these can be neglected as strong generators. It can be obtained by a simple counting argument.

Lemma 4.5.2. *The generating function for the number of fields $\Sigma_{a_1, \dots, a_N}^{(l)\pm}$ with respect to conformal weight is*

$$q^{N+\frac{l^2}{2}} \prod_{i=1}^{N-l} \frac{1}{1-q^i} \prod_{j=1}^l \frac{1}{1-q^j}.$$

Proof. The weight of the field $\Sigma_{a_1, \dots, a_N}^{(l)\pm}$ equals $N + \frac{1}{2}l + a + b$ where $a = \sum_{i=1}^{N-l} a_i$ and $b = \sum_{i=N-l+1}^N a_i$. Due to (anti-)commutativity of the fields in the set $\{\partial^i J^+, \partial^i G^+\}_{i=0}^\infty$ the number of fields at a given weight is determined by the number of partitions of a with at most $N - l$ parts and the number of partitions of b with exactly l and with exactly $l - 1$ parts such that all summands are distinct in both cases. The latter condition exists since at most one of the coefficients in the set $\{a_{N-i+1}, \dots, a_N\}$ can be zero, otherwise $\Sigma_{a_1, \dots, a_N}^{(i)\pm}$ vanishes. \square

Let $V_{(l)}$ be the subspace of the Fock space of $V^k(\mathfrak{n}_4)$ spanned by the vectors in the set $\{\sigma_{a_1, \dots, a_N}^{(l)\pm}\}_{a_1, \dots, a_N=0}^\infty$. It is obvious that $d \in \text{Der}(V_{(l)})$ and that it increases the conformal weight by 1. It follows that there are decoupling relations for vectors in the spanning set of $V_{(l)}$ when taking the action of the derivation into account. Dropping vectors in the set $\{\sigma_{a_1, \dots, a_N}^{(l)\pm}\}_{a_1, \dots, a_N=0}^\infty$ which decouple for this reason amounts to multiplying the generating function from Lemma 4.5.2 by $(1 - q)$.

We will now focus on the case $N = 2$. Multiplying the generating function from the previous Lemma by $(1 - q)$ yields

$$l = 0 : \sum_{i=0}^{\infty} q^{2n+2} \quad , \quad l = 1 : \sum_{i=0}^{\infty} q^{n+\frac{5}{2}} \quad , \quad l = 2 : \sum_{i=0}^{\infty} q^{2n+4} .$$

Thus, Lemma 4.5.1 can be improved upon as a set of strong generators of $V^k(\mathfrak{n}_4)^{\mathbb{Z}/2\mathbb{Z}}$ can be given by the strong generators of $V_k(\mathfrak{n}_4)^{U(1)}$ and the fields in the set

$\{\Sigma_{2n,0}^{(0)\pm}, \Sigma_{n,0}^{(1)\pm}, \Sigma_{2n+1,0}^{(2)\pm}\}_{n=0}^\infty$. The following Lemma will set up the proof of Theorem 4.5.4.

Lemma 4.5.3. *Let $n \in \mathbb{N}$ and $c_n^{(i)}, d_n^{(i)} \in \mathbb{Q}$ for $i = 0, 1, 2$.*

$$\begin{aligned} & : U_{2n,0} \Sigma_{0,0}^{(0)+} - U_{0,0} \Sigma_{2n,0}^{(0)+} : = p_n^{(0)}(k) \Sigma_{2n+2,0}^{(0)+} + \sum_{i=0}^{2n} c_i^{(0)} : \partial^i H \Sigma_{2n-i,0}^{(0)+} : \\ & : U_{2n-1,0} \Sigma_{1,0}^{(0)+} - U_{0,0} \Sigma_{2n-1,1}^{(0)+} : = q_n^{(0)}(k) \Sigma_{2n+2,0}^{(0)+} + \sum_{i=0}^{2n} d_i^{(0)} : \partial^i H \Sigma_{2n-i,0}^{(0)+} : \\ & : \Sigma_{n,0}^{(0)+} B_{0,0} - U_{n,0} \Sigma_{0,0}^{(1)+} + \Sigma_{n+1,0}^{(0)+} Q^+ : = p_n^{(1)}(k) \Sigma_{n+2,0}^{(1)+} + \sum_{i=0}^{n+1} c_i^{(1)} : \partial^i H \Sigma_{n+1-i,0}^{(1)+} : \end{aligned}$$

$$\begin{aligned}
& : \Sigma_{n,0}^{(0)+} B_{0,0} - U_{0,0} \Sigma_{n,0}^{(1)+} + \Sigma_{n,1}^{(0)+} Q^+ : = q_n^{(1)}(k) \Sigma_{n+2,0}^{(1)+} + \sum_{i=0}^{n+1} d_i^{(1)} : \partial^i H \Sigma_{n+1-i,0}^{(1)+} : \\
& : B_{0,0} \Sigma_{2n-1,0}^{(1)+} - \Sigma_{2n-1,1}^{(1)+} Q^+ : = p_n^{(2)}(k) \Sigma_{2n+1,0}^{(2)+} + \sum_{i=0}^{2n} c_i^{(2)} : \partial^i H \Sigma_{2n-i,0}^{(2)+} : \\
& : B_{2n-1,0} \Sigma_{0,0}^{(1)+} + \frac{1}{2n} \Sigma_{0,2n}^{(1)+} Q^+ : = q_n^{(2)}(k) \Sigma_{2n+1,0}^{(2)+} + \sum_{i=0}^{2n} d_i^{(2)} : \partial^i H \Sigma_{2n-i,0}^{(2)+} :
\end{aligned}$$

$$\begin{aligned}
p_n^{(0)}(k) &= -\frac{8n}{2n+1} + \frac{n}{2n+2}k & q_n^{(0)}(k) &= \frac{4}{3} + \frac{7+2n}{6(2n+1)}k \\
p_n^{(1)}(k) &= \frac{2}{n+2} \left[\left(n - \frac{1+(-1)^n}{n+1} \right) + (-1)^n \frac{k}{2} \right] & q_n^{(1)}(k) &= \frac{1}{n+2} \left[\frac{(-1)}{n+1} [4n+5+(-1)^{n+1}(2n+1)] + \frac{k}{2} [n+2(-1)^n] \right] \\
p_n^{(2)}(k) &= \frac{2+k}{2n+1} & q_n^{(2)}(k) &= \frac{2-k}{2n+1}
\end{aligned}$$

Proof. Computing the left hand side of each of these equations is straightforward and leads directly to the right hand side by using only the definition of normal ordering and the commutation relations of \mathfrak{n}_4 which are equivalent to the OPEs as stated in (4.2.1). \square

Theorem 4.5.4. *Let $k \neq -2, 0, 4, 16$. The vertex algebra $V^k(\mathfrak{n}_4)^{\mathbb{Z}/2\mathbb{Z}}$ is of type $\mathcal{W}(1, \frac{3^2}{2}, 2^4, \frac{5^4}{2}, 3^2, \frac{7^4}{2}, 4^5)$. A set of minimal strong generators is*

$$\mathcal{S} = \{H, Q^\pm, T, U_{i,0}, A_{i,0}, B_{i,0}, V_{i,0}, \Sigma_{0,0}^{(i)\pm}, \Sigma_{2,0}^{(0)\pm}, \Sigma_{1,0}^{(1)\pm}, \Sigma_{1,0}^{(2)\pm}\}_{i=0}^1.$$

At level $k = 4$ a set of minimal strong generators is $\mathcal{S} \cup \{\Sigma_{2,0}^{(1)\pm}\}$. At level $k = 16$ a set of minimal strong generators is $\mathcal{S} \cup \{U_{2,0}, A_{2,0}, B_{2,0}\}$.

Proof. By Lemma 4.5.1 and the discussion thereafter the vertex algebra $V^k(\mathfrak{n}_4)^{\mathbb{Z}/2\mathbb{Z}}$ is strongly generated by the strong generators of $V^k(\mathfrak{n}_4)^{U(1)}$ and the fields in the set $\{\Sigma_{2n,0}^{(0)\pm}, \Sigma_{n,0}^{(1)\pm}, \Sigma_{2n+1,0}^{(2)\pm}\}_{n=0}^\infty$. Observe in Lemma 4.5.3 that for any $i \in \{0, 1, 2\}$ the roots of $p_n^{(i)}$ and $q_n^{(i)}$ are distinct for all $n \in \mathbb{N}$. It follows that the fields $\{\Sigma_{2n+4,0}^{(0)+}, \Sigma_{n+3,0}^{(1)+}, \Sigma_{2n+3,0}^{(2)+}\}_{n=0}^\infty$ decouple at any level k . Let $\theta \in \text{Aut}(V^k(\mathfrak{n}_4))$ such that it restricts to an involution on the strong generators of $V^k(\mathfrak{sl}_2)$ given by

$$\theta(J^+) = J^- \quad , \quad \theta(J) = -J \quad , \quad \theta(J^-) = J^+$$

and such that $\theta(G^+) = G^-$. These requirements fix the action on the remaining strong generators. The map θ is an automorphism of $V^k(\mathfrak{n}_4)^{\mathbb{Z}/2\mathbb{Z}}$ a fortiori

and acting on the decoupling relations of Lemma 4.5.3 shows that the fields $\{\Sigma_{2n+4,0}^{(0)-}, \Sigma_{n+3,0}^{(1)-}, \Sigma_{2n+3,0}^{(2)-}\}_{n=0}^{\infty}$ decouple at any level k as well. The existence of decoupling relations for all remaining strong generators can be checked directly.² The field $\Sigma_{2,0}^{(1)+}$ decouples at all levels $k \neq 4$ (see (A.1.2)). Acting on the decoupling relation of $\Sigma_{2,0}^{(1)+}$ with the automorphism θ shows that $\Sigma_{2,0}^{(1)+}$ also decouples at all levels $k \neq 4$. Furthermore, the fields $U_{2,0}$, $A_{2,0}$ and $B_{2,0}$ decouple at all levels $k \neq 16$ (see (A.1.3)-(A.1.5)) and $V_{2,0}$ decouples at all levels $k \neq 0$ (see (A.1.6) and (A.1.7)). These exhaust all decoupling relations for the minimal strong generators which proves the Theorem. \square

4.6 Structure of the vertex algebras

We will now look at sub-structures and simple quotients of the two orbifolds of $V^k(\mathfrak{n}_4)$ that were constructed in the previous section. It will be helpful to define the following: Let $R^i \in V^k(\mathfrak{n}_4)$ be a field and define

$$C^i = Q^+ \circ_0 R^i \quad , \quad D^i = -Q^- \circ_0 R^i \quad , \quad S^i = \frac{1}{2} (Q^- \circ_0 C^i + Q^+ \circ_0 D^i) .$$

It is immediate that the fields in the set $\mathcal{S}^i = \{R^i, C^i, D^i, S^i\}$ and their derivatives span a $V^k(\mathfrak{n}_2)$ -module. Taking $R^0 = H$ and $R^{i+1} = U_{i,0}$ for $i = 0, 1, 2$ it can be checked that the fields in the set $\mathcal{S} = \{\mathcal{S}^i\}_{i=0}^3$ strongly generate $V^k(\mathfrak{n}_4)^{U(1)}$ at all levels $k \neq 0, -2$ (cf. Theorem 4.4.11). Furthermore, taking $R^3 = \Sigma_{0,0}^{(0)+}$ and $R^4 = \Sigma_{1,0}^{(1)+}$ as well as $R^{n+2} = \theta(R^n)$ for $n = 3, 4$ with θ being the automorphism defined in the proof of Theorem 4.5.4 we see that the set $\{\mathcal{S}^i\}_{i=0}^6$ contains all minimal strong generators of $V^k(\mathfrak{n}_4)^{\mathbb{Z}/2\mathbb{Z}}$ at levels $k \neq -2, 0, 4, 16$ (cf. Theorem 4.5.4).

Theorem 4.6.1. *Let $k \neq 0, -2$. For all but finitely many levels k the simple quotient of $V^k(\mathfrak{n}_4)^{U(1)}$ is of type $\mathcal{W}(1, \frac{3}{2}, 2^2, \frac{5}{2}, 3^2, \frac{7}{2}, 4^2, \frac{9}{2}, 5)$. The full list of exceptions is stated in the following table.*

²In our case this was done using Thielemans' MathematicaTM package [Thi91].

level k	central charge	type
$-\frac{5}{2}$	-15	$\mathcal{W}(1, \frac{3^2}{2}, 2^2, \frac{5^2}{2}, 3^2, \frac{7^2}{2}, 4)$
$-\frac{3}{2}$	-9	$\mathcal{W}(1, \frac{3^2}{2}, 2, 4, \frac{9^2}{2}, 5)$
$-\frac{4}{3}$	-8	$\mathcal{W}(1, \frac{3^2}{2}, 2^2, \frac{5^2}{2}, 3^2, \frac{7^2}{2}, 4)$
$-\frac{2}{3}$	-4	$\mathcal{W}(1, \frac{3^2}{2}, 2^2, \frac{5^2}{2}, 3^2, \frac{7^2}{2}, 4^2, \frac{9^2}{2})$
$-\frac{1}{2}$	-3	$\mathcal{W}(1, \frac{3^2}{2}, 2^2, \frac{5^2}{2}, 3)$
1	6	$\mathcal{W}(1, \frac{3^2}{2}, 2)$
2	12	$\mathcal{W}(1, \frac{3^2}{2}, 2^2, \frac{5^2}{2}, 3)$
3	18	$\mathcal{W}(1, \frac{3^2}{2}, 2^2, \frac{5^2}{2}, 3^2, \frac{7^2}{2}, 4)$
4	24	$\mathcal{W}(1, \frac{3^2}{2}, 2^2, \frac{5^2}{2}, 3^2, \frac{7^2}{2}, 4^2, \frac{9^2}{2})$

Proof. It is straightforward to establish a level dependent basis for the vector space of singular fields at a fixed weight using [Thi91]. Let $S \in V^k(\mathfrak{n}_4)^{U(1)}$. A singular field of the form $S + \dots$ where the ellipsis indicate a sum of normally ordered products induces a decoupling relation for the field S in the simple quotient. The type of the simple quotient can therefore be determined by obtaining all possible levels which contain singular fields of the form $S + \dots$ for which the field S is a minimal strong generator. All relevant singular fields are listed in Appendix A.2. Note that the $V^k(\mathfrak{n}_2)$ -module structure induces decoupling relations for further minimal strong generators. \square

Theorem 4.6.2. *Let $k \neq 0, -2$. For all but finitely many levels k the simple quotient of $V^k(\mathfrak{n}_4)^{\mathbb{Z}/2\mathbb{Z}}$ is of type $\mathcal{W}(1, \frac{3^2}{2}, 2^4, \frac{5^4}{2}, 3^2, \frac{7^4}{2}, 4^5)$. The full list of exceptions is stated in the following table.*

level k	central charge	type
$-\frac{5}{2}$	-15	$\mathcal{W}(1, \frac{3^2}{2}, 2^4, \frac{5^4}{2}, 3^2, \frac{7^4}{2}, 4^4)$
$-\frac{3}{2}$	-9	$\mathcal{W}(1, \frac{3^2}{2}, 2^3, \frac{5^2}{2})$
$-\frac{4}{3}$	-8	$\mathcal{W}(1, \frac{3^2}{2}, 2^4, \frac{5^4}{2}, 3^2, \frac{7^4}{2}, 4^4)$
$-\frac{1}{2}$	-3	$\mathcal{W}(1, \frac{3^2}{2}, 2^4, \frac{5^4}{2}, 3)$
1	6	$\mathcal{W}(1, \frac{3^2}{2}, 2, \frac{7^2}{2}, 4^4)$
2	12	$\mathcal{W}(1, \frac{3^2}{2}, 2^4, \frac{5^4}{2}, 3, \frac{7^2}{2}, 4^2)$
3	18	$\mathcal{W}(1, \frac{3^2}{2}, 2^4, \frac{5^4}{2}, 3^2, \frac{7^4}{2}, 4^3)$
4	24	$\mathcal{W}(1, \frac{3^2}{2}, 2^4, \frac{5^4}{2}, 3^2, \frac{7^4}{2}, 4^5, \frac{9^2}{2})$
16	96	$\mathcal{W}(1, \frac{3^2}{2}, 2^4, \frac{5^4}{2}, 3^2, \frac{7^4}{2}, 4^6, \frac{9^2}{2})$

Proof. The proof is analogous to the proof of Theorem 4.6.1. All relevant singular fields are listed in Appendix A.3. Again, note that the $V^k(\mathfrak{n}_2)$ -module

structure as well as the action of the automorphism θ as defined in the proof of Theorem 4.5.4 induce decoupling relations for further minimal strong generators. \square

Given that the proofs of Theorems 4.6.1 and 4.6.2 are purely computational some remarks are in order:

Remark 4.6.3. It is apparent from the singular fields in appendix A.3 that for $k = 1$ $V_k(\mathfrak{n}_4)$ admits an action of the simple vertex algebra $L_k(\mathfrak{sl}_2)$ at level $k = 1$. This statement can also be seen using free field realizations of $V_1(\mathfrak{n}_4)$, see [CH14, Lemma 3.4]. For positive integer n $L_{n+1}(\mathfrak{sl}_2)$ embeds in $L_n(\mathfrak{sl}_2) \otimes L_1(\mathfrak{sl}_2)$ and since \mathfrak{n}_4 is a Lie superalgebra also a homomorphic image of $V^{n+1}(\mathfrak{n}_4)$ embeds into $V_n(\mathfrak{n}_4) \otimes V_1(\mathfrak{n}_4)$. It thus follows that this homomorphic image of $V^{n+1}(\mathfrak{n}_4)$ contains a copy of $L_{n+1}(\mathfrak{sl}_2)$ and so especially the simple quotient $V_{n+1}(\mathfrak{n}_4)$ contains a copy of $L_{n+1}(\mathfrak{sl}_2)$

Remark 4.6.4. Also the level $-\frac{1}{2}(3 + 2n)$ for positive integer n are special. We will see in Theorem 4.7.5 that at these levels an orbifold of a coset of $V_k(\mathfrak{n}_4)$ is a principal \mathcal{W} -algebra of type A . The special cases $k = -\frac{1}{2}$ and $k = -\frac{3}{2}$ are already well understood. Namely $V_{-\frac{1}{2}} \cong (\mathcal{A}(1) \otimes \mathcal{S}(1))^{\mathbb{Z}/2\mathbb{Z}}$ by [CKL19, Thm 4.14]. Here $\mathcal{A}(1)$ is the rank one symplectic fermion algebra and $\mathcal{S}(1)$ the rank one $\beta\gamma$ system. The construction of $V_k(\mathfrak{n}_4)$ at level $k = -\frac{3}{2}$ is first given in [Ada16] and in [CGL18] it is then shown that

$$V_{-\frac{3}{2}} \cong \bigoplus_{n=0}^{\infty} V_{-\frac{3}{2}}(n) \otimes \rho_n$$

as $V_{-\frac{3}{2}}(\mathfrak{sl}_2) \otimes SU(2)$ -module.

Let us also note that this series of special points is suggested in [BMR19] to be subalgebras of the chiral algebras of certain four-dimensional super Yang-Mills theories.

Corollary 4.6.5. *The vertex algebra of global sections of the chiral de Rham complex on a complex Enriques surface is of type $\mathcal{W}(1, \frac{3}{2}, 2, \frac{7}{2}, 4^4)$. It is strongly generated by the fields*

$$J(z), Q^\pm(z), T(z), \Sigma_{1,0}^{(1)\pm}(z), \Sigma_{2,0}^{(0)\pm}(z), \Sigma_{1,0}^{(2)\pm}(z).$$

4.7 Coset of $V^k(\mathfrak{n}_4)$ by its affine subalgebra

In this section, we study the coset

$$\mathcal{C}^k = \text{Com}(V^k(\mathfrak{sl}_2), V^k(\mathfrak{n}_4)), \quad (4.7.1)$$

and we regard $V^k(\mathfrak{n}_4)$ as an extension of $V^k(\mathfrak{sl}_2) \otimes \mathcal{C}^k$. In Theorem 5.4 of [ACKL17], \mathcal{C}^k was incorrectly stated to be of type $\mathcal{W}(2, 3^3, 4, 5^3, 6, 7^3, 8)$. In this section, we give the correct description as well as some more details about its structure.

As in Section 4 of [ACKL17], if we rescale the generators of $V^k(\mathfrak{n}_4)$ by $\frac{1}{\sqrt{k}}$, there is a well-defined limit as $k \rightarrow \infty$, and

$$\lim_{k \rightarrow \infty} V^k(\mathfrak{n}_4) \cong \mathcal{H}(3) \otimes \mathcal{T} \otimes \mathcal{G}_{\text{odd}}(4).$$

In this notation, $\mathcal{H}(3)$ is the rank 3 Heisenberg vertex algebra, \mathcal{T} is a generalized free field algebra with one even generator T satisfying $T(z)T(w) \sim 6(z-w)^{-4}$, and $\mathcal{G}_{\text{odd}}(4)$ is the generalized free field algebra with odd generators G^\pm, Q^\pm satisfying

$$G^+(z)G^-(w) \sim 2(z-w)^{-3}, \quad Q^+(z)Q^-(w) \sim 2(z-w)^{-3}.$$

Note that our normalizations of the generator differ slightly from those in [ACKL17], but this does not change the above result. Note that the action of the inner automorphism group $G_{\text{Inn}} \cong SL_2$ on $V^k(\mathfrak{n}_4)$ coming from integrating the zero-mode action of \mathfrak{sl}_2 gives rise to action of SL_2 on $\mathcal{G}_{\text{odd}}(4)$, such that $\{G^+, Q^+\}$ and $\{G^-, Q^-\}$ both transform as copies of the standard module \mathbb{C}^2 .

As shown in [ACKL17] right before Theorem 4.12, \mathcal{C}^k has a well-defined limit as $k \rightarrow \infty$, and

$$\lim_{k \rightarrow \infty} \mathcal{C}^k \cong \mathcal{T} \otimes (\mathcal{G}_{\text{odd}}(4))^{SL_2}.$$

Moreover, structure of $(\mathcal{G}_{\text{odd}}(4))^{SL_2}$ can be worked out using classical invariant theory. First, we have the infinite generating set

$$\begin{aligned} m^j &= : G^+ \partial^j Q^+ : + : \partial^j G^+ Q^+ :, \\ p^j &= : G^- \partial^j Q^- : + : \partial^j G^- Q^- :, \\ w^j &= : G^+ \partial^j G^- : + : Q^+ \partial^j Q^- :. \end{aligned} \quad (4.7.2)$$

for $j \geq 0$. Note that m^j, p^j, w^j each have weight $j + 3$. It is straightforward to check that $\{w^0, m^0, p^0\}$ generates the algebra $(\mathcal{G}_{\text{odd}}(4))^{SL_2}$, and that the set $\{w^i, m^j, p^j \mid i = 0, 1, 2, 3, j = 0, 2\}$ close under OPE, and hence strongly generates the algebra. We obtain

Theorem 4.7.1. $(\mathcal{G}_{\text{odd}}(4))^{SL_2}$ is of type $\mathcal{W}(3^3, 4, 5^3, 6)$, so that \mathcal{C}^k is of type $\mathcal{W}(2, 3^3, 4, 5^3, 6)$ for generic values of k .

In [ACKL17], it was also stated (correctly) that \mathcal{C}^k contains a vertex subalgebra of type $\mathcal{W}(2, 3, 4, 5, 6, 7, 8)$. We now give more details about this subalgebra. First of all, inside the outer automorphism group $G_{\text{Out}} \cong SL_2$, there is a copy of $U(1)$, and a corresponding outer action of the one-dimensional abelian Lie algebra \mathfrak{t} . Note that the fields w^i, m^j, p^j are eigenvectors with eigenvalue $0, -2, 2$ under this action, respectively. It follows that the orbifold $(\mathcal{C}^k)^{U(1)}$ is strongly generated by the fields $\{w^i \mid i \geq 0\}$ together with all monomials

$$: \partial^{a_1} p^{j_1} \partial^{a_2} p^{j_2} \dots \partial^{a_s} p^{j_s} \partial^{b_1} m^{k_1} \partial^{b_2} m^{k_2} \dots \partial^{b_s} m^{k_s} :,$$

where a_i, b_i, j_i, k_i are nonnegative integers, and $s \geq 1$.

Moreover, one can verify by computer calculation that the fields $\{w^i \mid i = 0, 1, 2, 3, 4, 5\}$ close under OPE, and that for $a, b \geq 0$ and $j, k = 0, 2$, the field $: \partial^a p^j \partial^b m^k :$ lies in the subalgebra generated by $\{w^i \mid i = 0, 1, 2, 3, 4, 5\}$. From this observation, and by induction on s , we obtain

Theorem 4.7.2. $(\mathcal{C}^k)^{U(1)}$ is strongly generated by fields $\{w^i \mid i = 0, 1, 2, 3, 4, 5\}$, and hence is of type $\mathcal{W}(2, 3, 4, 5, 6, 7, 8)$.

We may take the weight 3 field w^0 to be primary of weight 3 and we normalize it so that its sixth order pole with itself is $\frac{k(3+2k)}{2+k} = \frac{c}{3}$. Following the notation in [Lin17], we denote this field by W^3 ; it has the explicit form

$$W^3 = \frac{1}{\sqrt{8+4k}} (: G^+ G^- : + : Q^+ Q^- : - \partial T).$$

Moreover, it is not difficult to verify that it generates $(\mathcal{C}^k)^{U(1)}$. Following the convention of [Lin17], we may take the strong generating set for $(\mathcal{C}^k)^{U(1)}$ to be $\{L, W^i \mid i = 3, 4, 5, 6, 7, 8\}$, where $W^i = W_{(1)}^3 W^{i-1}$ for $i = 4, 5, 6, 7, 8$.

It is readily verified that the hypotheses of Theorem 6.4 of [Lin17] are satisfied, so that $(\mathcal{C}^k)^{U(1)}$ can be realized as a quotient of $\mathcal{W}(c, \lambda)$ of the form $\mathcal{W}_I(c, \lambda) = \mathcal{W}^I(c, \lambda)/\mathcal{I}$. In this notation, $I \subseteq \mathbb{C}[c, \lambda]$ is some prime ideal in the ring of parameters $\mathbb{C}[c, \lambda]$, and \mathcal{I} is the maximal proper graded ideal of $\mathcal{W}^I(c, \lambda) = \mathcal{W}(c, \lambda)/I \cdot \mathcal{W}^I(c, \lambda)$.

By computing the third order pole of W^3 with itself, it is straightforward to verify that I is the ideal $(\lambda + \frac{1}{16})$. Rather surprisingly, this same vertex algebra was studied in Section 11 of [Lin17]. Combining this calculation with Corollary 10.3 of [Lin17], We obtain

Theorem 4.7.3. $(\mathcal{C}^k)^{U(1)}$ is isomorphic to the coset

$$\text{Com}(V^\ell(\mathfrak{sl}_2), V^{\ell+1}(\mathfrak{sl}_2) \otimes \mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}})),$$

where the parameters k and ℓ are related by $k = -\frac{\ell+1}{\ell+2}$.

Remark 4.7.4. This Theorem nicely relates to the coset realization of $V^k(\mathfrak{n}_4)$ of [CFL19]. Let $V^k(n)$ denote the irreducible highest-weight module of $\widehat{\mathfrak{sl}}_2$ of highest weight $n\omega$ at level k . ω is the fundamental weight of \mathfrak{sl}_2 and ρ_m denotes the irreducible highest-weight module of $SU(2)$ of highest weight $m\omega$. Also let \bar{n} be equal to 0 if n is even and 1 otherwise. We have the following list of isomorphisms

1. In [CKLR19, Section 5] diagonal Heisenberg cosets of rank n $\beta\gamma$ system times rank m bc system where studied. These cosets were denoted by $C(n, m)$ and $C(2, 0) \cong \mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}})$ [CKLR19, Remark 5.3] and $C(2, 2) \cong L_1(\mathfrak{sl}(2|2))$ [CKLR19, Theorem 5.5]. Moreover $C(0, 2)$ is nothing but the lattice VOA $L_1(\mathfrak{sl}_2)$ and so we have that $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}}) \subset \text{Com}(L_1(\mathfrak{sl}_2), L_1(\mathfrak{sl}(2|2)))$ and by passing to the simple quotient $L_1(\mathfrak{psl}(2|2))$ of $L_1(\mathfrak{sl}(2|2))$ we also have $\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}}) \subset \text{Com}(L_1(\mathfrak{sl}_2), L_1(\mathfrak{psl}(2|2)))$. The branching rules [Cre17, Cor. 5.3] and [CG17, Rem. 9.11]

$$\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}}) \cong \bigoplus_{n=0}^{\infty} V^{-1}(2m) \quad (4.7.3)$$

and

$$L_1(\mathfrak{psl}(2|2)) \cong \bigoplus_{n=0}^{\infty} V^{-1}(n) \otimes \rho_n \otimes L_1(\bar{n})$$

tell us that

$$\mathcal{W}_{-5/2}(\mathfrak{sl}_4, f_{\text{rect}}) \cong \text{Com}(L_1(\mathfrak{sl}_2), L_1(\mathfrak{psl}(2|2)))^{U(1)}.$$

2. In [CFL19] a vertex superalgebra $Y(\lambda)$ that is very closely related to $L_1(\mathfrak{d}(2, 1; -\lambda)) \otimes L_1(\mathfrak{psl}(2|2))$ has been constructed. It satisfies

$$Y(\lambda)^{\mathbb{Z}/2\mathbb{Z}} \cong \text{Com}(L_1(\mathfrak{sl}_2 \otimes L_1(\mathfrak{sl}_2, L_1(\mathfrak{d}(2, 1; -\lambda))) \otimes L_1(\mathfrak{psl}(2|2))).$$

3. $Y(\lambda)$ decomposes

$$Y(\lambda) := \bigoplus_{n,m=0}^{\infty} V^{k_1}(n) \otimes V^{k_2}(n) \otimes V^{-1}(m) \otimes \rho_m$$

as $V^{k_1}(\mathfrak{sl}_2) \otimes V^{k_2}(\mathfrak{sl}_2) \otimes V^{-1}(\mathfrak{sl}_2) \otimes SU(2)$ -module for generic complex λ . Here $k_1 = \lambda^{-1} - 1, k_2 = \lambda - 1$. Then by [CFL19] and for irrational λ we have that

$$V^{k_2}(\mathfrak{n}_4) \cong \text{Com}(V^{k_1-1}(\mathfrak{sl}_2), Y(\lambda)).$$

4. Putting all these together we get

$$\begin{aligned} (\mathcal{C}^{k_2})^{U(1)} &\cong \text{Com}(V^{k_2} \otimes V^{k_1-1}(\mathfrak{sl}_2), Y(\lambda))^{U(1)} \\ &\cong \text{Com}\left(V^{k_2} \otimes V^{k_1-1}(\mathfrak{sl}_2), \bigoplus_{n,m=0}^{\infty} V^{k_1}(n) \otimes V^{k_2}(n) \otimes V^{-1}(m) \otimes \rho_m\right)^{U(1)} \\ &\cong \text{Com}\left(V^{k_1-1}(\mathfrak{sl}_2), \bigoplus_{m=0}^{\infty} V^{k_1}(\mathfrak{sl}_2) \otimes V^{-1}(m) \otimes \rho_m\right)^{U(1)} \\ &\cong \text{Com}\left(V^{k_1-1}(\mathfrak{sl}_2), \bigoplus_{m=0}^{\infty} V^{k_1}(\mathfrak{sl}_2) \otimes V^{-1}(2m)\right). \end{aligned}$$

which using (4.7.3) and noticing that $k_2 = -\frac{k_1}{k_1+1}$ nicely compares to the Theorem.

We now present some consequences of the identification of $(\mathcal{C}^k)^{U(1)}$ with a quotient of $\mathcal{W}(c, \lambda)$. Recall from Section 10 of [Lin17], we can obtain coincidences between the simple quotient of $(\mathcal{C}^k)^{U(1)}$ with various other algebras arising as quotients of $\mathcal{W}(c, \lambda)$ by finding the intersection points on their truncation curves.

Recall that if we regard \mathcal{C}^k as a one-parameter vertex algebra, with k a formal variable, the specialization of \mathcal{C}^k at a complex number $k = k_0$ need not coincide with the actual coset, but this can only fail when $k_0 + 2 \in \mathbb{Q}_{\leq 0}$. This property is inherited by the orbifold $(\mathcal{C}^k)^{U(1)}$ if we also omit the point $k_0 = 0$. By abuse of notation, in the results below, $(\mathcal{C}^{k_0})^{U(1)}$ will always refer to the specialization of the one-parameter vertex algebra $(\mathcal{C}^k)^{U(1)}$ at the point $k = k_0$, even if is strictly larger than the actual algebra $\text{Com}(V^{k_0}(\mathfrak{sl}_2), V^{k_0}(\mathfrak{n}_4))^{U(1)}$. We also denote by $(\mathcal{C}_{k_0})^{U(1)}$ the simple quotient of $(\mathcal{C}^{k_0})^{U(1)}$.

The next result follows immediately from Theorem 4.7.3 and Theorem 11.5 of [Lin17]

Theorem 4.7.5. *For $n \geq 3$, aside from the critical levels $k = -2$ and $\ell = -n$, and the degenerate cases given by Theorem 10.1 of [Lin17], all isomorphism $(\mathcal{C}^k)^{U(1)} \cong \mathcal{W}_\ell(\mathfrak{sl}_n, f_{\text{prin}})$ appear on the following list.*

$$k = -\frac{1}{2}(n+2), \quad k = -\frac{2(n-1)}{n-2}, \quad \ell = -n + \frac{n-2}{n}, \quad \ell = -n + \frac{n}{n-2}, \quad (4.7.4)$$

which has central charge $c = -\frac{3(n-1)(n+2)}{n-2}$.

Next, in the terminology of [Lin17], recall the *generalized parafermion algebra*

$$\mathcal{G}^\ell(n) = \text{Com}(V^\ell(\mathfrak{gl}_n), V^\ell(\mathfrak{sl}_{n+1})),$$

and its simple quotient $\mathcal{G}_\ell(n)$. By Theorem 8.3 of [Lin17], this also arises a quotient of $\mathcal{W}(c, \lambda)$ and the corresponding truncation curve is given explicitly by (8.4) of [Lin17]. Additionally, by (8.5) of [Lin17], this curve has the following rational parametrization using the level ℓ as parameter:

$$\lambda(\ell) = \frac{(n+\ell)(1+n+\ell)}{(\ell-2)(2n+\ell)(2+2n+3\ell)}, \quad c(\ell) = \frac{n(\ell-1)(1+n+2\ell)}{(n+\ell)(1+n+\ell)}. \quad (4.7.5)$$

Theorem 4.7.6. *For $n \geq 3$, aside from the critical levels $k = -2$, $\ell = -n$, and $\ell = -n - 1$, and the degenerate cases given by Theorem 10.1 of [Lin17], all isomorphisms $(\mathcal{C}_k)^{U(1)} \cong \mathcal{G}_\ell(n)$ appear on the following list.*

1. $k = n$, $k = -\frac{3+2n}{n+2}$, $\ell = -2(1+n)$,
 which has central charge $c = \frac{3n(3+2n)}{2+n}$.
2. $k = -\frac{n-3}{n-2}$, $k = -\frac{n}{n-1}$, $\ell = -2$,
 which has central charge $c = -\frac{3n(n-3)}{(n-2)(n-1)}$.
3. $k = \frac{1}{3}(n-3)$, $k = -\frac{3+2n}{3+n}$, $\ell = -\frac{2n}{3}$,
 which has central charge $c = \frac{(n-3)(3+2n)}{3+n}$.

Proof. We first exclude the values $\ell = 2$, $-2n$, $-\frac{1}{3}(2m+2)$ which are poles of function $\lambda(\ell)$ given by (4.7.5). As explained in [Lin17], at these points, $\mathcal{G}^\ell(n)$ is not obtained as a quotient of $\mathcal{W}(c, \lambda)$ at these points. Note that the truncation curve for $(\mathcal{C}_k)^{U(1)}$ has parametrization

$$c(k) = \frac{3k(3+2k)}{2+k}, \quad \lambda = -\frac{1}{16},$$

and since the pole $k = -2$ has already been excluded, there are no additional points where $(\mathcal{C}_k)^{U(1)}$ cannot be obtained as a quotient of $\mathcal{W}(c, \lambda)$. By Corollary 10.2 of [Lin17], aside from the cases $c = 0, -2$, all remaining isomorphisms $(\mathcal{C}_k)^{U(1)} \cong \mathcal{G}_\ell(n)$ correspond to intersection points on the curves $V(K_m)$ and $V(I)$, where K_m is given by (8.4) of [Lin17], and $I = (\lambda + \frac{1}{16})$, as above. For each $n \geq 2$, there are exactly three intersection points (c, λ) , namely,

$$\left(\frac{3n(3+2n)}{2+n}, -\frac{1}{16} \right), \quad \left(-\frac{3n(n-3)}{(n-2)(n-1)}, -\frac{1}{16} \right), \quad \left(\frac{(n-3)(3+2n)}{3+n}, -\frac{1}{16} \right).$$

It is immediate that the above isomorphisms all hold, and that our list is complete except for possible coincidences at the excluded points $\ell = 2$, $-2n$, $-\frac{1}{3}(2n+2)$.

At $\ell = 2$, $\mathcal{G}_\ell(n)$ has central charge $c = \frac{n(5+n)}{(2+n)(3+n)}$ and the weight 3 field is singular. However, the weight 3 field in $(\mathcal{C}_k)^{U(1)}$ is not singular at this central charge, so there is no coincidence at this point. Similarly, at $\ell = -2n$ and $\ell = -\frac{1}{3}(2n+2)$, $\mathcal{G}_\ell(n)$ has central charge $c = \frac{(2n+1)(3n-1)}{n-1}$ and $c = \frac{n(2n+5)}{n-2}$, respectively, and has a singular vector in weight 3, but

at these central charges, $(\mathcal{C}_k)^{U(1)}$ does not. Therefore there are no additional coincidences at these points. \square

Remark 4.7.7. The first family (1) in Theorem 4.7.6 is of particular interest since it concerns the case where k is a positive integer n . By Remark 4.6.3, the map $V^n(\mathfrak{sl}_2) \rightarrow V^n(\mathfrak{n}_4)$ descends to a map of simple vertex algebras $L_n(\mathfrak{sl}_2) \rightarrow V_n(\mathfrak{n}_4)$. By Corollary 2.2 of [ACKL17], the coset $\text{Com}(L_n(\mathfrak{sl}_2), V_n(\mathfrak{n}_4))$ is simple, and hence coincides with the simple quotient \mathcal{C}_n of \mathcal{C}^n . Moreover, by [DLM96], the simplicity of \mathcal{C}_n is preserved by taking the $U(1)$ -orbifold. It follows that for all $n \in \mathbb{N}$,

$$\text{Com}(L_n(\mathfrak{sl}_2), V_n(\mathfrak{n}_4)) \cong \mathcal{G}_{-2(1+n)}(n).$$

In the case $n = 1$, note that $\mathcal{G}_{-4}(1)$ is just the parafermion algebra $N_{-4}(\mathfrak{sl}_2) = \text{Com}(\mathcal{H}, L_{-4}(\mathfrak{sl}_2))$. Therefore $V_1(\mathfrak{n}_4)$ may be regarded as an extension of $L_1(\mathfrak{sl}_2) \otimes N_{-4}(\mathfrak{sl}_2)$. Likewise, $V_1(\mathfrak{n}_4)^{\mathbb{Z}/2\mathbb{Z}}$, which is isomorphic to the global section algebra of the chiral de Rham complex of an Enriques surface, is an extension of $\mathcal{H} \otimes N_{-4}(\mathfrak{sl}_2)$, where \mathcal{H} is the Heisenberg algebra generated by J .

Consider the coset $\text{Com}(V^{k+2}(\mathfrak{sl}_n) \otimes \mathcal{H}(1), L_k(\mathfrak{sl}_{n+1}) \otimes \mathcal{E}(2n))$ where $\mathcal{E}(2n)$ denotes the bc -system of rank $2n$ and the Heisenberg algebra action is taken to be the diagonal one in such a way that this coset has four odd dimension $3/2$ fields. Its weight one subspace is $\mathcal{H}(1) \otimes L_n(\mathfrak{sl}_2)$ and if we specialize to $k = -2(n+1)$ then it is easy to check that the $\mathcal{H}(1)$ becomes central and so by uniqueness of minimal \mathcal{W} -superalgebras [ACKL17, Thm. 3.1] at this level the coset contains $V_n(\mathfrak{n}_4)$ as subalgebra. This fits into the observation of the first family (1) in Theorem 4.7.6 as this coset also obviously contains $\mathcal{G}_{-2(1+n)}(n)$.

This observation somehow extends to negative levels and thus connects Theorems 4.7.5 and 4.7.6. For this consider the coset

$$\text{Com}(V^{k-2}(\mathfrak{sl}_n) \otimes \mathcal{H}(1), L_k(\mathfrak{sl}_{n|1}) \otimes \mathcal{S}(2n)).$$

The rank $2n$ $\beta\gamma$ -system $\mathcal{S}(2n)$ carries an action of $V^{-n}(\mathfrak{sl}_2) \otimes V^{-2}(\mathfrak{sl}_n) \otimes \mathcal{H}(1)$ and in the commutant we choose the Heisenberg diagonally so that the coset has four dimension $3/2$ fields. As in the previous case the weight one subspace is $\mathcal{H}(1) \otimes V^{-n}(\mathfrak{sl}_2)$ and if we specialize to $k = -2(n-1)$ then it

is easy to check that the $\mathcal{H}(1)$ becomes central and so by uniqueness of minimal \mathcal{W} -superalgebras [ACKL17, Thm. 3.1] at this level the coset contains a homomorphic image of $V^{-n}(\mathfrak{n}_4)$ as subalgebra. This coset also contains $\mathcal{SG}^k(n) := \text{Com}(V^k(\mathfrak{sl}_n) \otimes \mathcal{H}(1), L_k(\mathfrak{sl}_{n|1}))$ as subalgebra, and its central charge for $k = -2(n-1)$ is precisely $-\frac{3(2n^2 - 3n)}{n-2}$, which is the central charge of $(\mathcal{C}_{-n})^{U(1)} \cong \mathcal{W}_\ell(\mathfrak{sl}_{2(n-1)}, f_{\text{prin}})$ at $\ell = -2(n-1) + \frac{n-2}{n-1}$. This observation actually leads us to a new level-rank duality that we will introduce in section 4.8.

Remark 4.7.8. There is another interesting family of vertex algebras that are expected to arise as quotients of $\mathcal{W}(c, \lambda)$, namely, the cosets

$$\mathcal{D}^\ell(n) = \text{Com}(\mathcal{H}, \mathcal{W}^\ell(\mathfrak{sl}_n, f_{\text{subreg}}));$$

see Conjecture 9.4 of [Lin17]. The explicit truncation curve was given in Conjecture 9.6 of [Lin17], and these conjectures were proven in the first nontrivial case $n = 4$. Conjecture 9.6 of [Lin17] has the following consequence.

Conjecture 4.7.9. *For $n \geq 3$, aside from the critical levels $k = -2$ and $\ell = -n$, and the degenerate cases given by Theorem 10.1 of [Lin17], all isomorphisms $(\mathcal{C}_k)^{U(1)} \cong \mathcal{D}_\ell(n)$ appear on the following list.*

1. $k = -\frac{n}{1+n}, k = -\frac{3+n}{2+n}, \ell = -n + \frac{2+n}{1+n}$, which has central charge $c = -\frac{3n(3+n)}{(1+n)(2+n)}$.
2. $k = -n, k = \frac{3-2n}{-2+n}, \ell = -n + \frac{n-2}{n-1}$, which has central charge $c = -\frac{3n(2n-3)}{n-2}$.
3. $k = -\frac{1}{3}(3+n), k = \frac{3-2n}{n-3}, \ell = -n + \frac{n}{n-3}$, which has central charge $c = -\frac{(3+n)(2n-3)}{n-3}$.

The proof that this conjecture follows from Conjecture 9.6 of [Lin17] is similar to the proof of Theorem 4.7.6, and is left to the reader.

4.8 Level-rank dualities

We will now explain that the central charge agreement observed in Remark 4.7.7 is not a coincidence and it fits into the following bigger picture. Firstly, the central charge of the cosets $\mathcal{SG}^{-m}(n)$ and of

$$\text{Com} \left(V^{-n+1}(\mathfrak{sl}_m), L_{-n}(\mathfrak{sl}_m) \otimes L_1(\mathfrak{sl}_m) \right) \quad (4.8.1)$$

are both equal to

$$c = m - 1 - \frac{m(m^2 - 1)}{(m - n)(m - n + 1)}.$$

On the other hand recall that the simple quotient of the coset in (4.8.1) is isomorphic to $\mathcal{W}_\ell(\mathfrak{sl}_m, f_{\text{prin}})$ at level $\ell = -m + \frac{m-n}{m-n+1}$ by the main Theorem of [ACL19].

We can be more general, namely consider now

$$\mathcal{SG}^{-m}(n|r) := \text{Com} \left(V^{-m}(\mathfrak{sl}_n) \otimes V^m(\mathfrak{sl}_r) \otimes \mathcal{H}(1), L_{-m}(\mathfrak{sl}_{n|r}) \right)$$

together with

$$\text{Com} \left(V^{-n+r}(\mathfrak{sl}_m), L_{-n}(\mathfrak{sl}_m) \otimes L_r(\mathfrak{sl}_m) \right)$$

and again there central charges turn out to coincide, i.e. they are equal to

$$c = \frac{(m^2 - 1)nr(n - r - 2m)}{(m - n)(m + r)(m + r - n)}.$$

This observation can be lifted to a new type of level-rank duality. For this consider $\mathcal{E}(mn) \otimes \mathcal{S}(\ell n)$ and recall that we denote by $\mathcal{E}(m)$ the bc -system of rank m and by $\mathcal{S}(m)$ the $\beta\gamma$ -system of rank m . The vertex superalgebra $\mathcal{E}(mn) \otimes \mathcal{S}(\ell n)$ is viewed as the $bc\beta\gamma$ -system for $\mathbb{C}^n \otimes \mathbb{C}^{m|\ell}$, i.e. for the tensor product of the standard representations of \mathfrak{gl}_n and $\mathfrak{sl}_{m|\ell}$. It thus carries a commuting action of $V^{m-\ell}(\mathfrak{gl}_n) \cong V^{m-\ell}(\mathfrak{sl}_n) \otimes \mathcal{H}(1)$ and $V^n(\mathfrak{sl}_{m|\ell})$. We normalize the Heisenberg field to have norm one, so that the b, c, β and γ all have Heisenberg weight $\mu = \frac{1}{\sqrt{n(m-\ell)}}$. The conformal weight Δ of the module $V^{m-\ell}(\omega_1) \otimes \pi_\mu \otimes V^n(\omega_1)$ of $V^{m-\ell}(\mathfrak{sl}_n) \otimes \mathcal{H}(1) \otimes V^n(\mathfrak{sl}_{m|\ell})$ is

$$\Delta = \frac{(n^2 - 1)}{2(n + m - \ell)} + \frac{1}{2n(m - \ell)} + \frac{((m - \ell)^2 - 1)}{2(n + m - \ell)} = \frac{1}{2}$$

so that [AKM⁺16, Corollary 2.2] applies, i.e. there is a conformal embedding of $V^{m-\ell}(\mathfrak{sl}_n) \otimes \mathcal{H}(1) \otimes V^n(\mathfrak{sl}_{m|\ell})$ in $\mathcal{E}(mn) \otimes \mathcal{S}(\ell n)$. We set

$$A^n(\mathfrak{sl}_{m|\ell}) := \text{Com}(V^{m-\ell}(\mathfrak{gl}_n), \mathcal{E}(mn) \otimes \mathcal{S}(\ell n))$$

and if $m = 0$, then we write $A^{-n}(\mathfrak{sl}_\ell)$ for $A^n(\mathfrak{sl}_{0|\ell})$. For descriptions of some of the cosets of these types see [LSS15]. We also need [OS14, Thm. 4.1], i.e.

$$\text{Com}(L_n(\mathfrak{gl}_m), \mathcal{E}(mn)) \cong L_m(\mathfrak{sl}_n). \quad (4.8.2)$$

With this notation and information we can slightly modify the argument of [ACL19, Thm. 13.1] to get

$$\begin{aligned} \text{Com}(V^{-n+r}(\mathfrak{sl}_m), A^{-n}(\mathfrak{sl}_m) \otimes L_r(\mathfrak{sl}_m)) &\cong \\ &\cong \text{Com}(V^{-n+r}(\mathfrak{sl}_m), \text{Com}(V^{-m}(\mathfrak{gl}_n) \otimes L_m(\mathfrak{gl}_r), \mathcal{S}(mn) \otimes \mathcal{E}(mr))) \\ &\cong \text{Com}(V^{-n+r}(\mathfrak{sl}_m) \otimes V^{-m}(\mathfrak{gl}_n) \otimes L_m(\mathfrak{gl}_r), \mathcal{S}(mn) \otimes \mathcal{E}(mr)) \\ &\cong \text{Com}(V^{-n+r}(\mathfrak{gl}_m) \otimes V^{-m}(\mathfrak{sl}_n) \otimes L_m(\mathfrak{sl}_r) \otimes \mathcal{H}(1), \mathcal{S}(mn) \otimes \mathcal{E}(mr)) \\ &\cong \text{Com}(V^{-m}(\mathfrak{sl}_n) \otimes L_m(\mathfrak{sl}_r) \otimes \mathcal{H}(1), \text{Com}(V^{-n+r}(\mathfrak{gl}_m), \mathcal{S}(mn) \otimes \mathcal{E}(mr))) \\ &\cong \text{Com}(V^{-m}(\mathfrak{sl}_n) \otimes L_m(\mathfrak{sl}_r) \otimes \mathcal{H}(1), A^m(\mathfrak{sl}_{r|n})) \end{aligned}$$

We thus have proven the level-rank duality theorem

Theorem 4.8.1. *Let r, n, m be positive integers, then there exist vertex algebra extensions $A^{-n}(\mathfrak{sl}_m)$ and $A^m(\mathfrak{sl}_{r|n})$ of homomorphic images $\tilde{V}^{-n}(\mathfrak{sl}_m)$ and $\tilde{V}^m(\mathfrak{sl}_{r|n})$ of $V^{-n}(\mathfrak{sl}_m)$ and $V^m(\mathfrak{sl}_{r|n})$ such that the level-rank duality*

$$\begin{aligned} \text{Com}(V^{-n+r}(\mathfrak{sl}_m), A^{-n}(\mathfrak{sl}_m) \otimes L_r(\mathfrak{sl}_m)) &\cong \\ &\text{Com}(V^{-m}(\mathfrak{sl}_n) \otimes L_m(\mathfrak{sl}_r) \otimes \mathcal{H}(1), A^m(\mathfrak{sl}_{r|n})) \end{aligned}$$

holds.

Remark 4.8.2. It is natural to ask if the statement of the Theorem can be improved, i.e. one could ask for a level-rank duality of the form

$$\begin{aligned} \text{Com}(V^{-n+r}(\mathfrak{sl}_m), \tilde{V}^{-n}(\mathfrak{sl}_m) \otimes L_r(\mathfrak{sl}_m)) &\stackrel{??}{\cong} \\ &\text{Com}(V^{-m}(\mathfrak{sl}_n) \otimes L_m(\mathfrak{sl}_r) \otimes \mathcal{H}(1), \tilde{V}^m(\mathfrak{sl}_{r|n})). \end{aligned}$$

Remark 4.8.3. One might wonder if there are other levels k for which the coset $\text{Com}(V^k(\mathfrak{sl}_n) \otimes V^{-k}(\mathfrak{sl}_r) \otimes \mathcal{H}(1), V^k(\mathfrak{sl}_{n|r}))$ coincides with a \mathcal{W} -algebra and indeed there are indications that these cosets sometimes coincide with rectangular \mathcal{W} -algebras of type A [CH19b, Appendix D].

Chapter 5

Lattice VOAs and vertex algebras of Odake type

Let K be an integral lattice. The associated vertex algebra \mathcal{V}_K may be extended by a simple current $J(z)$. Denote the resulting lattice obtained by this extension by L and let $L_+ \subset L$ denote its maximal even sublattice. The lattice L is required to be integral in order to have a well-defined vertex algebra structure. From this it can be seen that the only input data is given by the tuple (K, δ) where δ has to satisfy certain restrictions in order for L to be integral. Let K be of rank r and generated by $\{\mathbf{x}^i\}$ such that $(\mathbf{x}^i, \mathbf{x}^j)_K = k\delta_{i,j}$ for $k \in \mathbb{N}$. In this chapter we assume that $J(z)$ is a diagonal simple current. In other words, K is isomorphic to $(\sqrt{k}\mathbb{Z})^r$, the extended lattice L can be defined by a free resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f_\delta} \delta\mathbb{Z} \oplus K \longrightarrow L \longrightarrow 0,$$

and the syzygy coming from the image of f_δ can be given by

$$-\ell\delta + \sum_{i=1}^r \mathbf{x}^i = 0.$$

In order for L to be integral it is required that $\frac{kr}{\ell^2}, \frac{k}{\ell} \in \mathbb{Z}$. The case $\frac{1}{\ell} \in \mathbb{Z}$ is excluded as δ would span only a sublattice and so $K \cong L$. The conditions are equivalent to $\ell|k$ and $\ell|r$, or $\ell^2|k$. It is immediate that if k is prime then $r = kd$ for $d \in \mathbb{N}$.

In what follows we will restrict to the case $k = 3$. Recall that the vertex algebra $\mathcal{V}_{\sqrt{3}\mathbb{Z}}$ has a $\mathcal{N} = 2$ superconformal structure and is, as is well known,

isomorphic to $\mathcal{V}^1(\mathbf{n}_2)$. Hence, \mathcal{V}_L is in particular an extension of $\mathcal{V}^1(\mathbf{n}_2) \subset \mathcal{V}_K$. This will be the reason for the restriction of k . We may express the generators of K via a Euclidean basis $\mathbf{x}^i = \sqrt{3}\mathbf{e}_i$ and write $\delta = \frac{1}{\sqrt{3}}(1, \dots, 1)$. The important associated lattices are listed below.¹ Note that D_s denotes the lattice of type D of rank s .

$$\begin{aligned}
L_+ &= \bigcup_{n=-1}^1 (2n\delta + \sqrt{3}D_{3d}) \\
L^* &= \left\{ \frac{1}{\sqrt{3}}\lambda \in \frac{1}{\sqrt{3}}\mathbb{Z}^{3d} \mid 2 \sum_{i=1}^{3d} \lambda_i \equiv 0 \pmod{3} \right\} \\
L_+^* &= \left\{ \frac{1}{\sqrt{3}}\lambda \in \frac{1}{\sqrt{3}}D_{3d}^* \mid 2 \sum_{i=1}^{3d} \lambda_i \equiv 0 \pmod{3} \right\}
\end{aligned} \tag{5.0.1}$$

5.1 Embedding VOAs of Odake type

In this section we provide a definition of the Odake vertex algebra \mathcal{O}_d and show that it embeds into the lattice vertex superalgebra \mathcal{V}_L in any dimension d . Before moving on to the general case we first give a proof of this for $d = 1, 2, 3$. This will be proved by constructing a sub vertex algebra of \mathcal{V}_L that is isomorphic to \mathcal{O}_d .

Consider the vertex algebra $\mathcal{V} = \mathcal{H}(2d) \otimes \mathcal{E}(d)$. Let the sets of strong generators of the Heisenberg sub vertex algebra and the bc -system be given by $\{x^i(z), y^i(z)\}_{i=1}^d$ and $\{b^i(z), c^i(z)\}_{i=1}^d$, respectively. Define a $SU(d)$ -module structure on \mathcal{V} as follows: Let the group action on

$$(x^1(z), \dots, x^d(z)) \quad \text{and} \quad (b^1(z), \dots, b^d(z))$$

be given by the standard representation and the action on

$$(y^1(z), \dots, y^d(z)) \quad \text{and} \quad (c^1(z), \dots, c^d(z))$$

be given by the dual representation. Let S_n be the symmetric group of degree n .

¹Note that the definition of L^* contains a multiple of 2 which is unnecessary but will make the decomposition L^*/L more apparent.

Proposition 5.1.1. *The vertex algebra $\mathcal{V}^{SU(d)}$ is strongly generated by the fields*

$$tr_{m,n}(X, Y) = \sum_{i=1}^d : \partial^m X^i(z) \partial^n Y^i(z) : \quad (5.1.1)$$

for $X \in \{x, b\}$ and $Y \in \{y, c\}$ and

$$det_{m_1, \dots, m_d}(Z_{(1)}, \dots, Z_{(d)}) = \sum_{g \in S_d} g(: \partial^{m_1} Z_{(1)}^1(z) \cdots \partial^{m_d} Z_{(d)}^d(z) :) \quad (5.1.2)$$

for $Z_{(1)}, \dots, Z_{(d)} \in \{x, b\}$ and $Z_{(1)}, \dots, Z_{(d)} \in \{y, c\}$ where

$$: \partial^{m_1} Z_{(1)}^1(z) \cdots \partial^{m_d} Z_{(d)}^d(z) :$$

is an element of the non-trivial 1-dimensional S_d -module with the obvious group action.

Proof. Let $\mathcal{V} = (W, \mathbf{1}, T, \mathcal{Y})$. Furthermore, let V and V^* be the $SU(d)$ -modules which correspond to the standard and dual representation, respectively. Compatibility of W as a $SU(d)$ -module with the vertex algebra structure demands that the group action is equivariant with respect to the map \mathcal{Y} and the translation operator T . It follows that

$$(\partial^n z^1(z), \dots, \partial^n z^d(z)), \quad \text{equivalently} \quad (z_{-n-1}^1, \dots, z_{-n-1}^d),$$

is isomorphic to V as a $SU(d)$ -module if $z \in \{x, b\}$, and isomorphic to V^* as a $SU(d)$ -module if $z \in \{y, c\}$ for all $n \in \mathbb{N}_0$. The space of states W is linearly isomorphic to

$$\begin{aligned} & \mathbb{C}[x_{-1}^1, \dots, x_{-1}^d, x_{-2}^1, \dots, x_{-2}^d, \dots] \otimes \mathbb{C}[y_{-1}^1, \dots, y_{-1}^d, y_{-2}^1, \dots, y_{-2}^d, \dots] \otimes \\ & \otimes \Lambda[b_0^1, \dots, b_0^d, b_{-1}^1, \dots, b_{-1}^d, \dots] \otimes \Lambda[c_{-1}^1, \dots, c_{-1}^d, c_{-2}^1, \dots, c_{-2}^d, \dots] \end{aligned}$$

Let $\mathbb{V} = \bigoplus_{n=1}^{\infty} V$. It is clear that the space of states W as a $SU(d)$ -module can now be identified with

$$\mathcal{S}(\mathbb{V}) \otimes \mathcal{S}(\mathbb{V}^*) \otimes \bigwedge(\mathbb{V}) \otimes \bigwedge(\mathbb{V}^*)$$

where \mathbb{V}^* is understood to be the restricted dual, i.e. $\mathbb{V}^* = \bigoplus_{n=1}^{\infty} V^*$. The definition of the symmetric and of the antisymmetric algebra makes it obvious that W is a $SU(d)$ -submodule of

$$\mathcal{T}(\mathbb{V}) \stackrel{\text{def}}{=} T(\mathbb{V}) \otimes T(\mathbb{V}) \otimes T(\mathbb{V}^*) \otimes T(\mathbb{V}^*)$$

where $T(C)$ denotes the usual tensor algebra of a module C . It follows that $W^{SU(d)} \subset \mathcal{T}(\mathbb{V})^{SU(d)}$. The $SU(d)$ -invariants of $\mathcal{T}(\bar{\mathbb{V}})$ for $\bar{\mathbb{V}} = \bigoplus_{i=1}^n V$ have been computed in [Wey46] and are well known to be given by traces and determinants. Weyl's proof can be extended to the countably infinite setting at hand to yield the same invariants. \square

Definition 5.1.2. The *Odake vertex algebra* $\mathcal{O}_d \subset \mathcal{V}^{SU(d)}$ is the sub vertex algebra strongly generated by the fields

$$\begin{aligned} tr_{0,0}(x, y), \quad tr_{0,0}(x, c), \quad tr_{0,0}(b, y), \quad tr_{0,0}(b, c) \\ det_{0,\dots,0}(b, \dots, b), \quad det_{0,\dots,0}(x, b, \dots, b) \\ det_{0,\dots,0}(c, \dots, c), \quad det_{0,\dots,0}(y, c, \dots, c) \end{aligned}$$

Note that the Odake vertex algebra is by definition a sub vertex algebra of $\mathcal{H}(2d) \otimes \mathcal{E}(d)$ and thus a sub vertex algebra of a $(bc - \beta\gamma)$ -system of rank d . Recall from section 2.2 that over an affine open subset U of a scheme X of finite type over \mathbb{C} the vertex algebra associated to $\Gamma(U, \Omega^{ch})$ is isomorphic to a $(bc - \beta\gamma)$ -system of rank $\dim_{\mathbb{C}}(X)$. If X is a Calabi-Yau manifold then the vertex algebra associated to $\Gamma(X, \Omega^{ch})$ contains a $\mathcal{N} = 2$ structure and the holonomy group is $SU(\dim_{\mathbb{C}}(X))$ by definition. Observe that \mathcal{O}_d is isomorphic to an extension of $\mathcal{V}^{3d}(\mathfrak{n}_2)$ and defined as a sub vertex algebra of the $SU(d)$ -invariant sub vertex algebra of a $(bc - \beta\gamma)$ -system of rank d . We formulate the following

Conjecture 5.1.3. *Let X be a Calabi-Yau d -fold. Then*

$$\mathcal{O}_d \cong H^0(X, \Omega^{ch}).$$

Remark 5.1.4. It is easy to show that \mathcal{O}_2 is isomorphic to the small $\mathcal{N} = 4$ vertex algebra at central charge 6. The main result of [Son16] is that this vertex algebra is isomorphic to $H^0(X, \Omega^{ch})$ for any complex K3 surface X .

Remark 5.1.5. It was shown in [EHKZ13] that \mathcal{O}_3 is a sub vertex algebra of $H^0(X, \Omega^{ch})$ for any Calabi-Yau threefold X .

We now state the OPEs of a vertex algebra that is readily seen to be isomorphic to the Odake vertex algebra in any dimension d . This will be done by giving a free field construction using the conventional notation of a $(bc - \beta\gamma)$ -system.

$$\begin{aligned}
H(z) &= \sum_{i=1}^d : b^i(z)c^i(z) : \\
G^+(z) &= \sum_{i=1}^d : \beta^i(z)b^i(z) : \\
G^-(z) &= \sum_{i=1}^d : \partial\gamma^i(z)c^i(z) : \\
T(z) &= \frac{1}{2} \sum_{i=1}^d (2 : \beta^i(z)\partial\gamma^i(z) : + : \partial b^i(z)c^i(z) : - : b^i(z)\partial c^i(z) :)
\end{aligned}$$

It is easy to check that these fields strongly generate the vertex algebra $\mathcal{V}^k(\mathbf{n}_2)$ at central charge $c = 3d$ with $c = 6k$. The fields corresponding to the holomorphic and antiholomorphic volume forms are (the signs are chosen for later convenience)

$$V^+(z) =: b^1(z) \cdots b^d(z) : \quad V^-(z) = (-1)^d : c^1(z) \cdots c^d(z) :$$

The OPEs between all fields stated above do not close. The remaining fields that need to be added are

$$\begin{aligned}
W^+(z) &= \frac{1}{(d-1)!} \sum_{n_1, \dots, n_d=1}^d \epsilon_{n_1 \dots n_d} : \partial\gamma^{n_1}(z)b^{n_2}(z) \cdots b^{n_d}(z) : \\
W^-(z) &= \frac{(-1)^d}{(d-1)!} \sum_{n_1, \dots, n_d=1}^d \epsilon_{n_1 \dots n_d} : \beta^{n_1}(z)c^{n_2}(z) \cdots c^{n_d}(z) :
\end{aligned}$$

where ϵ is the totally antisymmetric symbol. We now restrict to the cases $d = 1, 2, 3$. Apart from the $\mathcal{N} = 2$ structure some OPEs can be stated more generally. They are

$$\begin{aligned}
H(z)V^\pm(w) &\sim \pm d \frac{V^\pm(w)}{z-w} \\
G^\pm(z)V^\mp(w) &\sim \frac{W^\mp(w)}{z-w}
\end{aligned}$$

$$\begin{aligned}
T(z)V^\pm(w) &\sim \frac{d}{2} \frac{V^\pm(w)}{(z-w)^2} + \frac{\partial V^\pm(w)}{z-w} \\
H(z)W^\pm(w) &\sim \pm(d-1) \frac{W^\pm(w)}{z-w} \\
G^\pm(z)W^\pm(w) &\sim d \frac{V^\pm(w)}{(z-w)^2} + \frac{\partial V^\pm(w)}{z-w} \\
T(z)W^\pm(w) &\sim \frac{d+1}{2} \frac{W^\pm(w)}{(z-w)^2} + \frac{\partial W^\pm(w)}{z-w}
\end{aligned}$$

The remaining OPEs are between the fields $V^+(z)$, $V^-(z)$, $W^+(z)$, and $W^-(z)$.

The non-regular ones are stated below.

$d = 1.$

$$\begin{aligned}
V^\pm(z)V^\mp(w) &\sim -\frac{1}{z-w} \\
W^\pm(z)W^\mp(w) &\sim -\frac{1}{(z-w)^2}
\end{aligned}$$

$d = 2.$

$$\begin{aligned}
V^\pm(z)V^\mp(w) &\sim -\frac{1}{(z-w)^2} \mp \frac{H(z)}{z-w} \\
V^\pm(z)W^\mp(w) &\sim \frac{G^\pm(z)}{z-w} \\
W^\pm(z)W^\mp(w) &\sim \frac{2}{(z-w)^3} \mp \frac{H(z)}{(z-w)^2} + \frac{T(z) \pm \frac{1}{2}\partial H(z)}{z-w}
\end{aligned}$$

$d = 3.$

$$\begin{aligned}
V^\pm(z)V^\mp(w) &\sim \frac{1}{(z-w)^3} \pm \frac{H(z)}{(z-w)^2} + \frac{1}{2} \frac{H(z)H(z) : \pm \partial H(z)}{z-w} \\
V^\pm(z)W^\mp(w) &\sim \frac{G^\pm(z)}{(z-w)^2} \pm \frac{: H(z)G^\pm(z) :}{z-w} \\
W^+(z)W^-(w) &\sim \frac{3}{(z-w)^4} + \frac{2H(z)}{(z-w)^3} + \frac{T(z) + \partial H(z) + \frac{1}{2} : H(z)H(z) :}{(z-w)^2} \\
&\quad + \frac{\partial T(z) + : T(z)H(z) : - : G^+(z)G^-(z) : + \frac{1}{2} : \partial H(z)H(z) :}{z-w} \\
W^-(z)W^+(w) &\sim \frac{3}{(z-w)^4} - \frac{2H(z)}{(z-w)^3} + \frac{T(z) - \partial H(z) + \frac{1}{2} : H(z)H(z) :}{(z-w)^2}
\end{aligned}$$

$$+ \frac{\partial T(z) - : T(z)H(z) : - : G^-(z)G^+(z) : + \frac{1}{2} : \partial H(z)H(z) :}{z - w}.$$

In order to compute the OPEs of the lattice vertex superalgebra explicitly it is required to choose a non-trivial element $\epsilon \in H^2(L, \mathbb{C}^\times)$ that satisfies (2.7). The following Lemma constructs such an element explicitly.

Lemma 5.1.6. *An element $\epsilon \in H^2(L, \mathbb{C}^\times)$ satisfying (2.7) can be chosen to satisfy the following conditions:*

1.

$$\epsilon(\mathbf{x}^i, \mathbf{x}^j) = \begin{cases} -1 & \text{if } i < j \\ 1 & \text{if } i \geq j \end{cases}$$

2.

$$\epsilon(\mathbf{x}^i, \delta) = (-1)^{d+1} \epsilon(\delta, \mathbf{x}^i) = (-1)^{d+i}$$

3.

$$\epsilon(\delta, \delta) = (-1)^{\frac{d}{2}(9d+5)} = \begin{cases} -1 & \text{if } d \equiv 1, 2 \pmod{4} \\ 1 & \text{if } d \equiv 0, 3 \pmod{4} \end{cases}$$

Proof. Recall from the construction of L that there exists an injective map $f : K \hookrightarrow L$ and K is freely generated over the set $\{\mathbf{x}^i\}$. In Remark 5.5a in [Kac98] a 2-cocycle with values in $\{\pm 1\}$ which in addition satisfies condition (2.7) was constructed explicitly. Moreover, this function is bimultiplicative. We set ϵ to equal this function under pullback $f_*(\epsilon)$. Part 1 is the result of this construction.

Part 2 follows from condition (2.5). Note that this also implies the equality $\epsilon(\alpha, -\beta) = \epsilon(\alpha, \beta)$ for $\alpha, \beta \in L$.

$$1 = \epsilon(\alpha, 0) = \epsilon(\alpha, 3\delta - \sum_{i=1}^{3d} \mathbf{x}^i) = \epsilon(\alpha, \delta) \prod_{i=1}^{3d} \epsilon(\alpha, \mathbf{x}^i)$$

Using the same argument on $\epsilon(0, \alpha)$ it now follows from part 1 that $\epsilon(\mathbf{x}^i, \delta) = (-1)^{i+1}$. Some thought yields the equality $\epsilon(\mathbf{x}^i, \delta) = (-1)^{d+1} \epsilon(\delta, \mathbf{x}^i)$. This proves part 2.

The first equality in part 3 is an easy consequence of part 2. The second equality follows from basic modular arithmetic. \square

Proposition 5.1.7. *Let \mathcal{V}_L be the vertex superalgebra associated to the lattice L of the previous section. There exists a vertex algebra embedding*

$$\iota : \mathcal{O}_d \hookrightarrow \mathcal{V}_L$$

in dimensions 1, 2 and 3.

Proof. The fields

$$\begin{aligned} J(z) &= \frac{1}{3} \sum_{i=1}^{3d} Y(\mathbf{x}_{-1}^i \otimes 1, z) = \frac{1}{3} \sum_{i=1}^{3d} \mathbf{x}^i(z) = \delta(z) \\ Q^\pm(z) &= \frac{1}{\sqrt{3}} \sum_{i=1}^{3d} Y(1 \otimes e^{\pm \mathbf{x}^i}, z) = \frac{1}{\sqrt{3}} \sum_{i=1}^{3d} V_{\pm \mathbf{x}^i}(z) \\ L(z) &= \frac{1}{6} \sum_{i=1}^{3d} : Y(\mathbf{x}_{-1}^i \otimes 1, z) Y(\mathbf{x}_{-1}^i \otimes 1, z) : = \frac{1}{6} \sum_{i=1}^{3d} : \mathbf{x}^i(z) \mathbf{x}^i(z) : \end{aligned}$$

are a representation of the universal affine vertex algebra $\mathcal{V}(\mathbf{n}_2)$ at central charge $c = 3d$ with $c = 6k$. The simple currents

$$A^\pm(z) = Y(1 \otimes e^{\pm \delta}, z) = V_{\pm \delta}(z) \quad (5.1.3)$$

are primary fields of conformal weight $\frac{d}{2}$. Using the identity $\partial V_\alpha(z) =: \alpha(z) V_\alpha(z) :$ for $\alpha \in L$ the OPEs with the above fields are

$$\begin{aligned} J(z)A^\pm(w) &\sim \pm d \frac{A^\pm(w)}{(z-w)^2} \\ Q^\pm(z)A^\mp(w) &\sim \frac{B^\mp(w)}{z-w} \\ L(z)A^\pm(w) &\sim \frac{d}{2} \frac{A^\pm(w)}{(z-w)^2} + \frac{\partial A^\pm(w)}{z-w} \\ A^\pm(z)A^\mp(w) &\sim \frac{\epsilon(\delta, \delta)}{(z-w)^d} \sum_{\substack{k_1+2k_2+\dots=n \\ n, k_i \in \mathbb{N}_0}} \frac{(z-w)^n (\pm 1)^{\sum_i k_i}}{(1!)^{k_1} k_1! (2!)^{k_2} k_2! \dots} : J(w)^{k_1} (\partial J(w))^{k_2} \dots : \end{aligned}$$

We see that in order for the OPEs to close a new field has to be introduced.

$$B^\pm(z) = \frac{1}{\sqrt{3}} \sum_{i=1}^{3d} \epsilon(\mathbf{x}^i, \delta) V_{\pm(\delta - \mathbf{x}^i)}(z)$$

The OPEs with the strong generators of $\mathcal{V}(\mathbf{n}_2)$ are

$$J(z)B^\pm(w) \sim \pm(d-1) \frac{B^\pm(w)}{(z-w)^2}$$

$$\begin{aligned}
Q^\pm(z)B^\pm(w) &\sim d \frac{A^\pm(w)}{(z-w)^2} + \frac{\partial A^\pm(w)}{z-w} \\
L(z)B^\pm(w) &\sim \frac{d+1}{2} \frac{B^\pm(w)}{(z-w)^2} + \frac{\partial B^\pm(w)}{z-w}.
\end{aligned}$$

The remaining OPEs that need to be checked are

$$A^\pm(z)B^\mp(w) \quad A^\pm(z)B^\pm(w) \quad B^\pm(z)B^\pm(w) \quad B^\pm(z)B^\mp(w).$$

Note that $(\delta|\delta - \mathbf{x}^i) = d - 1$ for all $i = 1, \dots, 3d$. Since the highest order pole that can appear in the OPE $V_\alpha(z)V_\beta(w)$ for $\alpha, \beta \in L$ is of order $-(\alpha|\beta)$ it follows that

$$A^\pm(z)B^\pm(w) \sim 0$$

and for the same reason

$$\begin{aligned}
A^\pm(z)B^\mp(w) &\sim 0 \quad \text{if } d = 1 \\
B^\pm(z)B^\pm(w) &\sim 0 \quad \text{if } d > 1.
\end{aligned}$$

Setting $d = 1$ shows that the OPE

$$\begin{aligned}
B^\pm(z)B^\pm(w) &\sim \frac{1}{3} \frac{1}{(z-w)} \sum_{i \neq j} \epsilon(\delta - \mathbf{x}^i, \delta - \mathbf{x}^j) V_{\pm(2\delta - \mathbf{x}^i - \mathbf{x}^j)}(w) \\
&= \frac{1}{3} \frac{(-1)^{d+1} \epsilon(\delta, \delta)}{(z-w)} \sum_{i \neq j} (-1)^{i+j} \epsilon(\mathbf{x}^i, \mathbf{x}^j) V_{\pm(2\delta - \mathbf{x}^i - \mathbf{x}^j)}(w) \\
&= 0
\end{aligned}$$

is regular for all $d \in \mathbb{N}$. The remaining OPEs are

$$\begin{aligned}
A^\pm(z)B^\mp(w) &\sim \frac{1}{\sqrt{3}} \frac{\epsilon(\delta, \delta)(-1)^{d+1}}{(z-w)^{d-1}} \\
&\quad \sum_{\substack{k_1+2k_2+\dots=n \\ n, k_i \in \mathbb{N}_0}} \frac{(z-w)^n (\pm 1)^{\sum_i k_i}}{(1!)^{k_1} k_1! (2!)^{k_2} k_2! \dots} : J(w)^{k_1} (\partial J(w))^{k_2} \dots Q^\pm(w) : \\
B^-(z)B^+(w) &\sim \frac{1}{3} \frac{\epsilon(\delta, \delta)(-1)^{d+1}}{(z-w)^{d+1}} \\
&\quad \left\{ \sum_{i=1}^{3d} \sum_{\substack{k_1+2k_2+\dots=n \\ n, k_i \in \mathbb{N}_0}} \frac{(z-w)^n}{(1!)^{k_1} k_1! (2!)^{k_2} k_2! \dots} : (\mathbf{x}^i - \delta)(w)^{k_1} (\partial(\mathbf{x}^i - \delta)(w))^{k_2} \dots : \right. \\
&\quad \left. + (z-w)^3 \sum_{i \neq j} (-1)^{i+j} \epsilon(\mathbf{x}^i, \mathbf{x}^j) \right.
\end{aligned}$$

$$\left. \sum_{\substack{k_1+2k_2+\dots=n \\ n, k_i \in \mathbb{N}_0}} \frac{(z-w)^n}{(1!)^{k_1} k_1! (2!)^{k_2} k_2! \dots} : (\mathbf{x}^i - \delta)(w)^{k_1} (\partial(\mathbf{x}^i - \delta)(w))^{k_2} \dots V_{\pm(\mathbf{x}^i - \mathbf{x}^j)}(w) : \right\}$$

From this expression the OPE $B^+(z)B^-(w)$ can be readily deduced. Note that $\frac{1}{3} \sum_{i=1}^{3d} (\mathbf{x}^i - \delta)(z) = (1-d)J(z)$. Restricting to $d = 1, 2, 3$ it can easily be seen that this vertex subalgebra is isomorphic to \mathcal{O}_d . An explicit vertex algebra embedding in these cases $\iota : \mathcal{O}_d \hookrightarrow \mathcal{V}_L$ can be given by

$$\begin{aligned} \iota(H(z)) &= J(z) & \iota(G^\pm(z)) &= Q^\pm(z) & \iota(T(z)) &= L(z) \\ \iota(V^\pm(z)) &= A^\pm(z) & \iota(W^\pm(z)) &= B^\pm(z) \end{aligned}$$

□

The last Proposition can easily be improved upon.

Theorem 5.1.8. *Let $d \in \mathbb{N}$. There exists a vertex algebra embedding*

$$\iota : \mathcal{O}_d \hookrightarrow \mathcal{V}_L.$$

Proof. The isomorphism $\mathcal{V}^{c=1}(\mathfrak{n}_2) \cong \mathcal{V}_{\sqrt{3}\mathbb{Z}}$ is well known. It is clear from the proof of Proposition 5.1.7 that this isomorphism can be used to show existence of a diagonal embedding $\mathcal{V}^{c=3d}(\mathfrak{n}_2) \hookrightarrow \mathcal{V}_K$ where $K = \oplus_{i=3d} \sqrt{3}\mathbb{Z}$. It was shown previously that the fields

$$tr_{0,0}(x, y), \quad tr_{0,0}(x, c), \quad tr_{0,0}(b, y), \quad tr_{0,0}(b, c)$$

as given in Definition 5.1.2 strongly generate a vertex algebra that is isomorphic to $\mathcal{V}^{c=3d}(\mathfrak{n}_2)$ and a sub vertex algebra of a $(bc - \beta\gamma)$ -system of rank d . Extending this vertex algebra by the fields

$$\begin{aligned} det_{0,\dots,0}(b, \dots, b) &=: b^1(z) \dots b^d(z) : \\ det_{0,\dots,0}(c, \dots, c) &=: c^1(z) \dots c^d(z) : \end{aligned}$$

requires the additional fields

$$det_{0,\dots,0}(x, b, \dots, b) \quad \text{and} \quad det_{0,\dots,0}(y, c, \dots, c)$$

in order for the OPEs to close as can be readily seen by applying the Wick theorem. The isomorphism $\mathcal{E}(d) \cong \mathcal{V}_{\mathbb{Z}^d}$, a.k.a. *boson-fermion correspondence*, is also well known. It is clear that the identities

$$\begin{aligned}\iota(: b^1(z) \cdots b^d(z) :) &= V_{+\delta}(z) \\ \iota(: c^1(z) \cdots c^d(z) :) &= V_{-\delta}(z)\end{aligned}$$

hold for the fields given in (5.1.3). This proves the Theorem. \square

5.2 Elliptic genera

Note that the factor 2 in the condition for L^* in (5.0.1) is unnecessary but it makes the decomposition L_+^*/L^* more apparent which can be seen as follows: Recall from chapter 3 that

$$D_{3d}^* = \bigcup_{\mathbf{i}=0}^3 ([\mathbf{i}]_D + D_{3d}) \quad \text{and} \quad \mathbb{Z}^{3d} = D_{3d} \cup ([\mathbf{2}]_D + D_{3d})$$

where the coset representatives are stated in (3.2.2). We repeat them here for the convenience of the reader:

$$\begin{aligned}[\mathbf{0}] &:= (0, \dots, 0, 0), & [\mathbf{1}] &:= \frac{1}{2}(1, \dots, 1, 1), \\ [\mathbf{2}] &:= (0, \dots, 0, 1), & [\mathbf{3}] &:= \frac{1}{2}(1, \dots, 1, -1).\end{aligned}\tag{5.2.1}$$

Seeing that $0 \neq \frac{1}{\sqrt{3}}[\mathbf{1}]_D = \frac{1}{2}\delta \in L_+^*/L^*$ we can write

$$L_+^* = L^* \cup \left(\frac{1}{2}\delta + L^* \right).$$

As $0 \in L^*$ we see that the Ramond sector R is given by $\frac{1}{2}\delta + L^*$. Let R_+ and R_- denote the subsets of the Ramond sector which contain all elements of R with positive and negative parity, respectively. Moreover, let $\varepsilon \in L \setminus L_+$. The elliptic genus reads as follows.

$$\begin{aligned}\mathcal{E}(\tau, u) &= \text{str}_{\mathbb{H}_R} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} z^{J_0} \right) \\ &= \text{tr}_{\mathbb{H}_R} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} z^{J_0} (-1)^{J_0 - \bar{J}_0} \right) \\ &= (\eta(q)\eta(\bar{q}))^{-\text{rank}(L)} \sum_{\alpha \in R/L_+} \vartheta_\alpha(u, \tau) (\vartheta_\alpha(0, \bar{\tau}) - \vartheta_{\alpha+\varepsilon}(0, \bar{\tau}))\end{aligned}\tag{5.2.2}$$

As usual, $q = \exp(2\pi i\tau)$ and $z = \exp(2\pi iu)$ where $\tau, u \in \mathbb{C}$ such that $\Im(\tau) > 0$. The elliptic genus is known to be independent of \bar{q} , hence we look at all terms in the sums which cancel the contributions coming from $\eta(\bar{q})^{-3d} = \bar{q}^{-\frac{d}{8}} \prod_j (1 - q^j)^{3d}$. Here we used that $\bar{c} = \text{rank}(N) = 3d$. The bilinear form on the Ramond sector is positive definite and so all exponents of \bar{q} in any theta function above are non-negative. Thus, the only terms to determine in the theta functions $\vartheta_\beta(0, \bar{\tau}) = \sum_{\gamma \in L_+} \bar{q}^{\frac{1}{2}(\beta+\gamma)^2}$ are the constant terms and those which cancel the negative exponents in the aforementioned prefactor. Thus, we look for elements in the Ramond sector $\mu = \frac{1}{2}\delta + \gamma \in \frac{1}{2}\delta + L^* = R$ such that

$$\frac{d}{8} - n = \frac{1}{2}\mu^2 \iff \delta\gamma + \gamma^2 = -2n \quad (5.2.3)$$

for $n \in \mathbb{N}_0$. Setting $\gamma = \frac{1}{\sqrt{3}}\tau \in L^* \subset \frac{1}{\sqrt{3}}\mathbb{Z}^{3d}$ yields the condition

$$\sum_{i=1}^{3d} \tau_i(1 + \tau_i) = -6n$$

It is easy to see that there exist no solutions to this Diophantine equation for $n > 0$ and so the only solutions are $\tau_i \in \{0, -1\}$ for all $i = 1, \dots, 3d$. The number of solutions equals 2^{3d} , however, since $\gamma \in L^*$ the number reduces to $\sum_{k=0}^d \binom{3d}{3k}$. We henceforth denote the set of solutions of eq. (5.2.3) containing exactly $3k$ entries equal to -1 (up to an overall factor of $\frac{1}{\sqrt{3}}$) by $[k]$ and the associated class of modules by $[\mathbf{k}] = [\mathbf{k}]_0 \oplus [\mathbf{k}]_1$. The parity of any element in $[k]$ is given by $k \pmod{2}$.

Before we proceed let us make two observations. Note that $\delta^2 = d$ which shows that

$$\delta \in \begin{cases} L_+ & \text{if } 2|d \\ L_+ + \varepsilon & \text{otherwise.} \end{cases} \quad (5.2.4)$$

It is apparent from this that the parity is given by the dimension d . Note that the set $[d]$ contains exactly one element. We will abuse notation and denote this element by the same symbol. Observe that

$$[\mathbf{d}]_{\mathbf{d} \pmod{2}} \ni \frac{1}{2}\delta + [d] \stackrel{L_+}{\sim} \begin{cases} \frac{1}{2}\delta & \text{if } 2|d \\ \frac{1}{2}\delta + \varepsilon & \text{otherwise} \end{cases}$$

which leads to the following

Observation 1.

$$[\mathbf{d}] = [\mathbf{0}] \quad (5.2.5)$$

For the second observation note that comparing the Jacobi forms

$$\begin{aligned} \vartheta_{\frac{1}{2}\delta+\gamma}(u, \tau) &= \sum_{\gamma \in L^*} e^{2\pi i(x, \frac{1}{2}\delta+\gamma)} q^{\frac{1}{2}(\frac{1}{2}\delta+\gamma)^2} = q^{\frac{d}{8}} z^{\frac{d}{2}} \sum_{\gamma \in L^*} e^{2\pi i(x, \gamma)} q^{\frac{1}{2}(\delta\gamma+\gamma^2)} \\ \vartheta_{\frac{1}{2}\delta-\gamma-\delta}(u, \tau) &= \sum_{\gamma \in L^*} e^{2\pi i(x, -\frac{1}{2}\delta-\gamma)} q^{\frac{1}{2}(-\frac{1}{2}\delta-\gamma)^2} = q^{\frac{d}{8}} z^{-\frac{d}{2}} \sum_{\gamma \in L^*} e^{-2\pi i(x, \gamma)} q^{\frac{1}{2}(\delta\gamma+\gamma^2)} \end{aligned}$$

shows

$$\vartheta_{-\frac{1}{2}\delta-\gamma}(u, \tau) = \vartheta_{\frac{1}{2}\delta+\gamma}(-u, \tau).$$

Observation 2. For $\mathbf{k} = \mathbf{0}, \dots, \mathbf{d}$ and $i \in \mathbb{Z}/2\mathbb{Z}$, it follows from (5.2.4) that

$$\vartheta_{[\mathbf{k}]_i}(u, \tau) = \begin{cases} \vartheta_{-[\mathbf{k}]_i}(-u, \tau) = \vartheta_{[\mathbf{d}-\mathbf{k}]_i}(-u, \tau) & \text{if } 2|d \\ \vartheta_{-[\mathbf{k}]_{i+1}}(-u, \tau) = \vartheta_{[\mathbf{d}-\mathbf{k}]_{i+1}}(-u, \tau) & \text{otherwise.} \end{cases} \quad (5.2.6)$$

Proposition 5.2.1. The elliptic genus equals

$$\begin{aligned} \mathcal{E}(\tau, u) &= \eta(q)^{-3d} \sum_{k=0}^d \binom{3d}{3k} \{ \vartheta_{[\mathbf{k}]_{k \bmod 2}}(u, \tau) - \vartheta_{[\mathbf{k}]_{k \bmod 2+1}}(u, \tau) \} \\ &= \eta(q)^{-3d} \sum_{k=0}^d (-1)^k \binom{3d}{3k} \{ \vartheta_{[\mathbf{k}]_0}(u, \tau) - \vartheta_{[\mathbf{k}]_1}(u, \tau) \}. \end{aligned}$$

Proof. The expression in (5.2.2) can be rewritten to

$$\mathcal{E}(\tau, u) = (\eta(q)\eta(\bar{q}))^{-\text{rank}(L)} \sum_{\alpha \in R/L_+} (\vartheta_{\alpha}(u, \tau) - \vartheta_{\alpha+\varepsilon}(u, \tau)) \vartheta_{\alpha}(0, \bar{\tau}).$$

The result now follows from the condition stated in (5.2.3) and the proceeding discussion, and the fact that $\vartheta_{\alpha}(u, \tau) = \vartheta_{\beta}(u, \tau)$ for $\alpha, \beta \in [\mathbf{k}]_0$ or $\alpha, \beta \in [\mathbf{k}]_1$ for $\mathbf{k} = 1 \dots, d$. \square

Note that due to the first observation (see (5.2.5)) the contributions of the modules $[\mathbf{0}]$ and $[\mathbf{d}]$ to the elliptic genus always cancel in odd dimensions. Because of the second observation (see (5.2.6)) a further consequence of the last Proposition is that

$$\mathcal{E}(\tau, -u) = \mathcal{E}(\tau, u). \quad (5.2.7)$$

Remark 5.2.2. On a compact complex manifold of dimension d with vanishing first Chern class the elliptic genus is a Jacobi form of weight 0 and index $\frac{d}{2}$ (see Prop. 1.2 in [Gri00]). A consequence of this is that the elliptic genus is invariant under the $\mathbb{Z}/2\mathbb{Z}$ -action induced by $-id \in SL_2(\mathbb{Z})$ which yields the relation $\mathcal{E}(\tau, -u) = \mathcal{E}(\tau, u)$. Hence, in order for the vertex algebra \mathcal{V}_L to be defined on a compact manifold with vanishing first Chern class (5.2.7) is a necessary condition.

Example 5.2.3. Let $d = 1$. Recall our first observation which implies $[\mathbf{0}]_k = [\mathbf{1}]_k$ for $k \in \mathbb{Z}/2\mathbb{Z}$. It follows that the elliptic genus vanishes

$$\mathcal{E}(\tau, u) = \eta(q)^{-3} \{ \vartheta_{[\mathbf{0}]_0}(u, \tau) - \vartheta_{[\mathbf{0}]_1}(u, \tau) + \vartheta_{[\mathbf{0}]_1}(u, \tau) - \vartheta_{[\mathbf{0}]_0}(u, \tau) \} = 0.$$

The following Jacobi functions are provided for later reference.

$$\begin{aligned} \vartheta_{[\mathbf{0}]_0}(u, \tau) &= \vartheta_{\frac{1}{2}\delta}(u, \tau) = q^{\frac{1}{8}} z^{+\frac{1}{2}} (1 + q(6 + z^{-2}) + \mathcal{O}(q^2)) \\ \vartheta_{[\mathbf{0}]_1}(u, \tau) &= \vartheta_{\frac{1}{2}\delta+\varepsilon}(u, \tau) = q^{\frac{1}{8}} z^{-\frac{1}{2}} (1 + q(z^2 + 6) + \mathcal{O}(q^2)) \end{aligned}$$

Example 5.2.4. Let $d = 2$ and $\gamma_k \in [k]$. Since the elliptic genus is a weak Jacobi form of weight 0 it is uniquely determined by its q^0 -term if its index is less than 6 or equal to $\frac{13}{2}$ (see Corollary 1.7 in [Gri00]). The relevant Jacobi forms are

$$\begin{aligned} \vartheta_{[\mathbf{0}]_0}(u, \tau) &= \vartheta_{\frac{1}{2}\delta}(u, \tau) = q^{\frac{1}{4}} (z + z^{-1} + \mathcal{O}(q^2)) \\ \vartheta_{[\mathbf{0}]_1}(u, \tau) &= \vartheta_{\frac{1}{2}\delta+\varepsilon}(u, \tau) = q^{\frac{1}{4}} (0 + 12q + \mathcal{O}(q^2)) \\ \vartheta_{[\mathbf{1}]_1}(u, \tau) &= \vartheta_{\frac{1}{2}\delta+\gamma_1}(u, \tau) = q^{\frac{1}{4}} (1 + q(z^2 + 6 + z^{-2}) + \mathcal{O}(q^2)) \\ \vartheta_{[\mathbf{1}]_0}(u, \tau) &= \vartheta_{\frac{1}{2}\delta+\gamma_1+\varepsilon}(u, \tau) = q^{\frac{1}{4}} (0 + 7q(z + z^{-1}) + \mathcal{O}(q^2)) \end{aligned}$$

All of the above theta functions satisfy $\vartheta_{[\mathbf{i}]_k}(u, \tau) = \vartheta_{[\mathbf{i}]_k}(-u, \tau)$ which is a consequence of the first and second observation. The elliptic genus equals

$$\begin{aligned} \mathcal{E}(\tau, u) &= \eta(q)^{-6} \{ 2(\vartheta_{[\mathbf{0}]_0} - \vartheta_{[\mathbf{0}]_1}) + 20(\vartheta_{[\mathbf{1}]_1} - \vartheta_{[\mathbf{1}]_0}) \} \\ &= 2(z + 10 + z^{-1} + q(10z^2 - 64z + 108 - 64z^{-1} + 10z^{-2}) + \mathcal{O}(q^2)) \\ &= 2\phi_{0,1}. \end{aligned}$$

Example 5.2.5. Let $d = 3$ and $\gamma_k \in [k]$. The Jacobi forms are

$$\begin{aligned}\vartheta_{[\mathbf{0}]_0}(u, \tau) &= \vartheta_{\frac{1}{2}\delta}(u, \tau) = q^{\frac{3}{8}}z^{-\frac{1}{2}}(z^2 + 9q + \mathcal{O}(q^2)) \\ \vartheta_{[\mathbf{0}]_1}(u, \tau) &= \vartheta_{\frac{1}{2}\delta+\varepsilon}(u, \tau) = q^{\frac{3}{8}}z^{+\frac{1}{2}}(z^{-2} + 9q + \mathcal{O}(q^2)) \\ \vartheta_{[\mathbf{1}]_1}(u, \tau) &= \vartheta_{\frac{1}{2}\delta+\gamma_1}(u, \tau) = q^{\frac{3}{8}}z^{+\frac{1}{2}}(1 + q(1 + 3z^{-2}) + \mathcal{O}(q^2)) \\ \vartheta_{[\mathbf{1}]_0}(u, \tau) &= \vartheta_{\frac{1}{2}\delta+\gamma_1+\varepsilon}(u, \tau) = q^{\frac{3}{8}}z^{-\frac{1}{2}}(0 + q(3z^2 + 9 + z^{-2}) + \mathcal{O}(q^2)) \\ \vartheta_{[\mathbf{2}]_0}(u, \tau) &= \vartheta_{\frac{1}{2}\delta+\gamma_2}(u, \tau) = q^{\frac{3}{8}}z^{-\frac{1}{2}}(1 + q(3z^2 + 1) + \mathcal{O}(q^2)) \\ \vartheta_{[\mathbf{2}]_1}(u, \tau) &= \vartheta_{\frac{1}{2}\delta+\gamma_2+\varepsilon}(u, \tau) = q^{\frac{3}{8}}z^{+\frac{1}{2}}(0 + q(z^2 + 9 + 3z^{-2}) + \mathcal{O}(q^2))\end{aligned}$$

and the elliptic genus can be determined uniquely as in the previous example.

$$\begin{aligned}\mathcal{E}(\tau, u) &= 84\eta(q)^{-9}\{\vartheta_{[\mathbf{1}]_1} - \vartheta_{[\mathbf{1}]_0} + \vartheta_{[\mathbf{2}]_0} - \vartheta_{[\mathbf{2}]_1}\} \\ &= 84(z^{\frac{1}{2}} + z^{-\frac{1}{2}} + q(-z^{\frac{5}{2}} + z^{\frac{1}{2}} + z^{-\frac{1}{2}} - z^{-\frac{5}{2}}) + \mathcal{O}(q^2)) \\ &= 84\phi_{0, \frac{3}{2}}.\end{aligned}$$

The examples above are summarized in Table 5.1.

d	Module Class	Number of Modules	$\mathcal{E}(\tau, u)$	χ
1	$[\mathbf{0}]_0$	1	0	0
	$[\mathbf{1}]_1$	1		
2	$[\mathbf{0}]_0$	1	$2\phi_{0,1}$	24
	$[\mathbf{1}]_1$	20		
	$[\mathbf{2}]_0$	1		
3	$[\mathbf{0}]_0$	1	$84\phi_{0, \frac{3}{2}}$	168
	$[\mathbf{1}]_1$	84		
	$[\mathbf{2}]_0$	84		
	$[\mathbf{3}]_1$	1		

Table 5.1: Summary of the preceding examples. The letter χ denotes the Euler characteristic.

Proposition 5.2.6. *The Euler characteristic χ equals*

$$2e^{id\pi} + \sum_{n=0}^d \binom{3d}{3n}.$$

Proof. The Euler characteristic can be recovered from the elliptic genus

$$\chi = \lim_{\tau \rightarrow i\infty} \mathcal{E}(\tau, 0).$$

In order to determine the Euler characteristic we proceed by determining all terms proportional to q^0 in the elliptic genus. Just as in the case of \bar{q}^0 which was discussed previously this leads to condition (5.2.3). We rewrite this condition as follows

$$\frac{1}{2} \left(\frac{1}{2} \delta + \gamma + \lambda \right)^2 = \frac{d}{8} - n \iff \gamma^2 + \lambda^2 + \delta\gamma + 2\lambda \left(\gamma + \frac{1}{2} \delta \right) = -2n$$

where $\gamma \in L^*/L_+$ and $\lambda \in L_+$ and $n \in \mathbb{N}_0$. As determined previously, this condition has solutions if and only if $n = 0$. It follows that $\gamma \in [k]$ and so no module in a class $[\mathbf{k}]_{k \bmod 2+1}$ contributes to the Euler characteristic. Additionally, since $\gamma \in [k]$ we have $\gamma^2 + \delta\gamma = 0$ and the last equation simplifies to

$$\lambda(\lambda + \delta + 2\gamma) = 0.$$

1. $\lambda = 0$. The coefficients of the terms proportional to $q^{\frac{d}{8}}$ of the Jacobi forms $\vartheta_{[\mathbf{k}]_{k \bmod 2}}$ are $e^{2\pi i(x, \frac{1}{2}\delta + \gamma)} = e^{2\pi i\sqrt{3}x(\frac{d}{2} + m)} = z^{\frac{d}{2} + m}$ where $m = -k$ if $\gamma \in [k]$.
2. $\lambda + \delta + 2\gamma = 0$. Hence, $\delta + 2\gamma \in L_+$. Recall from (5.2.4) that $\delta \in L_+$ in even dimensions. It follows that γ can only equal either $[0]$ or $[d]$. The same conclusion follows in odd dimensions. The coefficients of the terms proportional to $q^{\frac{d}{8}}$ of the Jacobi forms $\vartheta_{[\mathbf{k}]_{k \bmod 2}}$ are $e^{2\pi i(x, \frac{1}{2}\delta + \gamma - \delta - 2\gamma)} = e^{-2\pi i\sqrt{3}x(\frac{d}{2} + m)} = z^{-(\frac{d}{2} + m)}$ where $\gamma \in [0]$ and $m = 0$ or $\gamma \in [d]$ and $m = -d$. Thus, for even d , the condition $\delta + 2\gamma \in L_+$ suggests that only the module $[\mathbf{0}]_0 = [\mathbf{d}]_0$ yields non zero terms to the elliptic genus. For odd d the module $[\mathbf{0}]_1 = [\mathbf{d}]_1$ could contribute as well, however, contributions from the vacuum module always cancel if d is odd due to Observation 1 and Proposition 5.2.1.

In summary, every module in a class $[\mathbf{k}]_{k \bmod 2}$ for $\mathbf{k} = 1, \dots, d-1$ adds a factor of 1 to the Euler characteristic and in odd dimensions these are the only contributions. In even dimensions the only remaining contributions are from the modules $[\mathbf{0}]_0$ and $[\mathbf{d}]_0$ where each of which contributes a factor of 2. □

Corollary 5.2.7. *The elliptic genus associated to \mathcal{V}_L vanishes if and only if $d = 1$.* \square

Lemma 5.2.8. *The Euler characteristic χ is divisible by 24.*

Proof. Using the equality

$$\sum_{k=0}^d \binom{3d}{3k} = \frac{1}{3} (2^{3d} + 2e^{id\pi})$$

the Euler characteristic equals

$$\chi = \begin{cases} \frac{2^3}{3} (8^{d-1} + 1) & \text{for even } d \\ \frac{2^3}{3} (8^{d-1} - 1) & \text{for odd } d. \end{cases}$$

By induction, $8^{\text{even} \geq 0} = 1 \pmod{3^2}$. It follows easily that $8^{\text{odd} > 0} = -1 \pmod{3^2}$.

Thus, $24|\chi$. \square

d	χ
1	0
2	24
3	168
4	1368
5	10920
6	87384
7	699048
8	5592408
9	44739240
10	357913944
\vdots	\vdots

Table 5.2: Euler Characteristic for the lowest dimensions.

The definition of the elliptic genus may be extended by inserting a further Jacobi variable. Following [KT17, Wen19] let $\tilde{\mathbb{H}}_R = \{\phi \in \mathbb{H}_R | (\bar{L}_0 - \frac{\bar{c}}{24})\phi = 0\}$ and define the *Hodge-elliptic genus* (cf. Def. 2.2 in [Wen19])

$$\mathcal{E}_{\text{Hodge}}(\tau, u, v) = \text{tr}_{\tilde{\mathbb{H}}_R} \left(q^{L_0 - \frac{c}{24}} z^{J_0} y^{\bar{J}_0} (-1)^{J_0 - \bar{J}_0} \right).$$

For an element of an anti-chiral module $\gamma \in [k]$ we have $\bar{J}_0(\gamma) = -\sqrt{3}k$. Thus, the elliptic genus as given in Prop. 5.2.1 is refined to

$$\mathcal{E}_{\text{Hodge}}(\tau, u, v) = \eta(q)^{-3d} \sum_{k=0}^d y^{\frac{d}{2}-k} \binom{3d}{3k} \left\{ \vartheta_{[\mathbf{k}]_k \pmod{2}}(u, \tau) - \vartheta_{[\mathbf{k}]_k \pmod{2+1}}(u, \tau) \right\}$$

and connected to the elliptic genus via $\mathcal{E}(\tau, u) = \mathcal{E}_{\text{Hodge}}(\tau, u, 0)$. Up to signs the Poincaré polynomial can be inferred from the Hodge-elliptic genus in the case at hand. It is known that the Poincaré polynomial can be written as

$$(yz)^{\frac{c}{6}} \text{tr}_S(y^{\bar{J}_0} z^{J_0})$$

where the trace is taken over a subspace in the Ramond sector $S = \{\phi \in \mathbb{H}_R | (L_0 - \frac{c}{24})\phi = (\bar{L}_0 - \frac{\bar{c}}{24})\phi = 0\}$ (cf. eq. 2.21 in [LVW89]). One might wonder about possible cancellations due to the factor $(-1)^{J_0 - \bar{J}_0}$ appearing in the Hodge-elliptic genus which could prevent a comparison with the Poincaré polynomial, however, this turns out not to be the case as can be seen by the following Lemma.

Proposition 5.2.9. *The Poincaré polynomial equals*

$$(y^d + z^d) + \sum_{k=0}^d \binom{3d}{3k} (yz)^{d-k}.$$

Proof. We want to determine all terms proportional to q^0 in the Hodge-elliptic genus. Analogous to the proof of Prop. 5.2.6 it can be shown that there exists exactly one term in the theta function $\vartheta_{[\mathbf{k}]_{k \bmod 2}}(u, \tau)$ that contributes such a term and none in $\vartheta_{[\mathbf{k}]_{k \bmod 2+1}}(u, \tau)$ for $\mathbf{k} = \mathbf{1}, \dots, \mathbf{d} - \mathbf{1}$. For even d , the terms proportional to $q^{\frac{d}{8}}$ in the Jacobi form over the module $[\mathbf{0}]_0$ equals $z^{\frac{d}{2}} + z^{-\frac{d}{2}}$ and the same holds true for $[\mathbf{d}]_0 = [\mathbf{0}]_0$. For odd d , the Jacobi forms over the modules $[\mathbf{0}]_0$ and $[\mathbf{d}]_1$ contribute the terms $z^{\frac{d}{2}}$ and $z^{-\frac{d}{2}}$, respectively. Again, by Observation 1 $[\mathbf{d}]_1 = [\mathbf{0}]_1$ and $[\mathbf{0}]_0 = [\mathbf{d}]_0$ and so the contributions from the modules $[\mathbf{0}] = [\mathbf{d}]$ are $z^{\frac{d}{2}} - z^{-\frac{d}{2}}$. In conclusion, no cancellations occur and the term proportional to q^0 in the Hodge-elliptic genus is

$$\lim_{\tau \rightarrow i\infty} \mathcal{E}_{\text{Hodge}}(\tau, u, v) = (-1)^d \left(y^{\frac{d}{2}} z^{-\frac{d}{2}} + y^{-\frac{d}{2}} z^{\frac{d}{2}} \right) + \sum_{k=0}^d \binom{3d}{3k} (yz)^{\frac{d}{2}-k}.$$

The result follows after multiplication by $(yz)^{\frac{d}{2}}$ and dropping the factor $(-1)^d$. □

Remark 5.2.10. The Euler characteristic can be inferred from the Poincaré polynomial. Thus, Proposition 5.2.6 is a corollary of Proposition 5.2.9.

Remark 5.2.11. The Poincaré polynomial given in Proposition 5.2.9 yields a Hodge diamond for which the only non-vanishing Hodge numbers are $h^{d,0} = h^{0,d} = 1$ and $h^{k,k} = \binom{3d}{3k}$.

d	$\mathcal{E}_{\text{Hodge}}(\tau, u, v)$
1	$(y^{\frac{1}{2}} - y^{-\frac{1}{2}}) \left(z^{\frac{1}{2}} - z^{-\frac{1}{2}} - q(z^{\frac{3}{2}} - 9z^{\frac{1}{2}} + 9z^{-\frac{1}{2}} - z^{-\frac{3}{2}}) + \mathcal{O}(q^2) \right)$
2	$(y + y^{-1})(z + z^{-1} + 6q(z - 2 + z^{-1}) + \mathcal{O}(q^2)) + 20(1 + q(z^2 - 7z + 12 - 7z^{-1} + z^{-2}) + \mathcal{O}(q^2))$
3	$(y^{\frac{3}{2}} - y^{-\frac{3}{2}}) \left(z^{\frac{3}{2}} - z^{-\frac{3}{2}} + 9q(z^{\frac{3}{2}} - z^{\frac{1}{2}} + z^{-\frac{1}{2}} - z^{-\frac{3}{2}}) + \mathcal{O}(q^2) \right) + 84y^{\frac{1}{2}} \left(z^{\frac{1}{2}} - q(3z^{\frac{3}{2}} - 10z^{\frac{1}{2}} + 9z^{-\frac{1}{2}} - 3z^{-\frac{3}{2}} + z^{-\frac{5}{2}}) + \mathcal{O}(q^2) \right) + 84y^{-\frac{1}{2}} \left(z^{-\frac{1}{2}} - q(z^{\frac{5}{2}} - 3z^{\frac{3}{2}} + 9z^{\frac{1}{2}} - 10z^{-\frac{1}{2}} + 3z^{-\frac{3}{2}}) + \mathcal{O}(q^2) \right)$

Table 5.3: Expansions of the conformal field theoretic Hodge-elliptic genus in lowest dimensions. Setting $y = 1$ yields the elliptic genus $\mathcal{E}(\tau, u)$ (cf. Table 5.1).

d	$(zy)^{\frac{d}{6}} \lim_{\tau \rightarrow i\infty} \mathcal{E}_{\text{Hodge}}(\tau, u, v)$
1	$1 - y - z + yz$
2	$1 + y^2 + 20zy + z^2 + (yz)^2$
3	$1 + 84zy - y^3 - z^3 + 84(zy)^2 + (yz)^3$

Table 5.4: Changing all signs in $\lim_{\tau \rightarrow i\infty} (zy)^{\frac{d}{6}} \mathcal{E}_{\text{Hodge}}(\tau, u, v)$ to be positive yields the Poincaré polynomial (equivalently the Hodge diamond). Setting $z = y = 1$ yields the Euler characteristic.

Remark 5.2.12. The Poincaré polynomials inferred from Table 5.4 show that the target space in dimension 1 is a torus and in dimension 2 a $K3$ surface. In dimension 3 we see that the target space threefold is rigid, i.e. $h^{2,1}$ vanishes.

			1		
		0		0	
	0		84		0
1		0		0	1
	0		84		0
		0		0	
					1

Table 5.5: The Hodge diamond in dimension 3.

Chapter 6

On conformal embeddings of affine VOAs into rectangular \mathcal{W} -algebras

Let \mathfrak{g}' be a Lie algebra such that $\mathfrak{g}' \subset \mathfrak{g}$. Take the sequence of embeddings

$$\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}' \xrightarrow{\iota} \mathfrak{g}$$

with the first map being the principal embedding. Denote the image of the \mathfrak{sl}_2 -triple by $\{x, e, f\}$ where $[x, f] = -f$. It is clear that $\mathfrak{g}' \subset \mathfrak{g}^{\iota(f)}$. Hence, for any Lie algebra $\mathfrak{s} \subset \mathfrak{g}'$ it follows from the introduction above that

$$\mathcal{V}^\ell(\mathfrak{s}) \hookrightarrow \mathcal{W}^k(\mathfrak{g}, \iota(f)).$$

One may wonder which condition(s) allow(s) for a stronger statement, e.g. when this embedding is conformal or an isomorphism. A similar question has been asked in [AKM⁺18b] where all levels k were found for which the embedding of the maximal affine vertex algebra in a simple minimal \mathcal{W} -algebra $\mathcal{W}_k(\mathfrak{g}, \theta)$ is conformal, with \mathfrak{g} being a simple Lie super-algebra and $-\theta$ its minimal root. This investigation was continued in [AKM⁺17] where the decomposition of the minimal simple affine \mathcal{W} -algebra as a module over its maximal affine vertex algebra was discussed. It is worth mentioning that the authors showed in a closely related work [AKM⁺18a] that in case the conformal embedding is an isomorphism, the representation category of the affine VOA is semi-simple even at non-admissible levels.

In this chapter we find obstructions for the levels k and ℓ for when this embedding cannot be conformal for $\mathfrak{s} \subset \mathfrak{g}^{\mathfrak{g}'}$ maximal by comparing the central charges of the affine vertex algebra and the \mathcal{W} -algebra. This is done for some tuples $(\mathfrak{g}, \mathfrak{g}')$ where \mathfrak{g} is of type A , B , C or a subset of type D . The specific cases under investigation are summarized in Table 6.1. Furthermore, we find examples where the simple quotients of $\mathcal{W}^k(\mathfrak{g}, \iota(f))$ and $V^\ell(\mathfrak{s})$ are isomorphic.

\mathfrak{g}	\mathfrak{g}'	\mathfrak{s}
\mathfrak{sl}_{mn}	\mathfrak{sl}_n	\mathfrak{sl}_m
\mathfrak{so}_{mn}	\mathfrak{so}_n	\mathfrak{so}_m
\mathfrak{sp}_{2mn}	\mathfrak{sp}_{2n}	\mathfrak{so}_m
\mathfrak{so}_{4mn}	\mathfrak{sp}_{2n}	\mathfrak{sp}_{2m}

Table 6.1: Examples where \mathfrak{s} is the maximal Lie algebra in $\mathfrak{g}^{\mathfrak{g}'}$.

6.1 Background

Let \mathfrak{g} be a simple Lie algebra and $\hat{\mathfrak{g}}$ its affinization. For a principal admissible weight Λ we denote the image of the irreducible $\hat{\mathfrak{g}}$ -module $\rho_\lambda^{\hat{\mathfrak{g}}}$ at level k under the quantum Drinfeld-Sokolov reduction functor by $\mathcal{H}_{\text{DS}}(\lambda)$ and assume it to be non-zero. Let $\sigma \in \mathbb{Z}$. We call a level k *principal admissible* if

$$k + h^\vee \geq \frac{h^\vee}{\sigma} \quad \text{such that} \quad \gcd(h^\vee, \sigma) = \gcd(r^\vee, \sigma) = 1 \quad (6.1.1)$$

where h^\vee denotes the dual Coxeter number and r^\vee equals the largest number of edges between two nodes in the Dynkin diagram. Moreover, we call the level k *boundary principal admissible* if equality in (6.1.1) is satisfied.

Recall the standard theta-functions

$$\vartheta_{11}(\tau, z) = -iq^{\frac{1}{12}}u^{-\frac{1}{2}}\eta(\tau) \prod_{r=1}^{\infty} (1 - u^{-1}q^r)(1 - uq^{r-1})$$

$$\vartheta_{01}(\tau, z) = q^{-\frac{1}{24}}\eta(\tau) \prod_{r=1}^{\infty} (1 - u^{-1}q^{r-\frac{1}{2}})(1 - uq^{r-\frac{1}{2}})$$

with the definitions $u = \exp(2\pi iz)$ and $q = \exp(2\pi i\tau)$. We will make repeated use of the functions κ_{11} and κ_{01} defined as follows

$$\vartheta_{11}(\tau, z) = \eta(\tau)\kappa_{11}(\tau, z)$$

$$\vartheta_{01}(\tau, z) = \eta(\tau)\kappa_{01}(\tau, z).$$

Let \mathfrak{g} be a Lie algebra of rank l and let $\mathfrak{h} \subset \mathfrak{g}$ denote the Cartan algebra. Let $\tau, t \in \mathbb{C}$ with $\text{Im}(\tau) > 0$ and $z \in \mathfrak{h}^f$. The character for the vacuum module $\rho_{k\Lambda_0}^{\hat{\mathfrak{g}}}$ at boundary principal admissible level k reads (see Remark 1 in [KW17])

$$ch_{k\Lambda_0}(\tau, z, t) = e^{2\pi ikt} \left(\frac{\eta(\sigma\tau)}{\eta(\tau)} \right)^l \prod_{\alpha \in \Delta_+} \frac{\kappa_{11}(\sigma\tau, \alpha(z))}{\kappa_{11}(\tau, \alpha(z))}. \quad (6.1.2)$$

For a nilpotent element $f \in \mathfrak{g}$ the character of the irreducible $\mathcal{W}_k(\mathfrak{g}, f)$ -module $\mathcal{H}_{\text{DS}}(\lambda)$ at boundary principal admissible level is given in eq. (11) in [KW17] which we repeat here in a slightly altered form for the benefit of the reader. We adopt the notation and denote the (positive) roots of \mathfrak{g} by (Δ_+) Δ with the number of positive simple roots being l . As usual, we look at a Lie algebra embedding $\mathfrak{sl}_2 \xrightarrow{\pi} \mathfrak{g}$. Due to the Jacobson-Morozov theorem we may take $x, f \in \text{im}(\pi)$ such that x is an element in the Cartan subalgebra of $\mathfrak{sl}_2 \subset \mathfrak{g}$ which satisfies $[x, f] = -f$. Let $\mathfrak{g} = \bigoplus_j \mathfrak{g}_j$ be the eigenspace decomposition with respect to the adjoint representation of x . Furthermore, let $\Delta_+^0 = \{\alpha \in \Delta_+ | \alpha(x) = 0\}$ and $\Delta_{\frac{1}{2}} = \{\alpha \in \Delta | \alpha(x) = \frac{1}{2}\}$. Lastly, let β be an element of the dual root lattice and let y be an element of the Weyl group such that $(t_\beta y)\hat{\Pi}_\sigma \subset \hat{\Delta}_+$ where $\hat{\Pi}_\sigma = \{\sigma\delta - \theta, \alpha_1, \dots, \alpha_l\}$ with $\delta \in \hat{\mathfrak{h}}^*$ being the imaginary root and θ the highest root, and $t_\beta \in \text{End}(\hat{\mathfrak{h}}^*)$ a translation defined by

$$t_\beta(\lambda) = \lambda + \lambda(K)\beta - \delta(\lambda|\beta) - \frac{\delta}{2}\lambda(K)|\beta|^2.$$

The character reads

$$ch_{\mathcal{H}_{\text{DS}}(\lambda)}(\tau, z) = (-i)^{|\Delta_+|} q^{\frac{\hbar^\vee}{2\sigma}|\beta-x|^2} e^{2\pi i \frac{\hbar^\vee}{\sigma}(\beta|z)} \cdot \left(\frac{\eta(\sigma\tau)}{\eta(\tau)} \right)^l \frac{\prod_{\alpha \in \Delta_+} \kappa_{11}(\sigma\tau, y(\alpha)(z + \tau\beta - \tau x))}{\prod_{\alpha \in \Delta_+^0} \kappa_{11}(\tau, \alpha(z)) \left(\prod_{\alpha \in \Delta_{\frac{1}{2}}} \kappa_{01}(\tau, \alpha(z)) \right)^{\frac{1}{2}}}. \quad (6.1.3)$$

6.2 Obstructions to conformal embeddings of maximal affine sub vertex algebras

We adopt the convention that $1 < N \in \mathbb{N}$, take n to be a divisor of N , and write $N = mn$ throughout this section. Table 6.2 and Theorem 6.2.1 summarize the findings of this section.

\mathfrak{g}	$\mathfrak{g}' \otimes \mathfrak{s}$	\mathfrak{g} type	\mathfrak{g}' type	$k + h_{\mathfrak{g}}^{\vee}$	$\ell + h_{\mathfrak{s}}^{\vee}$
\mathfrak{sl}_{mn}	$\mathfrak{sl}_n \otimes \mathfrak{sl}_m$	A	A	$\frac{h_{\mathfrak{g}}^{\vee}+1}{n}$	$h_{\mathfrak{s}}^{\vee} + 1$
				$\frac{h_{\mathfrak{g}}^{\vee}}{n+1}$	$\frac{h_{\mathfrak{s}}^{\vee}}{n+1}$
				$\frac{h_{\mathfrak{g}}^{\vee}-1}{n}$	$h_{\mathfrak{s}}^{\vee} - 1$
\mathfrak{so}_{mn}	$\mathfrak{so}_n \otimes \mathfrak{so}_m$	B	B	$\frac{h_{\mathfrak{g}}^{\vee}+2}{n+1}$	$\frac{h_{\mathfrak{s}}^{\vee}+2}{n+1}$
				$\frac{h_{\mathfrak{g}}^{\vee}+1}{n}$	$h_{\mathfrak{s}}^{\vee} + 1$
				$\frac{h_{\mathfrak{g}}^{\vee}}{n}$	$h_{\mathfrak{s}}^{\vee}$
\mathfrak{sp}_{2mn}	$\mathfrak{so}_n \otimes \mathfrak{sp}_{2m}$	C	B	$\frac{h_{\mathfrak{g}}^{\vee}}{n}$	$h_{\mathfrak{s}}^{\vee}$
				$\frac{h_{\mathfrak{g}}^{\vee}-\frac{1}{2}}{n}$	$h_{\mathfrak{s}}^{\vee} - \frac{1}{2}$
	$\mathfrak{sp}_{2n} \otimes \mathfrak{so}_m$		C	$\frac{h_{\mathfrak{g}}^{\vee}-1}{n+1}$	$\frac{h_{\mathfrak{s}}^{\vee}-1}{n+1}$
				$\frac{h_{\mathfrak{g}}^{\vee}}{2n+1}$	$\frac{h_{\mathfrak{s}}^{\vee}}{2n+1}$
\mathfrak{so}_{4mn}	$\mathfrak{sp}_{2n} \otimes \mathfrak{sp}_{2m}$	D	C	$\frac{h_{\mathfrak{g}}^{\vee}-\frac{1}{2}}{2n}$	$h_{\mathfrak{s}}^{\vee} - \frac{1}{2}$
				$\frac{h_{\mathfrak{g}}^{\vee}}{2n+1}$	$\frac{h_{\mathfrak{s}}^{\vee}}{2n+1}$
				$\frac{h_{\mathfrak{g}}^{\vee}-m}{2n-1}$	$\frac{2mn-h_{\mathfrak{s}}^{\vee}}{2n-1}$
				$\frac{h_{\mathfrak{g}}^{\vee}-2m}{2n-1}$	$\frac{2mn-h_{\mathfrak{s}}^{\vee}}{2n-1}$

Table 6.2: Levels at which the central charges of the rectangular \mathcal{W} -algebra $\mathcal{W}_k(\mathfrak{g}, \iota(f))$ and the affine vertex algebra $\mathcal{V}_{\ell}(\mathfrak{s})$ coincide. Note that \mathfrak{s} is the maximal Lie algebra in \mathfrak{g}' . For the rectangular vertex algebra $\mathcal{W}_k(\mathfrak{g}, \iota(f))$, \mathfrak{sl}_2 is chosen to be principally embedded into \mathfrak{g}' .

Theorem 6.2.1. *For any tuple of Lie algebras $(\mathfrak{g}, \mathfrak{s})$ at boundary principal admissible level k as stated in Table 6.2 there exists an isomorphism of vertex*

algebras

$$\mathcal{W}_k(\mathfrak{g}, \iota(f)) \xrightarrow{\sim} \mathcal{V}_\ell(\mathfrak{s})$$

if ℓ is either boundary principal admissible or zero.

6.2.1 Type A

Take a sequence of embeddings

$$\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_n \xrightarrow{\iota} \mathfrak{sl}_N \quad (6.2.1)$$

and let the first map be the principal and the map ι be the diagonal embedding. Denote the image of the \mathfrak{sl}_2 triplet under the first map by the symbols $\{x, e, f\}$ such that $[x, f] = -f$. In case of rectangular \mathcal{W} -algebras this datum is sufficient to determine the central charge. Before doing this, the following Lemma will be convenient later on. Recall that \mathfrak{sl}_m is the maximal Lie algebra in the centralizer of \mathfrak{sl}_n in \mathfrak{sl}_{mn} (see Table 6.1).

Lemma 6.2.2. *There exists a $\mathfrak{sl}_n \otimes \mathfrak{sl}_m$ -module isomorphism*

$$\mathfrak{sl}_N \cong \mathfrak{sl}_n \otimes \mathfrak{sl}_m \oplus \mathbf{1}^{\mathfrak{sl}_n} \otimes \mathfrak{sl}_m \oplus \mathfrak{sl}_n \otimes \mathbf{1}^{\mathfrak{sl}_m}. \quad (6.2.2)$$

Proof. We may take $\rho_{\omega_1}^{\mathfrak{sl}_N} \cong \rho_{\omega_1}^{\mathfrak{sl}_n} \otimes \rho_{\omega_1}^{\mathfrak{sl}_m}$ and $\rho_{\omega_{N-1}}^{\mathfrak{sl}_N} \cong \rho_{\omega_{n-1}}^{\mathfrak{sl}_n} \otimes \rho_{\omega_{m-1}}^{\mathfrak{sl}_m}$. Recall that $\rho_{\omega_1 + \omega_{N-1}}^{\mathfrak{sl}_N}$ is the adjoint representation. A simple computation

$$\begin{aligned} \rho_{\omega_1 + \omega_{N-1}}^{\mathfrak{sl}_N} \oplus \mathbf{1}^{\mathfrak{sl}_N} &\cong \rho_{\omega_1}^{\mathfrak{sl}_N} \otimes \rho_{\omega_{N-1}}^{\mathfrak{sl}_N} \\ &\cong \rho_{\omega_1 + \omega_{n-1}}^{\mathfrak{sl}_n} \otimes \rho_{\omega_1 + \omega_{m-1}}^{\mathfrak{sl}_m} \oplus \mathbf{1}^{\mathfrak{sl}_n} \otimes \rho_{\omega_1 + \omega_{m-1}}^{\mathfrak{sl}_m} \\ &\quad \oplus \rho_{\omega_1 + \omega_{n-1}}^{\mathfrak{sl}_n} \otimes \mathbf{1}^{\mathfrak{sl}_m} \oplus \mathbf{1}^{\mathfrak{sl}_n} \otimes \mathbf{1}^{\mathfrak{sl}_m} \end{aligned}$$

yields the result. □

Lemma 6.2.3. *The central charge of $\mathcal{W}_k(\mathfrak{sl}_N, \iota(f))$ equals*

$$\frac{k(N^2 - 1)}{k + N} - kN(n^2 - 1) - Nm(n - 1)(n^2 - n - 1).$$

Proof. Restricting (6.2.2) to a \mathfrak{sl}_n -module isomorphism and requiring that the embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_n$ is principal leads to an isomorphism of \mathfrak{sl}_2 -modules

$$\mathfrak{sl}_N \cong (m^2 - 1)\rho_1 \oplus m^2 \bigoplus_{j=2}^n \rho_{2i-1}. \quad (6.2.3)$$

More explicitly, for the second embedding we may take

$$\iota(f) = \sum_{j=0}^{m-1} \sum_{i=1}^{n-1} e_{jn+i+1, jn+i} \quad , \quad \iota(x) = \frac{1}{2} \sum_{j=0}^{m-1} \sum_{i=1}^n (n+1-2i) e_{jn+i, jn+i}.$$

Applying eq. (2.10) then yields the result. The only contributions which are not straight forward are the ones from the charged and the neutral free fermions. Using the notation of *loc. cit.*, observe that $\mathfrak{g}_{\frac{1}{2}} = \emptyset$ as can be seen from (6.2.3) since no \mathfrak{sl}_2 -module of even dimension appears in the decomposition. The contribution from the charged free fermions can be inferred from (6.2.3) and equals

$$-2m^2 \sum_{i=1}^{n-1} (n-i)(6i^2 - 6i + 1) = -Nm(n-1)(n^2 - n - 1).$$

□

Proposition 6.2.4. *All levels k at which the central charges of $V^\ell(\mathfrak{sl}_m)$ and $\mathcal{W}^k(\mathfrak{sl}_N, \iota(f))$ coincide are*

$$k + N = \frac{N}{n+1} \quad \text{and} \quad k + N = \frac{N \pm 1}{n}.$$

Proof. Observe from Lemma 6.2.2 that there are copies of $\mathcal{V}^k(\mathfrak{sl}_m)$ in the \mathcal{W} -algebra due to the submodule $\mathbf{1}^{\mathfrak{sl}_n} \otimes \mathfrak{sl}_m \subset \mathfrak{sl}_N$. Furthermore, each set of ghosts shifts the level by a factor that depends on the representation of \mathfrak{sl}_m . All ghosts that appear are either in the trivial or the adjoint representation of \mathfrak{sl}_m . The former does not contribute to the level shift. This leaves the submodule $\mathfrak{sl}_n \otimes \mathfrak{sl}_m \subset \mathfrak{sl}_N$ where the number of ghosts needs to be determined, i.e. the order of the set $\bigoplus_{i \neq 0} (\mathfrak{sl}_n)_i$ under the induced grading of \mathfrak{sl}_N under ad_x . One is quick to see that

$$\left| \bigoplus_{i \neq 0} (\mathfrak{sl}_n)_i \right| = n(n-1).$$

The dependence of the levels can now be seen to be

$$\ell = kn + mn(n-1).$$

The result follows after equating the central charge of $V^\ell(\mathfrak{sl}_m)$ with the central charge of $\mathcal{W}^k(\mathfrak{sl}_N, \iota(f))$ which is given by Lemma 6.2.3. □

The levels appearing in Prop. 6.2.4 expressed in terms of the level of the affine vertex algebra are

$$\ell + m = \frac{m}{n+1} \quad \text{and} \quad \ell = \pm 1,$$

respectively. Observe that the former level is a boundary (principal admissible) level iff m and $n+1$ are co-prime. This condition implies that k is boundary principal admissible if and only if ℓ is.

Theorem 6.2.5. *Let $\mathfrak{g} = \mathfrak{sl}_{mn}$ and $\mathfrak{s} = \mathfrak{sl}_m$. At boundary principal admissible levels*

$$k + h_{\mathfrak{g}}^{\vee} = \frac{h_{\mathfrak{g}}^{\vee}}{n+1} \quad \text{and} \quad \ell + h_{\mathfrak{s}}^{\vee} = \frac{h_{\mathfrak{s}}^{\vee}}{n+1}$$

there exists an isomorphism

$$\mathcal{V}_{\ell}(\mathfrak{s}) \xrightarrow{\sim} \mathcal{W}_k(\mathfrak{g}, \iota(f)).$$

Proof. There exists an injective map $\kappa : \mathcal{V}_{\ell}(\mathfrak{s}) \hookrightarrow \mathcal{W}_k(\mathfrak{g}, \iota(f))$. It is shown in Appendix B.1 that the characters $ch_{\mathcal{W}_k(\mathfrak{g}, \iota(f))}(\tau, z)$ and $ch_{\mathcal{V}_{\ell}(\mathfrak{s})}(\tau, z, t)$ are equal up to multiplication by $aq^b u^c$ for $a, b, c \in \mathbb{C}$. Since both vertex algebras share the same vacuum vector it follows that the characters are equal, thereby showing that κ is surjective. The statement follows. \square

6.2.2 Type B

We look at the sequence of embeddings

$$\mathfrak{sl}_2 \hookrightarrow \mathfrak{so}_n \xrightarrow{\iota} \mathfrak{so}_N$$

with the requirement that the first map be the principal and ι the diagonal embedding. The following Lemma will be helpful for the remaining subsection.

Lemma 6.2.6. *There exists a $\mathfrak{so}_n \otimes \mathfrak{so}_m$ -module isomorphism*

$$\mathfrak{so}_N \cong \mathfrak{so}_n \otimes \text{Sym}^2(\rho_{\omega_1}^{\mathfrak{so}_m}) \oplus \text{Sym}^2(\rho_{\omega_1}^{\mathfrak{so}_n}) \otimes \mathfrak{so}_m. \quad (6.2.4)$$

Proof. It is immediate that the tensor product $\rho_{\omega_1}^{\mathfrak{so}_N} \otimes \rho_{\omega_1}^{\mathfrak{so}_N}$ splits into the direct sum of its symmetric and anti-symmetric part with the latter being isomorphic to the adjoint representation. This together with the $\mathfrak{so}_n \otimes \mathfrak{so}_m$ -module isomorphism $\rho_{\omega_1}^{\mathfrak{so}_N} \cong \rho_{\omega_1}^{\mathfrak{so}_n} \otimes \rho_{\omega_1}^{\mathfrak{so}_m}$ yields the result. \square

In the remaining subsection we assume that \mathfrak{so}_n is of type B and write $n = 2d + 1$.

Lemma 6.2.7. *The central charge of $W_k(\mathfrak{so}_N, \iota(f))$ equals*

$$\frac{1}{2} \frac{kN(N-1)}{k+N-2} - \frac{kN}{2} (n^2 - 1) - m^2 d(2d+1)(4d^2 + 2d - 1) + md(8d^2 + 6d - 1).$$

Proof. Restrict the isomorphism from Lemma 6.2.6 to a \mathfrak{so}_n -module isomorphism. Under the principal embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{so}_n$ this induces an \mathfrak{sl}_2 -module isomorphism

$$\mathfrak{so}_N \cong \frac{1}{2} m(m+1) \bigoplus_{\substack{i=3 \\ \text{step 4}}}^{4d-1} \rho_i \oplus \frac{1}{2} m(m-1) \bigoplus_{\substack{i=1 \\ \text{step 4}}}^{4d+1} \rho_i.$$

The element $\iota(x)$ can be defined as in the proof of Lemma 6.2.3. Observe from the above decomposition that $\mathfrak{g}_{\frac{1}{2}} = \emptyset$. and the central charge is obtained in the same way as in the case for type A . The ghost contributions from the modules

$$\bigoplus_{\substack{i=3 \\ \text{step 4}}}^{4d-1} \rho_i \quad \text{and} \quad \bigoplus_{\substack{i=1 \\ \text{step 4}}}^{4d+1} \rho_i$$

respectively equal $2d^2(4d^2 - 3)$ and $2d(4d^3 + 8d^2 + 3d - 1)$. \square

Proposition 6.2.8. *All levels k at which the central charges of $V^\ell(\mathfrak{so}_m)$ and $\mathcal{W}^k(\mathfrak{so}_N, \iota(f))$ coincide are*

$$\frac{h_{\mathfrak{so}_N}^\vee}{n} - h_{\mathfrak{so}_N}^\vee, \quad \frac{h_{\mathfrak{so}_N}^\vee + 1}{n} - h_{\mathfrak{so}_N}^\vee \quad \text{and} \quad \frac{h_{\mathfrak{so}_N}^\vee + 2}{n+1} - h_{\mathfrak{so}_N}^\vee.$$

Proof. As in the case of type A we need to relate the levels of the vertex algebras $V^\ell(\mathfrak{so}_m)$ and $\mathcal{W}^k(\mathfrak{so}_N, \iota(f))$. Note that

$$\text{Sym}^2(\rho_{\omega_1}^{\mathfrak{so}_n}) \cong \mathbf{1}^{\mathfrak{so}_n} \oplus \rho_{2\omega_1}^{\mathfrak{so}_n}$$

and thus there exists a submodule $\mathbf{1}^{\mathfrak{so}_n} \otimes \mathfrak{so}_m \subset \mathfrak{so}_N$ as can be seen from (6.2.4). Using the proof of Lemma 6.2.7 the isomorphism of Lemma 6.2.6 induces a $\mathfrak{sl}_2 \otimes \mathfrak{so}_m$ -module isomorphism

$$\mathfrak{so}_N \cong \bigoplus_{\substack{i=3 \\ \text{step 4}}}^{4d-1} \rho_i \otimes \text{Sym}^2(\rho_{\omega_1}^{\mathfrak{so}_m}) \oplus \bigoplus_{\substack{i=1 \\ \text{step 4}}}^{4d+1} \rho_i \otimes \mathfrak{so}_m.$$

From this it is easy to determine that the number of ghosts that are tensored with the \mathfrak{so}_m -modules $\text{Sym}^2(\rho_{\omega_1}^{\mathfrak{so}_m})$ and \mathfrak{so}_m respectively are $\frac{1}{2}(n-1)^2$ and $\frac{1}{2}(n^2-1)$. Each ghost tensored with the representation $\text{Sym}^2(\rho_{\omega_1}^{\mathfrak{so}_m})$ or \mathfrak{so}_m shifts the level by a factor of $m+2$ or $h_{\mathfrak{so}_m}^\vee = m-2$, respectively. Thus, the levels of the affine vertex algebra and the \mathcal{W} -algebra are related as follows

$$\ell = kn + (m+2)\frac{(n-1)^2}{2} + (m-2)\frac{(n^2-1)}{2}.$$

Equating the central charge of $V^\ell(\mathfrak{so}_m)$ and of $\mathcal{W}^k(\mathfrak{so}_N, \iota(f))$ as it appears in Lemma 6.2.7 yields the result. \square

Observe that among the levels appearing in the previous Lemma is again a level that is boundary principal admissible, provided that $(h_{\mathfrak{so}_N}^\vee, n) = (mn-2, n) = 1$. At this level - boundary principal admissible or otherwise - the level of the affine vertex algebra is $\ell = 0$. The following Theorem therefore trivially implies that $\mathcal{W}_k(\mathfrak{so}_N, \iota(f))$ and $\mathcal{V}_\ell(\mathfrak{so}_m)$ are isomorphic.

Theorem 6.2.9. *Let $\mathfrak{g} = \mathfrak{so}_N$ be of type B and let n be a divisor of N . At boundary principal admissible level $k + h^\vee = \frac{h^\vee}{n}$ the vertex algebra $\mathcal{W}_k(\mathfrak{g}, \iota(f))$ is trivial.*

Proof. It is shown in Appendix B.2 that the character $ch_{\mathcal{W}_k(\mathfrak{g}, \iota(f))}(\tau, z)$ is equal to $aq^b u^c$ for some $a, b, c \in \mathbb{C}$ which implies the statement. \square

6.2.3 Type C

Observe from Table 6.1 that, contrary to the previous subsections, in case of a Lie algebra \mathfrak{g} of type C there exists the option to embed \mathfrak{sl}_2 into Lie algebras of different type. The next two subsections discuss implications of these different embeddings, i.e. they each discuss a sequence of embeddings

$$\mathfrak{sl}_2 \hookrightarrow \mathfrak{g}' \xrightarrow{\iota} \mathfrak{sp}_{2mn}$$

for \mathfrak{g}' either \mathfrak{so}_n or \mathfrak{sp}_{2n} where the first map is the principal and ι is the diagonal embedding. The following Lemma will be helpful for that.

Lemma 6.2.10. *There exists an isomorphism of $\mathfrak{so}_n \otimes \mathfrak{sp}_{2m}$ -modules*

$$\mathfrak{sp}_{2mn} \cong \mathfrak{so}_n \otimes \bigwedge^2 (\rho_{\omega_1}^{\mathfrak{sp}_{2m}}) \oplus \text{Sym}^2(\rho_{\omega_1}^{\mathfrak{so}_n}) \otimes \mathfrak{sp}_{2m}. \quad (6.2.5)$$

Proof. It is immediate that the tensor product $\rho_{\omega_1}^{\mathfrak{sp}_{2mn}} \otimes \rho_{\omega_1}^{\mathfrak{sp}_{2mn}}$ splits into the direct sum of its symmetric and anti-symmetric part with the former being isomorphic to the adjoint representation. This together with the $\mathfrak{so}_n \otimes \mathfrak{sp}_{2m}$ -module isomorphism $\rho_{\omega_1}^{\mathfrak{sp}_{2mn}} \cong \rho_{\omega_1}^{\mathfrak{so}_n} \otimes \rho_{\omega_1}^{\mathfrak{sp}_{2m}}$ yields the result. \square

Embedding into \mathfrak{so}_n .

Take $\mathfrak{g}' = \mathfrak{so}_n$ to be of type B and let $n = 2d + 1$.

Lemma 6.2.11. *The central charge of $\mathcal{W}^k(\mathfrak{sp}_{2mn}, \iota(f))$ equals*

$$\frac{kmn(2mn+1)}{k+mn+1} - 2kmn(n^2-1) - 4m^2d(8d^3+8d^2-1) - 2md(8d^2+6d-1).$$

Proof. Under the restriction of the isomorphism (6.2.5) to a \mathfrak{so}_n -module isomorphism the principal embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{so}_n$ induces a \mathfrak{sl}_2 -module isomorphism

$$\mathfrak{sp}_{2mn} \cong m(2m-1) \bigoplus_{\substack{i=3 \\ \text{step 4}}}^{4d-1} \rho_i \oplus m(2m+1) \bigoplus_{\substack{i=1 \\ \text{step 4}}}^{4d+1} \rho_i.$$

The element $\iota(x)$ can be defined as in the proof of Lemma 6.2.7 from which the ghost contributions can be inferred as well. \square

Proposition 6.2.12. *All levels k at which the central charges of $V^\ell(\mathfrak{sp}_{2m})$ and $\mathcal{W}^k(\mathfrak{sp}_{2N}, \iota(f))$ coincide are*

$$\frac{h_{\mathfrak{sp}_{2N}}^\vee}{n} - h_{\mathfrak{sp}_{2N}}^\vee, \quad \frac{h_{\mathfrak{sp}_{2N}}^\vee - \frac{1}{2}}{n} - h_{\mathfrak{sp}_{2N}}^\vee \quad \text{and} \quad \frac{h_{\mathfrak{sp}_{2N}}^\vee - 1}{n+1} - h_{\mathfrak{sp}_{2N}}^\vee.$$

Proof. It is known that $\text{Sym}^2(\rho_{\omega_1}^{\mathfrak{so}_n}) \cong \mathbf{1}^{\mathfrak{so}_n} \oplus \rho_{2\omega_1}^{\mathfrak{so}_n}$ thus showing the existence of a submodule $\mathbf{1}^{\mathfrak{so}_n} \otimes \mathfrak{sp}_{2m} \subset \mathfrak{sp}_{2mn}$ as can be seen from (6.2.5). The principal embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{so}_n$ induces a $\mathfrak{sl}_2 \otimes \mathfrak{sp}_{2m}$ -module isomorphism from Lemma 6.2.10

$$\mathfrak{sp}_{2mn} \cong \bigoplus_{\substack{i=3 \\ \text{step 4}}}^{4d-1} \rho_i \otimes \bigwedge^2 (\rho_{\omega_1}^{\mathfrak{sp}_{2m}}) \oplus \bigoplus_{\substack{i=1 \\ \text{step 4}}}^{4d+1} \rho_i \otimes \mathfrak{sp}_{2m}$$

as can be inferred from the proof of Lemma 6.2.11. A ghost tensored with the representations $\bigwedge^2(\rho_{\omega_1}^{\mathfrak{sp}_{2m}})$ and \mathfrak{sp}_{2m} shifts the level by a factor of $m - 1$ and $h_{\mathfrak{sp}_{2m}}^\vee = m + 1$ respectively. It follows that

$$\ell = kn + (m - 1)\frac{(n - 1)^2}{2} + (m + 1)\frac{(n^2 - 1)}{2}.$$

Equating the central charge of the affine vertex algebra $V^\ell(\mathfrak{sp}_{2m})$ and the central charge of Lemma 6.2.11 yields the result. \square

Observe that among the levels appearing in the previous Lemma is again a level that is boundary principal admissible, provided that $(h_{\mathfrak{sp}_{2mn}}^\vee, n) = (mn + 1, n) = 1$. At this level - boundary principal admissible or otherwise - the level of the affine vertex algebra is $\ell = 0$. The following Theorem therefore trivially implies that $\mathcal{W}_k(\mathfrak{sp}_{2mn}, \iota(f))$ and $\mathcal{V}_\ell(\mathfrak{sp}_{2m})$ are isomorphic.

Theorem 6.2.13. *Let n be odd and a divisor of N . At boundary principal admissible level $k + h^\vee = \frac{h^\vee}{n}$ the vertex algebra $\mathcal{W}_k(\mathfrak{sp}_{2N}, \iota(f))$ is trivial.*

Proof. It is shown in Appendix B.3.1 that the character $ch_{\mathcal{W}_k(\mathfrak{sp}_{2N}, \iota(f))}(\tau, z)$ is equal to $aq^b u^c$ for some $a, b, c \in \mathbb{C}$ which implies the statement. \square

Embedding into \mathfrak{sp}_{2n} .

Let \mathfrak{g}' be \mathfrak{sp}_{2n} . For a uniform presentation of results we assume the labels m and n in Lemma 6.2.10 to be switched in this subsection.

Lemma 6.2.14. *The central charge of $\mathcal{W}^k(\mathfrak{sp}_{2mn}, \iota(f))$ equals*

$$\frac{kmn(2mn + 1)}{k + mn + 1} - 2kmn(4n^2 - 1) - m^2n(8n^3 - 8n^2 + 1) - mn(8n^2 - 6n - 1).$$

Proof. Under the principal embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sp}_{2n}$ the restriction of the isomorphism of Lemma 6.2.10 to a \mathfrak{sp}_{2n} -module isomorphism induces an isomorphism of \mathfrak{sl}_2 -modules

$$\mathfrak{sp}_{2mn} \cong \frac{1}{2}m(m - 1) \bigoplus_{\substack{i=1 \\ \text{step } 4}}^{4n-3} \rho_i \oplus \frac{1}{2}m(m + 1) \bigoplus_{\substack{i=3 \\ \text{step } 4}}^{4n-1} \rho_i.$$

The element $\iota(x)$ can be defined similarly as in the proof of Lemma 6.2.3. The remaining proof proceeds as the proof of Lemma 6.2.7 from which the ghost contribution to the central charge can also be inferred. \square

Proposition 6.2.15. *All levels k at which the central charges of $V^\ell(\mathfrak{so}_m)$ and $\mathcal{W}^k(\mathfrak{sp}_{2N}, \iota(f))$ coincide are*

$$\frac{h_{\mathfrak{sp}_{2N}}^\vee}{2n+1} - h_{\mathfrak{sp}_{2N}}^\vee, \quad \frac{h_{\mathfrak{sp}_{2N}}^\vee - \frac{1}{2}}{2n} - h_{\mathfrak{sp}_{2N}}^\vee \quad \text{and} \quad \frac{h_{\mathfrak{sp}_{2N}}^\vee - \frac{m}{2}}{2n-1} - h_{\mathfrak{sp}_{2N}}^\vee.$$

Proof. It is known that $\mathbf{1}^{\mathfrak{sp}_{2n}} \subset \bigwedge^2(\rho_{\omega_1}^{\mathfrak{sp}_{2n}})$ thus showing the existence of a submodule $\mathbf{1}^{\mathfrak{sp}_{2n}} \otimes \mathfrak{so}_m \subset \mathfrak{sp}_{2mn}$ as can be seen from (6.2.5). The principal embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sp}_{2n}$ induces a $\mathfrak{sl}_2 \otimes \mathfrak{so}_m$ -module isomorphism from Lemma 6.2.10

$$\mathfrak{sp}_{2mn} \cong \bigoplus_{\substack{i=1 \\ \text{step 4}}}^{4n-3} \rho_i \otimes \mathfrak{so}_n \oplus \bigoplus_{\substack{i=3 \\ \text{step 4}}}^{4n-1} \rho_i \otimes \text{Sym}^2(\rho_{\omega_1}^{\mathfrak{so}_n})$$

as can be inferred from the proof of Lemma 6.2.14. A ghost tensored with the representations $\text{Sym}^2(\rho_{\omega_1}^{\mathfrak{so}_m})$ and \mathfrak{so}_m shifts the level by a factor of $m+2$ and $h_{\mathfrak{so}_m}^\vee = m-2$ respectively. It follows that

$$\ell = kn + (m-2)2n(n-1) + (m+2)2n^2.$$

Equating the central charge of the affine vertex algebra $V^\ell(\mathfrak{so}_m)$ and the central charge of Lemma 6.2.14 yields the result. \square

As in the previous cases, observe that among the levels in Prop. 6.2.15 exists a boundary principal admissible level. Independent of whether this level is boundary principal admissible or not, the level ℓ of the affine vertex algebra in this case equals $\frac{h_{\mathfrak{so}_m}^\vee}{2n+1} - h_{\mathfrak{so}_m}^\vee$.

Theorem 6.2.16. *Let $\mathfrak{g} = \mathfrak{sp}_{2mn}$ and $\mathfrak{s} = \mathfrak{so}_m$. At boundary principal admissible levels*

$$k + h_{\mathfrak{g}}^\vee = \frac{h_{\mathfrak{g}}^\vee}{2n+1} \quad \text{and} \quad \ell + h_{\mathfrak{s}}^\vee = \frac{h_{\mathfrak{s}}^\vee}{2n+1}$$

there exists an isomorphism

$$\mathcal{V}_\ell(\mathfrak{s}) \xrightarrow{\sim} \mathcal{W}_k(\mathfrak{g}, \iota(f)).$$

Proof. There exists an injective map $\kappa : \mathcal{V}_\ell(\mathfrak{s}) \hookrightarrow \mathcal{W}_k(\mathfrak{g}, \iota(f))$. It is shown in Appendix B.3.2 that the characters $ch_{\mathcal{W}_k(\mathfrak{g}, \iota(f))}(\tau, z)$ and $ch_{\mathcal{V}_\ell(\mathfrak{s})}(\tau, z, t)$ are equal up to multiplication by $aq^b u^c$ for $a, b, c \in \mathbb{C}$. Since both vertex algebras share the same vacuum vector it follows that the characters are equal, thereby showing that κ is surjective. The statement follows. \square

6.2.4 Type D

We now look at the sequence of embeddings

$$\mathfrak{sl}_2 \hookrightarrow \mathfrak{sp}_{2n} \xrightarrow{\iota} \mathfrak{so}_{4mn}$$

where the first map is the principal and ι is the diagonal embedding.

Lemma 6.2.17. *There exists an isomorphism of $\mathfrak{sp}_{2n} \otimes \mathfrak{sp}_{2m}$ -modules*

$$\mathfrak{so}_{4mn} \cong \mathfrak{sp}_{2n} \otimes \bigwedge^2(\rho_{\omega_1}^{\mathfrak{sp}_{2m}}) \oplus \bigwedge^2(\rho_{\omega_1}^{\mathfrak{sp}_{2n}}) \otimes \mathfrak{sp}_{2m}.$$

Proof. It is immediate that the tensor product $\rho_{\omega_1}^{\mathfrak{so}_{4mn}} \otimes \rho_{\omega_1}^{\mathfrak{so}_{4mn}}$ decomposes into a direct sum of its symmetric and anti-symmetric part. Knowing that $\text{Sym}^2(\rho_{\omega_1}^{\mathfrak{sp}_{2n}}) \cong \mathfrak{sp}_{2n}$ and using the $\mathfrak{sp}_{2n} \otimes \mathfrak{sp}_{2m}$ -module isomorphism $\rho_{\omega_1}^{\mathfrak{so}_{4mn}} \cong \rho_{\omega_1}^{\mathfrak{sp}_{2n}} \otimes \rho_{\omega_1}^{\mathfrak{sp}_{2m}}$ yields the result. \square

Lemma 6.2.18. *The central charge of $\mathcal{W}^k(\mathfrak{so}_{4mn}, \iota(f))$ equals*

$$\frac{k2mn(4mn-1)}{k+4mn-2} - 2kmn(4n^2-1) - 4m^2n(8n^3-8n^2+1) + 2mn(8n^2-6n-1).$$

Proof. Restricting the isomorphism from Lemma 6.2.17 to a \mathfrak{sp}_{2n} -isomorphism and using the \mathfrak{sl}_2 -module decomposition used in the proof of Lemma 6.2.14 yields the \mathfrak{sl}_2 -isomorphism

$$\mathfrak{so}_{4mn} \cong m(2m-1) \bigoplus_{\substack{i=3 \\ \text{step 4}}}^{4n-1} \rho_i \oplus m(2m+1) \bigoplus_{\substack{i=1 \\ \text{step 4}}}^{4n-3} \rho_i.$$

The element $\iota(x)$ can be defined similarly as in Lemma 6.2.14 and the ghost contributions can be inferred from Lemma 6.2.7. \square

Proposition 6.2.19. *All levels k at which the central charges of $V^\ell(\mathfrak{sp}_{2m})$ and $\mathcal{W}^k(\mathfrak{so}_{4N}, \iota(f))$ coincide are*

$$\frac{h_{\mathfrak{so}_{4N}}^\vee}{2n+1} - h_{\mathfrak{so}_{4N}}^\vee, \quad \frac{h_{\mathfrak{so}_{4N}}^\vee + 1}{2n} - h_{\mathfrak{so}_{4N}}^\vee \quad \text{and} \quad \frac{h_{\mathfrak{so}_{4N}}^\vee - 2m}{2n-1} - h_{\mathfrak{so}_{4N}}^\vee.$$

Proof. Restricting the isomorphism of Lemma 6.2.17 to an $\mathfrak{sl}_2 \otimes \mathfrak{sp}_{2m}$ -module isomorphism under the principal embedding $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sp}_{2n}$ yields

$$\mathfrak{so}_{4mn} \cong \bigoplus_{\substack{i=3 \\ \text{step 4}}}^{4n-1} \rho_i \otimes \bigwedge^2(\rho_{\omega_1}^{\mathfrak{sp}_{2m}}) \oplus \bigoplus_{\substack{i=1 \\ \text{step 4}}}^{4n-3} \rho_i \otimes \mathfrak{sp}_{2m}.$$

Ghosts tensored with the representations $\bigwedge^2(\rho_{\omega_1}^{\mathfrak{sp}_{2m}})$ and \mathfrak{sp}_{2m} contribute a shift to the level by $m - 1$ and $h_{\mathfrak{sp}_{2m}}^\vee = m + 1$, respectively. The level of the affine vertex algebra can be written as

$$\ell = kn + (m - 1)2n^2 + (m + 1)2n(n - 1).$$

It is known that $\mathbf{1}^{\mathfrak{sp}_{2n}} \subset \bigwedge^2(\rho_{\omega_1}^{\mathfrak{sp}_{2n}})$ which shows that there exists a submodule $\mathbf{1}^{\mathfrak{sp}_{2n}} \otimes \mathfrak{sp}_{2m} \subset \mathfrak{so}_{4mn}$ thereby explaining the term kn . Equating the central charge of the affine vertex algebra $V^\ell(\mathfrak{sp}_{2m})$ with the central charge from Lemma 6.2.18 yields the result. \square

Theorem 6.2.20. *Let $\mathfrak{g} = \mathfrak{so}_{4mn}$ and $\mathfrak{s} = \mathfrak{sp}_{2m}$. At boundary principal admissible levels*

$$k + h_{\mathfrak{g}}^\vee = \frac{h_{\mathfrak{g}}^\vee}{2n + 1} \quad \text{and} \quad \ell + h_{\mathfrak{s}}^\vee = \frac{h_{\mathfrak{s}}^\vee}{2n + 1}$$

there exists an isomorphism

$$\mathcal{V}_\ell(\mathfrak{s}) \xrightarrow{\sim} \mathcal{W}_k(\mathfrak{g}, \iota(f)).$$

Proof. There exists an injective map $\kappa : \mathcal{V}_\ell(\mathfrak{s}) \hookrightarrow \mathcal{W}_k(\mathfrak{g}, \iota(f))$. It is shown in Appendix B.4 that the characters $ch_{\mathcal{W}_k(\mathfrak{g}, \iota(f))}(\tau, z)$ and $ch_{\mathcal{V}_\ell(\mathfrak{s})}(\tau, z, t)$ are equal up to multiplication by $aq^b u^c$ for $a, b, c \in \mathbb{C}$. Since both vertex algebras share the same vacuum vector it follows that the characters are equal, thereby showing that κ is surjective. The statement follows. \square

Chapter 7

Conclusion

This conclusion discusses possible ways of continuation of the results presented in this work. This is done mainly by posing questions - most of which arising as a natural consequence of these results - which can be addressed in future works.

Mathieu moonshine was discovered through the appearance of a natural decomposition of the elliptic genus of $K3$ surfaces into characters of modules of the $\mathcal{N} = 4$ vertex algebra. This vertex algebra is also the vertex algebra of global sections of the chiral de Rham complex. Given the close connection between $K3$ and Enriques surfaces and that the vertex algebra of global sections of the chiral de Rham complex on the latter was constructed in chapter 4, this begs the question about a further instance of a moonshine phenomenon. We also pose the question anticipated in Remark 4.8.2 and repeat Conjecture 4.7.9.

Question 1a. Does the elliptic genus of complex Enriques surfaces naturally decompose into characters of modules of the vertex algebra of global sections of the chiral de Rham complex such that it hints towards the existence of an infinite graded module of a finite group?

Question 1b. Can the level-rank duality of Theorem 4.8.1 be improved to

$$\text{Com}\left(V^{-n+r}(\mathfrak{sl}_m), \tilde{V}^{-n}(\mathfrak{sl}_m) \otimes L_r(\mathfrak{sl}_m)\right) \stackrel{??}{\cong} \text{Com}\left(V^{-m}(\mathfrak{sl}_n) \otimes L_m(\mathfrak{sl}_r) \otimes \mathcal{H}(1), \tilde{V}^m(\mathfrak{sl}_{r|n})\right)?$$

Conjecture 4.7.9. For $n \geq 3$, aside from the critical levels $k = -2$ and $\ell = -n$, and the degenerate cases given by Theorem 10.1 of [Lin17], all isomorphisms $(\mathcal{C}_k)^{U(1)} \cong \mathcal{D}_\ell(n)$ appear on the following list.

1. $k = -\frac{n}{1+n}$, $k = -\frac{3+n}{2+n}$, $\ell = -n + \frac{2+n}{1+n}$, which has central charge $c = -\frac{3n(3+n)}{(1+n)(2+n)}$.
2. $k = -n$, $k = \frac{3-2n}{-2+n}$, $\ell = -n + \frac{n-2}{n-1}$, which has central charge $c = -\frac{3n(2n-3)}{n-2}$.
3. $k = -\frac{1}{3}(3+n)$, $k = \frac{3-2n}{n-3}$, $\ell = -n + \frac{n}{n-3}$, which has central charge $c = -\frac{(3+n)(2n-3)}{n-3}$.

In chapter 5 a family of lattice vertex superalgebras is introduced. Under the assumption of existence of a conformal field theory interpretation associated to \mathcal{V}_L the following question seems not obvious from the exposition presented here but appears to be naturally motivated by physics.

Question 2a. Is there a Landau-Ginzburg correspondence to \mathcal{V}_L in dimension 3?

Furthermore, we also repeat a conjecture stated in the main text.

Conjecture 5.1.3. *Question 2b.* Let X be a Calabi-Yau d -fold. Then

$$\mathcal{O}_d \cong H^0(X, \Omega^{ch}).$$

Chapter 6 shows examples at which levels the central charge of a rectangular \mathcal{W} -algebra and its maximal affine vertex subalgebra coincide. These findings are summarized in Table 6.2. For these instances, it is further proved that rectangular \mathcal{W} -algebras at boundary principal admissible levels are isomorphic to affine vertex algebras. Note that not all possible rectangular \mathcal{W} -algebras appear in Table 6.2, for instance, an example not covered is when \mathfrak{g} is

of type C and \mathfrak{g}' is of type D . With that in mind we pose two general questions.

Question 3a. At what levels does a rectangular \mathcal{W} -algebra allow a conformal embedding of its maximal universal affine sub vertex algebra?

Question 3b. At what levels is the simple quotient of a rectangular \mathcal{W} -algebra isomorphic to the simple quotient of its maximal universal affine sub vertex algebra?

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Appendix A

Decoupling relations and singular fields

A.1 Decoupling relations

For the convenience of the reader we repeat the chosen conventions.

$$\begin{aligned} Q^+ &= G^{-,2} & Q^- &= G^{+,1} \\ G^+ &= G^{+,2} & G^- &= G^{-,1} \\ U_{n,0} &=: \partial^n J^+ J^- : & V_{n,0} &=: \partial^n G^+ G^- : \\ A_{n,0} &=: \partial^n J^+ G^- : & B_{n,0} &=: \partial^n J^- G^+ : \end{aligned}$$

$$\Sigma_{2n,0}^{(0)\pm} =: \partial^{2n} J^\pm J^\pm : \quad , \quad \Sigma_{n,0}^{(1)\pm} =: \partial^n J^\pm G^\pm : \quad , \quad \Sigma_{2n+1,0}^{(2)\pm} =: \partial^{2n+1} G^\pm G^\pm :$$

All relations below were verified by computer [Thi91]. Note that applying the automorphism θ specified in the proof of Theorem 4.5.4 on the decoupling relation for $\Sigma_{2,0}^{(1)+}$ (see A.1.2) yields a decoupling relation for $\Sigma_{2,0}^{(1)-}$ that holds

at all levels $k \neq 4$.

$$\begin{aligned}
0 = & \frac{k}{3}U_{3,0} - \frac{k}{2}V_{2,0} + \partial U_{2,0} - \frac{1}{2} : U_{2,0}J : + (1+k) \partial V_{1,0} + : V_{1,0}J : \\
& - : A_{1,0}Q^+ : - : B_{1,0}Q^- : - \frac{1}{2} \partial^2 U_{1,0} + : U_{1,0}T : - \frac{1}{2} : U_{1,0} \partial J : \\
& + \frac{1-k}{2} \partial^2 V_{0,0} - : \partial V_{0,0}J : - : V_{0,0}U_{0,0} : + : \partial A_{0,0}Q^+ : - : A_{0,0}B_{0,0} : \\
& + : A_{0,0} \partial Q^+ : + : \partial B_{0,0}Q^- : + : B_{0,0} \partial Q^- : + \frac{1}{6} \partial^3 U_{0,0} + : U_{0,0} \partial T : \\
& + \frac{1+k}{6} \partial^3 T + \frac{1}{2} : \partial^2 T J : - \frac{1}{2} : \partial^2 Q^+ Q^- : - : \partial Q^+ \partial Q^- : \\
& - \frac{1}{2} : Q^+ \partial^2 Q^- : + \frac{3+k}{24} \partial^4 J + \frac{1}{12} : \partial^3 J J : .
\end{aligned} \tag{A.1.1}$$

$$\begin{aligned}
0 = & (4-k) \Sigma_{2,0}^{(1)+} - 6 : J \Sigma_{1,0}^{(1)+} : - 2 : U_{0,0} \Sigma_{0,0}^{(1)+} : - 2 : \partial J \Sigma_{0,0}^{(1)+} : \\
& + 2 : \Sigma_{0,0}^{(0)+} B_{0,0} : + : \partial \Sigma_{0,0}^{(0)+} Q^+ :
\end{aligned} \tag{A.1.2}$$

$$\begin{aligned}
0 = & (16-k)U_{2,0} - (8+k) \partial U_{1,0} - 6 : J U_{1,0} : + \frac{2+k}{2} \partial^2 U_{0,0} + : J \partial U_{0,0} : \\
& - : U_{0,0} U_{0,0} : - : \partial J U_{0,0} : - \frac{k}{6} \partial^3 J - \frac{1}{2} : \partial^2 J J : + : \Sigma_{0,0}^{(0)+} \Sigma_{0,0}^{(0)-} : = 0
\end{aligned} \tag{A.1.3}$$

$$\begin{aligned}
0 = & \frac{16-k}{2} A_{2,0} - 3 : J A_{1,0} : - : U_{0,0} A_{0,0} : - : \partial J A_{0,0} : + 3 : U_{1,0} Q^- : \\
& - : \partial U_{0,0} Q^- : + \frac{1}{2} : \partial^2 J Q^- : + : \Sigma_{0,0}^{(0)+} \Sigma_{0,0}^{(1)-} :
\end{aligned} \tag{A.1.4}$$

$$\begin{aligned}
0 = & \frac{16-k}{2} B_{2,0} + 2(k-4) \partial B_{1,0} + 3 : J B_{1,0} : + (2-k) \partial^2 B_{0,0} - 2 : J \partial B_{0,0} : \\
& - : U_{0,0} B_{0,0} : + \frac{k-1}{3} \partial^3 Q^+ + : J \partial^2 Q^+ : + 2 : U_{0,0} \partial Q^+ : + 3 : U_{1,0} Q^+ : \\
& + : \Sigma_{0,0}^{(1)+} \Sigma_{0,0}^{(0)-} :
\end{aligned} \tag{A.1.5}$$

$$\begin{aligned}
& k \left((6-k)V_{2,0-} : A_{0,0}\partial Q^+ : + : \partial B_{0,0}Q^- : - \frac{\partial^2 Q^+ Q^-}{2} : + : \Sigma_{0,0}^{(1)+}\Sigma_{0,0}^{(1)-} : \right) \\
& + (k-4) \left(\partial U_{2,0+} : \partial T U_{0,0} : + \frac{(k+1)}{6} \partial^3 T^- : J\partial V_{0,0} : - : A_{0,0}B_{0,0} : \right. \\
& + \frac{1}{12} : \partial^3 J J : + \frac{1}{2} : \partial^2 T J : \left. \right) - 4 : J V_{1,0} : + \frac{20-13k}{2} \partial^2 U_{1,0} - 4 : T U_{1,0} : \\
& + (2-k) (: J U_{2,0} : + : \partial J U_{1,0} : + 2 : U_{0,0}V_{0,0} :) + (k^2 - 5k - 4)\partial V_{1,0} \\
& + \frac{4+4k-k^2}{2} \partial^2 V_{0,0} + \frac{7k-10}{3} \partial^3 U_{0,0} + (4-3k) (: A_{1,0}Q^+ : + : B_{1,0}Q^- :) \\
& + \frac{k^2 - 26k + 32}{24} \partial^4 J = 0
\end{aligned} \tag{A.1.6}$$

$$\begin{aligned}
& (32+3k) \left(\frac{k}{2} V_{2,0} - (1+k) \left(\partial V_{1,0} - \frac{\partial^2 V_{0,0}}{2} + \frac{\partial^3 T}{6} \right) - : J V_{1,0} : \right. \\
& - : T U_{1,0} : + : J\partial V_{0,0} : + : U_{0,0}V_{0,0} : + : B_{1,0}Q^- : + : A_{0,0}B_{0,0} : \\
& \left. - \frac{1}{2} : \partial^2 T J : + : A_{1,0}Q^+ : - \frac{5}{6} \partial^3 U_{0,0-} : \partial T U_{0,0} : \right) + \frac{32-7k}{2} : J U_{2,0} : \\
& - (32+11k+2k^2)\partial U_{2,0} + \frac{160+19k+2k^2}{2} \partial^2 U_{1,0} + \frac{32-k}{2} : \partial J U_{1,0} : \\
& + k \left(k : J\partial U_{1,0} : - 2 : U_{0,0}U_{1,0} : - : \partial^2 J U_{0,0} : + : \partial \Sigma_{0,0}^{(0)+}\Sigma_{0,0}^{(0)-} : \right) \\
& - \frac{(8+k)(5k-32)}{24} \partial^4 J - \frac{32+7k}{12} : \partial^3 J J : = 0
\end{aligned} \tag{A.1.7}$$

A.2 Singular fields in $V^k(\mathfrak{n}_4)^{U(1)}$

Level $k = -\frac{5}{2}$

$$\begin{aligned}
& U_{2,0} - V_{1,0} + \frac{1}{2} \partial V_{0,0-} : J V_{0,0} : - \partial U_{1,0} + \frac{3}{4} \partial^2 U_{0,0-} : U_{0,0}U_{0,0} : \\
& - : T U_{0,0} : + : \partial J U_{0,0} : - \frac{1}{2} : J J U_{0,0} : - \frac{5}{8} \partial^2 T + \frac{3}{4} : T T : \\
& + \frac{1}{2} : T \partial J : - \frac{1}{4} : T J J : - \frac{1}{8} \partial^3 J - \frac{1}{8} : \partial J \partial J : + \frac{1}{4} : \partial J J J : \\
& - \frac{1}{16} : J J J J : + \frac{3}{2} : \partial Q^+ Q^- : - \frac{3}{2} : Q^+ \partial Q^- : + : J Q^+ Q^- : \\
& - 2 : Q^+ A_{0,0} : + 2 : Q^- B_{0,0} :
\end{aligned} \tag{A.2.1}$$

Level $k = -\frac{3}{2}$

$$4U_{0,0} - 2T - 2\partial J + : JJ : \quad (\text{A.2.2})$$

$$-4U_{1,0} + 6\partial U_{0,0} + 4 : JU_{0,0} : - 3\partial T - 4 : TJ : + : JJJ : + 2 : Q^+ Q^- : \quad (\text{A.2.3})$$

Level $k = -\frac{4}{3}$

$$\begin{aligned} & \frac{1}{2}U_{2,0} + \frac{2}{3}V_{1,0} - \frac{1}{3}\partial V_{0,0} - \frac{1}{2} : JV_{0,0} : - \frac{1}{2}\partial U_{1,0} - \frac{1}{2}\partial^2 U_{0,0} \\ & -4 : U_{0,0}U_{0,0} : + \frac{16}{3} : TU_{0,0} : + 4 : \partial JU_{0,0} : - 2 : JJU_{0,0} : + \frac{5}{12}\partial^2 T \\ & - \frac{23}{18} : TT : - \frac{8}{3} : T\partial J : + \frac{4}{3} : TJJ : + \frac{37}{24}\partial^3 J - \frac{35}{12} : \partial^2 JJ : \quad (\text{A.2.4}) \\ & - \frac{11}{8} : \partial J\partial J : + \partial JJJ : - \frac{1}{4} : JJJJ : + \frac{4}{3} : \partial Q^+ Q^- : \\ & - \frac{4}{3} : Q^+ \partial Q^- : + \frac{1}{2} : JQ^+ Q^- : - : Q^+ A_{0,0} : + : Q^- B_{0,0} : \end{aligned}$$

Level $k = -\frac{2}{3}$

$$\begin{aligned}
& -\frac{200}{27}V_{2,0} - \frac{32}{3}\partial U_{2,0} + \frac{4}{3} : JU_{2,0} : -\frac{88}{27}\partial V_{1,0} - \frac{88}{9} : JV_{1,0} : \\
& +\frac{92}{9}\partial^2 U_{1,0} + \frac{20}{3} : J\partial U_{1,0} : +\frac{32}{3} : U_{0,0}U_{1,0} : -\frac{200}{9} : TU_{1,0} : \\
& +\frac{8}{3} : \partial JU_{1,0} : +\frac{8}{3} : JJU_{1,0} : -\frac{116}{9}\partial^2 V_{0,0} + \frac{116}{9} : J\partial V_{0,0} : \\
& -\frac{8}{3} : U_{0,0}V_{0,0} : +\frac{112}{9} : TV_{0,0} : +\frac{28}{3} : \partial JV_{0,0} : -\frac{14}{27}\partial^3 U_{0,0} \\
& -\frac{22}{3} : J\partial^2 U_{0,0} : -\frac{16}{3} : \partial U_{0,0}U_{0,0} : +\frac{28}{9} : T\partial U_{0,0} : +\frac{2}{3} : \partial J\partial U_{0,0} : \\
& -\frac{4}{3} : J\partial U_{0,0} : +16 : JU_{0,0}U_{0,0} - 16 : \partial TU_{0,0} : -\frac{112}{3} : TJU_{0,0} : \\
& +\frac{56}{3} : U_{0,0}Q^+Q^- : +\frac{292}{9} : \partial^2 JU_{0,0} : -16 : \partial JJU_{0,0} : \\
& +8 : JJJU_{0,0} : +\frac{116}{27}\partial^3 T - \frac{38}{3} : \partial^2 TJ : +\frac{56}{9} : \partial T\partial J : \\
& +\frac{56}{3} : TTJ : -\frac{112}{9} : TQ^+Q^- : -\frac{742}{27} : T\partial^2 J : +\frac{56}{3} : T\partial JJ : \\
& -\frac{28}{3} : TJJJ : +\frac{88}{9} : A_{1,0}Q^+ : +\frac{88}{9} : B_{1,0}Q^- : -\frac{112}{9} : \partial A_{0,0}Q^+ : \\
& +16 : A_{0,0}B_{0,0} : -\frac{112}{9} : A_{0,0}\partial Q^+ : -\frac{28}{3} : JA_{0,0}Q^+ : \\
& -\frac{112}{9} : \partial B_{0,0}Q^- : -\frac{112}{9} : B_{0,0}\partial Q^- : +\frac{28}{3} : JB_{0,0}Q^- : \\
& +\frac{56}{9} : \partial^2 Q^+Q^- : +\frac{112}{27} : \partial Q^+\partial Q^- : -\frac{56}{9} : J\partial Q^+Q^- : \\
& +\frac{56}{9} : Q^+\partial^2 Q^- : +\frac{56}{9} : JQ^+\partial Q^- : -\frac{28}{3} : \partial JQ^+Q^- : \\
& +\frac{89}{18}\partial^4 J - \frac{17}{9} : \partial^3 JJ : -\frac{113}{9} : \partial^2 J\partial J : +\frac{190}{9} : \partial^2 JJJ : \\
& +\frac{17}{3} : \partial J\partial JJ : -4 : \partial JJJJ : + : JJJJJ :
\end{aligned} \tag{A.2.5}$$

Level $k = -\frac{1}{2}$

$$\begin{aligned}
& 4U_{1,0} - 5V_{0,0} - 2\partial U_{0,0} + 8 : JU_{0,0} : -10 : TJ : \\
& +7\partial^2 J - 4 : \partial JJ : +2 : JJJ : +5 : Q^+Q^- :
\end{aligned} \tag{A.2.6}$$

$$\begin{aligned}
& -3U_{2,0} + 5V_{1,0} + 4\partial U_{1,0} + : JU_{1,0} : -\partial^2 U_{0,0} \\
& - : J\partial U_{0,0} : -8 : U_{0,0}U_{0,0} : +10 : TU_{0,0} : +5 : \partial JU_{0,0} : \\
& -2 : JJU_{0,0} : +\frac{2}{3}\partial^3 J - 4 : \partial^2 JJ : +5 : Q^+A_{0,0} :
\end{aligned} \tag{A.2.7}$$

Level $k = 1$

$$\Sigma_{0,0}^{(0)-} \circ_2 \Sigma_{0,0}^{(0)+} \propto -2U_{0,0} + \partial J + : JJ : \quad (\text{A.2.8})$$

$$\begin{aligned} \Sigma_{0,0,0}^{(0)-} \circ_3 \Sigma_{0,0,0}^{(0)+} &\propto 30U_{1,0} - 18\partial U_{0,0} - 12 : JU_{0,0} : \\ &+ : JJJ : + 9 : \partial JJ : + 4\partial^2 J \end{aligned} \quad (\text{A.2.9})$$

$$\begin{aligned} \Sigma_{0,0,0,0}^{(0)-} \circ_4 \Sigma_{0,0,0,0}^{(0)+} &\propto -480U_{2,0} + 564\partial U_{1,0} - 174\partial^2 U_{0,0} \\ &+ 36 : U_{0,0}U_{0,0} : + 288 : JU_{1,0} : \\ &- 180 : J\partial U_{0,0} : - 36 : JJU_{0,0} : \\ &- 72 : \partial JU_{0,0} : + : JJJJ : + 30 : \partial JJJ : \\ &+ 39 : \partial J\partial J : + 58 : \partial^2 JJ : + 21\partial^3 J \end{aligned} \quad (\text{A.2.10})$$

Level $k = 2$

$$\begin{aligned} \Sigma_{0,0,0}^{(0)-} \circ_3 \Sigma_{0,0,0}^{(0)+} &\propto 12U_{1,0} - 6\partial U_{0,0} - 6 : JU_{0,0} : \\ &+ \partial^2 J + 3 : \partial JJ : + : JJJ : \end{aligned} \quad (\text{A.2.11})$$

$$\begin{aligned} \Sigma_{0,0,0,0}^{(0)-} \circ_4 \Sigma_{0,0,0,0}^{(0)+} &\propto -252U_{2,0} + 276\partial U_{1,0} - 84\partial^2 U_{0,0} \\ &+ 18 : U_{0,0}, U_{0,0} : + 156 : JU_{1,0} : \\ &- 90 : J\partial U_{0,0} : - 24 : JJU_{0,0} : \\ &- 30 : \partial JU_{0,0} : + : JJJJ : \\ &+ 18 : \partial JJJ : + 15 : \partial J\partial J : \\ &+ 25 : \partial^2 JJ : + 11\partial^3 J \end{aligned} \quad (\text{A.2.12})$$

Level $k = 3$

$$\begin{aligned} \Sigma_{0,0,0,0}^{(0)-} \circ_4 \Sigma_{0,0,0,0}^{(0)+} &\propto -90U_{2,0} + 90\partial U_{1,0} + 60 : JU_{1,0} : - 27\partial^2 U_{0,0} \\ &- 30 : J\partial U_{0,0} : + 6 : U_{0,0}U_{0,0} : - 6 : \partial JU_{0,0} : \\ &- 12 : JJU_{0,0} : + 4\partial^3 J + 7 : \partial^2 JJ : \\ &+ 3\partial J\partial J : + 6 : \partial JJJ : + : JJJJ : \end{aligned} \quad (\text{A.2.13})$$

Level $k = 4$

Note that the field $U_{3,0}$ appears in the expression below. Using A.1.1 we see that the singular field induces a decoupling relation for the field $V_{2,0}$ in the simple quotient.

$$\begin{aligned}
\Sigma_{0,0,0,0,0}^{(0)-} \circ_4 \Sigma_{0,0,0,0,0}^{(0)+} \propto & 880U_{3,0} - 1320\partial U_{2,0} + 1080\partial^2 U_{1,0} - 280\partial^3 U_{0,0} \\
& - 120 : U_{1,0}U_{0,0} : + 60 : \partial U_{0,0}U_{0,0} : + 30 : JU_{0,0}U_{0,0} : \\
& - 660 : JU_{2,0} : + 660 : J\partial U_{1,0} : - 240 : J\partial^2 U_{0,0} : \\
& + 60 : \partial JU_{1,0} : - 60 : \partial J\partial U_{0,0} : + 180 : JJU_{1,0} : \\
& - 90 : JJ\partial U_{0,0} : - 20 : JJJU_{0,0} : - 30 : \partial JJU_{0,0} : \\
& + 10 : \partial^2 JU_{0,0} : + : JJJJJ : + 10 : \partial JJJJ : \\
& + 15 : \partial J\partial JJ : + 25 : \partial^2 JJJ : + 10 : \partial^2 J\partial J : \\
& + 55 : \partial^3 JJ : + 51\partial^4 J
\end{aligned} \tag{A.2.14}$$

A.3 Singular fields in $V^k(\mathfrak{n}_4)^{\mathbb{Z}/2\mathbb{Z}}$

Note that some singular fields in this section involve the strong generator $U_{2,0}$ which decouples at all levels $k \neq 16$ (see A.1.3).

Level $k = -\frac{5}{2}$

$$\begin{aligned}
U_{2,0} - V_{1,0} + \frac{1}{2}\partial V_{0,0} - : JV_{0,0} : - \partial U_{1,0} + \frac{3}{4}\partial^2 U_{0,0} - : U_{0,0}U_{0,0} : \\
- : TU_{0,0} : + : \partial JU_{0,0} : - \frac{1}{2} : JJU_{0,0} : - \frac{5}{8}\partial^2 T + \frac{3}{4} : TT : \\
+ \frac{1}{2} : T\partial J : - \frac{1}{4} : TJJ : - \frac{1}{8}\partial^3 J - \frac{1}{8} : \partial J\partial J : + \frac{1}{4} : \partial JJJ : \\
- \frac{1}{16} : JJJJ : + \frac{3}{2} : \partial Q^+ Q^- : - \frac{3}{2} : Q^+ \partial Q^- : + : JQ^+ Q^- : \\
- 2 : Q^+ A_{0,0} : + 2 : Q^- B_{0,0} :
\end{aligned} \tag{A.3.1}$$

Level $k = -\frac{3}{2}$

$$4U_{0,0} - 2T - 2\partial J + : JJ : \tag{A.3.2}$$

$$\begin{aligned}
-4U_{1,0} + 6\partial U_{0,0} + 4 : JU_{0,0} : - 3\partial T \\
- 4 : TJ : + : JJJ : + 2 : Q^+ Q^- :
\end{aligned} \tag{A.3.3}$$

$$-5\Sigma_{1,0}^{(1)+} + 3\partial\Sigma_{0,0}^{(1)+} + : J\Sigma_{0,0}^{(1)+} : + 2 : Q^+\Sigma_{0,0}^{(0)+} : \quad (\text{A.3.4})$$

Level $k = -\frac{4}{3}$

$$\begin{aligned} & -1811U_{2,0} + 12V_{1,0} - 6\partial V_{0,0} - 9 : JV_{0,0} : + 691\partial U_{1,0} \\ & + 630 : JU_{1,0} : - 18 : Q^+A_{0,0} : + 18 : Q^-B_{0,0} : - 44\partial^2U_{0,0} \\ & - 105 : J\partial U_{0,0} : + 33 : U_{0,0}U_{0,0} : + 96 : TU_{0,0} : + 177 : \partial JU_{0,0} : \\ & - 36 : JJU_{0,0} : + \frac{15}{2}\partial^2T - 23 : TT : - 48 : T\partial J : + 24 : TJJ : \quad (\text{A.3.5}) \\ & - 105 : \Sigma_{0,0}^{(0)+}\Sigma_{0,0}^{(0)-} : + \frac{53}{12}\partial^3J - \frac{99}{4} : \partial J\partial J : + 18 : \partial JJJ : \\ & - \frac{9}{2} : JJJJ : + 24 : \partial Q^+Q^- : - 24 : Q^+\partial Q^- : + 9 : JQ^+Q^- : \end{aligned}$$

Level $k = -\frac{1}{2}$

$$4U_{1,0} - 5V_{0,0} - 2\partial U_{0,0} + 8 : JU_{0,0} - 10 : TJ : \quad (\text{A.3.6})$$

$$+ 7\partial^2J - 4 : \partial JJ : + 2 : JJJ : + 5 : Q^+Q^- :$$

$$- 15\Sigma_{1,0}^{(1)+} + 8\partial\Sigma_{0,0}^{(1)+} + 2 : J\Sigma_{0,0}^{(1)+} : + 4 : Q^+\Sigma_{0,0}^{(0)+} : \quad (\text{A.3.7})$$

Level $k = 1$

$$\Sigma_{0,0}^{(0)+} \quad (\text{A.3.8})$$

$$\Sigma_{0,0}^{(0)-} \circ_2 \Sigma_{0,0}^{(0)+} \propto -2U_{0,0} + \partial J + : JJ : \quad (\text{A.3.9})$$

$$\begin{aligned} \Sigma_{0,0,0}^{(0)-} \circ_3 \Sigma_{0,0,0}^{(0)+} & \propto 30U_{1,0} - 18\partial U_{0,0} - 12 : JU_{0,0} : \\ & + : JJJ : + 9 : \partial JJ : + 4\partial^2J \end{aligned} \quad (\text{A.3.10})$$

Level $k = 2$

$$\begin{aligned} \Sigma_{0,0,0}^{(0)-} \circ_3 \Sigma_{0,0,0}^{(0)+} & \propto 12U_{1,0} - 6\partial U_{0,0} - 6 : JU_{0,0} : \\ & + \partial^2J + 3 : \partial JJ : + : JJJ : \end{aligned} \quad (\text{A.3.11})$$

$$\Sigma_{2,0}^{(0)+} - : U_{0,0}\Sigma_{0,0}^{(0)+} : + 2 : \partial J\Sigma_{0,0}^{(0)+} : \quad (\text{A.3.12})$$

Level $k = 3$

$$\begin{aligned} & 4\Sigma_{2,0}^{(0)+} - 2 : U_{0,0}\Sigma_{0,0}^{(0)+} : + 5\partial^2\Sigma_{0,0}^{(0)+} \\ & - 8 : J\partial\Sigma_{0,0}^{(0)+} : - : \partial J\Sigma_{0,0}^{(0)+} : + 3 : JJ\Sigma_{0,0}^{(0)+} : \end{aligned} \quad (\text{A.3.13})$$

Appendix B

Characters of rectangular \mathcal{W} -algebras at boundary principal admissible level

Some additional notation is used in this appendix: The symbol \propto is used to indicate equality up to multiplication by aq^bu^c for $a, b, c \in \mathbb{C}$ where $q = \exp(2\pi i\tau)$ and $u = \exp(2\pi iz)$. For basis-dependent definitions of \mathfrak{g} of any type we use the definitions as they are given in [GW09]. In particular, this allows one to set

$$\iota(x) = \frac{1}{2} \sum_{j=0}^{r-1} \sum_{i=1}^s (s+1-2i) e_{js+i, js+i} \quad (\text{B.0.1})$$

in all cases under consideration. Lastly, $I_d \in M_{d \times d}(\mathbb{C})$ denotes the d -dimensional identity matrix.

B.1 Type A

Using the standard basis $\{\varepsilon_i\}_{i=1}^N$ in \mathfrak{h}^* which acts on the unit matrices via $\varepsilon_i(e_{j,j}) = \delta_{i,j}$ the set of positive roots of \mathfrak{sl}_N is given by $\Delta_+ = \delta_1 \cup \delta_2$ where

$$\begin{aligned} \delta_1 &= \{ \varepsilon_{j_1 n + i_1} - \varepsilon_{j_2 n + i_2} \mid 1 \leq i_1, i_2 \leq n \wedge 0 \leq j_1 < j_2 < m \} \\ \delta_2 &= \{ \varepsilon_{jn+i_1} - \varepsilon_{jn+i_2} \mid 1 \leq i_1 < i_2 \leq n \wedge 0 \leq j < m \}. \end{aligned}$$

Denote the intersection of the centralizer of $\iota(f)$ and the Cartan algebra of \mathfrak{sl}_N by $\mathfrak{h}^{\iota(f)}$. One is quick to see that $\dim(\mathfrak{h}^{\iota(f)}) = m - 1$. Taking $z \in \mathfrak{h}^{\iota(f)}$ we

may write

$$z = \sum_{j=0}^{m-1} \sum_{i=1}^n c_j e_{jn+i, jn+i}$$

or equivalently

$$z = \text{diag}(c_0 I_n, c_1 I_n, \dots, c_{m-1} I_n)$$

under the condition that $\sum_{j=0}^{m-1} c_j = 0$. With this choice of notation it follows that for $\alpha \in \Delta_+$ as given above yields

$$\alpha(z) = c_{j_1} - c_{j_2} \quad \text{and} \quad \alpha(\iota(x)) = -i_1 + i_2$$

It follows that $\Delta_{\frac{1}{2}} = \emptyset$ and

$$\Delta_+^0 = \{ \varepsilon_{j_1 n+i} - \varepsilon_{j_2 n+i} \mid 1 \leq i \leq n \wedge 0 \leq j_1 < j_2 < m \}.$$

In order to compute the character of $\mathcal{W}_k(\mathfrak{g}, \iota(f))$ we use the formular given in 6.1.3 and proceed by determining the products over subsets of the positive roots. Let $x_j = \exp(2\pi i c_j)$.

$$\begin{aligned} & \prod_{\alpha \in \delta_1} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) \propto \\ & \propto \prod_{0 \leq j_1 < j_2 < m} \prod_{r=1}^{\infty} \prod_{i_1, i_2=1}^n \left(1 - \frac{x_{j_2}}{x_{j_1}} q^{\sigma r - i_1 + i_2} \right) \left(1 - \frac{x_{j_1}}{x_{j_2}} q^{\sigma(r-1) + i_1 - i_2} \right) \\ & = \prod_{0 \leq j_1 < j_2 < m} \prod_{r=1}^{\infty} \left\{ \prod_{i=0}^n \left(1 - \frac{x_{j_2}}{x_{j_1}} q^{\sigma r + n - i} \right)^i \left(1 - \frac{x_{j_1}}{x_{j_2}} q^{\sigma(r-1) - n + i} \right)^i \right. \\ & \quad \left. \cdot \prod_{i=1}^n \left(1 - \frac{x_{j_2}}{x_{j_1}} q^{\sigma r - i} \right)^{n-i} \left(1 - \frac{x_{j_1}}{x_{j_2}} q^{\sigma(r-1) + i} \right)^{n-i} \right\} \\ & \propto \prod_{0 \leq j_1 < j_2 < m} \prod_{r=1}^{\infty} \left\{ \prod_{i=1}^{n+1} \left(1 - \frac{x_{j_2}}{x_{j_1}} q^{\sigma r - i} \right)^{i-1} \left(1 - \frac{x_{j_1}}{x_{j_2}} q^{\sigma(r-1) + i} \right)^{i-1} \right. \\ & \quad \left. \cdot \prod_{i=1}^n \left(1 - \frac{x_{j_2}}{x_{j_1}} q^{\sigma r - i} \right)^{n-i} \left(1 - \frac{x_{j_1}}{x_{j_2}} q^{\sigma(r-1) + i} \right)^{n-i} \right\} \\ & \propto \prod_{0 \leq j_1 < j_2 < m} \kappa_{11}(\tau, c_{j_1} - c_{j_2})^{n-1} \kappa_{11}(\sigma\tau, c_{j_1} - c_{j_2}) \end{aligned}$$

$$\prod_{\alpha \in \delta_2} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) \propto$$

$$\begin{aligned}
& \propto \prod_{0 \leq j < m} \prod_{r=1}^{\infty} \prod_{1 \leq i_1 < i_2 \leq n} \left(1 - q^{\sigma r - i_1 + i_2}\right) \left(1 - q^{\sigma(r-1) + i_1 - i_2}\right) \\
& = \prod_{0 \leq j < m} \prod_{r=1}^{\infty} \prod_{i=1}^{n-1} \left(1 - q^{\sigma r + i}\right)^{n-i} \left(1 - q^{\sigma(r-1) - i}\right)^{n-i} \\
& \propto \prod_{0 \leq j < m} \prod_{r=1}^{\infty} \prod_{i=1}^n \left(1 - q^{\sigma r - i}\right)^{i-1} \left(1 - q^{\sigma(r-1) + i}\right)^{i-1} \\
& = \prod_{0 \leq j < m} \prod_{r=1}^{\infty} \prod_{i=1}^n \left(1 - q^{\sigma(r-1) + i}\right)^{n-1} \\
& \propto \left(\frac{\eta(\tau)}{\eta(\sigma\tau)}\right)^{m(n-1)}
\end{aligned}$$

$$\begin{aligned}
\prod_{\alpha \in \Delta_+^0} \kappa_{11}(\tau, \alpha(z)) & \propto \prod_{0 \leq j_1 < j_2 < m} \prod_{r=1}^{\infty} \prod_{i=1}^n \left(1 - \frac{x_{j_2}}{x_{j_1}} q^{\sigma r}\right) \left(1 - \frac{x_{j_1}}{x_{j_2}} q^{\sigma(r-1)}\right) \\
& \propto \prod_{0 \leq j_1 < j_2 < m} \kappa_{11}(\tau, c_{j_1} - c_{j_2})^n
\end{aligned}$$

Putting all together yields

$$ch_{\mathcal{W}_k(\mathfrak{g}, \iota(f))}(\tau, z) \propto \left(\frac{\eta(\sigma\tau)}{\eta(\tau)}\right)^{m-1} \prod_{0 \leq j_1 < j_2 < m} \frac{\kappa_{11}(\sigma\tau, c_{j_1} - c_{j_2})}{\kappa_{11}(\tau, c_{j_1} - c_{j_2})}.$$

Compare this result with (6.1.2) and notice that the right hand side equals $ch_{\mathcal{V}_\ell(\mathfrak{sl}_m)}(\tau, z, 0)$.

B.2 Type B

Let \mathfrak{so}_N be of type B_l . Recall that the set of positive roots is given by

$$\Delta_+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq l\} \cup \{\varepsilon_i \mid 1 \leq i \leq l\}.$$

Take n to be a divisor of N and write $N = nm$. Choose a partition of the set of positive roots $\Delta_+ = \delta \cup \varepsilon$ where

$$\delta = \delta_1^+ \cup \delta_1^- \cup \delta_2^+ \cup \delta_2^- \cup \delta_3^+ \cup \delta_3^- \cup \delta_4^+ \cup \delta_4^- \quad \text{and} \quad \varepsilon = \varepsilon_1 \cup \varepsilon_2$$

with the subsets being defined as follows:

$$\delta_1^\pm = \left\{ \varepsilon_{nj_1+i_1} \pm \varepsilon_{nj_2+i_2} \mid 1 \leq i_1, i_2 \leq n \wedge 0 \leq j_1 < j_2 < \left\lfloor \frac{m}{2} \right\rfloor \right\}$$

$$\begin{aligned}
\delta_2^\pm &= \left\{ \varepsilon_{nj+i_1} \pm \varepsilon_{nj+i_2} \mid 1 \leq i_1 < i_2 \leq n \wedge 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\delta_3^\pm &= \left\{ \varepsilon_{nj+i_1} \pm \varepsilon_{n\lfloor \frac{m}{2} \rfloor + i_2} \mid 1 \leq i_1 \leq n \wedge 1 \leq i_2 \leq \left\lfloor \frac{n}{2} \right\rfloor \wedge 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\delta_4^\pm &= \left\{ \varepsilon_{n\lfloor \frac{m}{2} \rfloor + i_1} \pm \varepsilon_{n\lfloor \frac{m}{2} \rfloor + i_2} \mid 1 \leq i_1 < i_2 \leq \left\lfloor \frac{n}{2} \right\rfloor \right\} \\
\varepsilon_1 &= \left\{ \varepsilon_{nj+i} \mid 1 \leq i \leq n \wedge 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\varepsilon_2 &= \left\{ \varepsilon_{n\lfloor \frac{m}{2} \rfloor + i} \mid 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}.
\end{aligned}$$

The element $\iota(x)$ can be set to equal (B.0.1) with $(r, s) = (m, n)$. Let $z \in \mathfrak{h}^{\iota(f)}$ and write

$$z = \text{diag}(c_0 I_n, \dots, c_{\frac{m-3}{2}} I_n, 0 I_n, -c_{\frac{m-3}{2}} I_n, \dots, -c_0 I_n).$$

Observe that

$$\begin{aligned}
\varepsilon_{nj+i}(\iota(x)) &= \frac{n+1}{2} - i \quad \text{for } i = 1, \dots, n \\
\varepsilon_{nj+i}(z) &= \begin{cases} c_j & \text{for } 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \\ 0 & \text{for } j = \left\lfloor \frac{m}{2} \right\rfloor \end{cases}
\end{aligned}$$

which determines the sets $\Delta_+^0 = \gamma_1^+ \cup \gamma_2^+ \cup \gamma_3^+ \cup \gamma_1^- \cup \gamma_2^- \cup \phi$ and $\Delta_{\frac{1}{2}} = \emptyset$ where the subsets are defined as follows:

$$\begin{aligned}
\gamma_1^+ &= \left\{ \varepsilon_{nj_1+i} + \varepsilon_{nj_2+n+1-i} \mid 1 \leq i \leq n \wedge 0 \leq j_1 < j_2 < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\gamma_2^+ &= \left\{ \varepsilon_{nj+i} + \varepsilon_{nj+n+1-i} \mid 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \wedge 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\gamma_3^+ &= \left\{ \varepsilon_{nj+n+1-i} + \varepsilon_{n\lfloor \frac{m}{2} \rfloor + i} \mid 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \wedge 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\gamma_1^- &= \left\{ \varepsilon_{nj_1+i} - \varepsilon_{nj_2+i} \mid 1 \leq i \leq n \wedge 0 \leq j_1 < j_2 < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\gamma_2^- &= \left\{ \varepsilon_{nj+i} - \varepsilon_{n\lfloor \frac{m}{2} \rfloor + i} \mid 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \wedge 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\phi &= \left\{ \varepsilon_{nj+\frac{n+1}{2}} \mid 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \right\}.
\end{aligned}$$

$$\begin{aligned}
&\prod_{\alpha \in \delta_1^-} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) \propto \\
&\propto \prod_{0 \leq j_1 < j_2 < \left\lfloor \frac{m}{2} \right\rfloor} \prod_{r=1}^{\infty} \prod_{i_1, i_2=1}^n \left(1 - \frac{x_{j_2}}{x_{j_1}} q^{\sigma r - i_1 + i_2}\right) \left(1 - \frac{x_{j_1}}{x_{j_2}} q^{\sigma(r-1) + i_1 - i_2}\right) \\
&= \prod_{0 \leq j_1 < j_2 < \left\lfloor \frac{m}{2} \right\rfloor} \prod_{r=1}^{\infty} \left\{ \prod_{i=0}^{n-1} \left(1 - \frac{x_{j_2}}{x_{j_1}} q^{\sigma r + i}\right)^{n-i} \left(1 - \frac{x_{j_1}}{x_{j_2}} q^{\sigma(r-1) - i}\right)^{n-i} \right\}
\end{aligned}$$

$$\begin{aligned} & \cdot \prod_{i=1}^n \left(1 - \frac{x_{j_2}}{x_{j_1}} q^{\sigma r - i}\right)^{n-i} \left(1 - \frac{x_{j_1}}{x_{j_2}} q^{\sigma(r-1)+i}\right)^{n-i} \Big\} \\ \propto & \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_{j_1} - c_{j_2})^n \end{aligned}$$

$$\begin{aligned} \prod_{\alpha \in \delta_1^+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) & \propto \\ \propto & \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i_1, i_2=1}^n (1 - (x_{j_1} x_{j_2})^{-1} q^{\sigma r + n + 1 - i_1 - i_2}) (1 - x_{j_1} x_{j_2} q^{\sigma(r-1) - n - 1 + i_1 + i_2}) \\ \propto & \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left\{ \prod_{i=1}^n (1 - (x_{j_1} x_{j_2})^{-1} q^{\sigma r - i})^i (1 - x_{j_1} x_{j_2} q^{\sigma(r-1)+i})^i \right. \\ & \cdot \left. \prod_{i=1}^n (1 - (x_{j_1} x_{j_2})^{-1} q^{\sigma r - i})^{n-i} (1 - x_{j_1} x_{j_2} q^{\sigma(r-1)+i})^{n-i} \right\} \\ \propto & \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_{j_1} + c_{j_2})^n \end{aligned}$$

$$\begin{aligned} \prod_{\alpha \in \delta_2^-} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) & \propto \\ \propto & \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{1 \leq i_1 < i_2 \leq n} (1 - q^{\sigma r - i_1 + i_2}) (1 - q^{\sigma(r-1) + i_1 - i_2}) \\ = & \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i=1}^{n-1} (1 - q^{\sigma r + i})^{n-i} (1 - q^{\sigma(r-1) - i})^{n-i} \\ \propto & \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i=1}^{n-1} (1 - q^{\sigma(r-1) + i})^{n-i} (1 - q^{\sigma(r-1) - i})^i \\ \propto & \left(\frac{\eta(\tau)}{\eta(\sigma\tau)} \right)^{n \lfloor \frac{m}{2} \rfloor} \end{aligned}$$

In the evaluation of the next product we write $n = 2d + 1$ for easier readability.

$$\begin{aligned} \prod_{\alpha \in \delta_2^+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) & \propto \\ \propto & \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{1 \leq i_1 < i_2 \leq n} (1 - x_j^{-2} q^{\sigma r + n + 1 - i_1 - i_2}) (1 - x_j^2 q^{\sigma(r-1) - n - 1 + i_1 + i_2}) \end{aligned}$$

$$\begin{aligned}
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i=1}^d \left\{ (1 - x_j^{-2} q^{\sigma r - 2i})^i (1 - x_j^2 q^{\sigma(r-1) + 2i})^i \right. \\
& \quad \cdot (1 - x_j^{-2} q^{\sigma r - (2i+1)})^i (1 - x_j^2 q^{\sigma(r-1) + 2i + 1})^i \\
& \quad \cdot (1 - x_j^{-2} q^{\sigma r - 2d - 2i})^{d+1-i} (1 - x_j^2 q^{\sigma(r-1) + 2d + 2i})^{d+1-i} \\
& \quad \left. \cdot (1 - x_j^{-2} q^{\sigma r - 2d - (2i+1)})^{d-i} (1 - x_j^2 q^{\sigma(r-1) + 2d + 2i + 1})^{d-i} \right\} \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, 2c_j)^{\lfloor \frac{n}{2} \rfloor}
\end{aligned}$$

$$\begin{aligned}
& \prod_{\alpha \in \delta_3^-} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) \propto \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i_1=1}^n \prod_{i_2=1}^{\lfloor \frac{n}{2} \rfloor} (1 - x_j^{-1} q^{\sigma r - i_1 + i_2}) (1 - x_j q^{\sigma(r-1) + i_1 - i_2}) \\
& = \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i_2=1}^{\lfloor \frac{n}{2} \rfloor} \left\{ (1 - x_j^{-1} q^{\sigma r - n + i_2}) (1 - x_j^{-1} q^{\sigma(r-1) + n - i_2}) \right. \\
& \quad \cdot \prod_{i_1=1}^{\lfloor \frac{n}{2} \rfloor} \left((1 - x_j^{-1} q^{\sigma r - i_1 + i_2}) (1 - x_j^{-1} q^{\sigma(r-1) + i_1 - i_2}) \right. \\
& \quad \left. \left. \cdot (1 - x_j^{-1} q^{\sigma r - \lfloor \frac{n}{2} \rfloor - i_1 + i_2}) (1 - x_j^{-1} q^{\sigma(r-1) + \lfloor \frac{n}{2} \rfloor + i_1 - i_2}) \right) \right\} \\
& = \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left\{ \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left((1 - x_j^{-1} q^{\sigma(r-1) + i}) (1 - x_j^{-1} q^{\sigma r - i}) \right. \right. \\
& \quad \cdot (1 - x_j^{-1} q^{\sigma r + i})^{\lfloor \frac{n}{2} \rfloor - i} (1 - x_j q^{\sigma(r-1) - i})^{\lfloor \frac{n}{2} \rfloor - i} \\
& \quad \cdot (1 - x_j^{-1} q^{\sigma r - \lfloor \frac{n}{2} \rfloor + i})^{\lfloor \frac{n}{2} \rfloor - i} (1 - x_j q^{\sigma(r-1) + \lfloor \frac{n}{2} \rfloor - i})^{\lfloor \frac{n}{2} \rfloor - i} \\
& \quad \left. \left. \cdot \prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} \left((1 - x_j^{-1} q^{\sigma r - i})^{\lfloor \frac{n}{2} \rfloor - i} (1 - x_j^{-1} q^{\sigma(r-1) + i})^{\lfloor \frac{n}{2} \rfloor - i} \right. \right. \right. \\
& \quad \left. \left. \left. \cdot (1 - x_j^{-1} q^{\sigma r - \lfloor \frac{n}{2} \rfloor - i})^{\lfloor \frac{n}{2} \rfloor - i} (1 - x_j^{-1} q^{\sigma(r-1) + \lfloor \frac{n}{2} \rfloor + i}) \right) \right)^{\lfloor \frac{n}{2} \rfloor - i} \right\} \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_j)^{\lfloor \frac{n}{2} \rfloor}
\end{aligned}$$

$$\begin{aligned}
& \prod_{\alpha \in \delta_3^+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) \propto \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i_1=1}^n \prod_{i_2=1}^{\lfloor \frac{n}{2} \rfloor} (1 - x_j^{-1} q^{\sigma r + n + 1 - i_1 - i_2}) (1 - x_j q^{\sigma(r-1) - n - 1 + i_1 + i_2}) \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left\{ (1 - x_j^{-1} q^{\sigma(r+1) - i})^i (1 - x_j q^{\sigma(r-2) + i})^i \right. \\
& \quad \cdot (1 - x_j^{-1} q^{\sigma(r+1) - \lfloor \frac{n}{2} \rfloor - i})^{\lfloor \frac{n}{2} \rfloor} (1 - x_j q^{\sigma(r-2) + \lfloor \frac{n}{2} \rfloor + i})^{\lfloor \frac{n}{2} \rfloor} \\
& \quad \left. \cdot (1 - x_j^{-1} q^{\sigma(r+1) - 2\lfloor \frac{n}{2} \rfloor - i})^{\lfloor \frac{n}{2} \rfloor + 1 - i} (1 - x_j q^{\sigma(r-2) + 2\lfloor \frac{n}{2} \rfloor + i})^{\lfloor \frac{n}{2} \rfloor + 1 - i} \right\} \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left\{ \prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} (1 - x_j^{-1} q^{\sigma r - i})^{\lfloor \frac{n}{2} \rfloor} (1 - x_j q^{\sigma(r-1) + i})^{\lfloor \frac{n}{2} \rfloor} \right. \\
& \quad \left. \cdot \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (1 - x_j^{-1} q^{\sigma r - \lfloor \frac{n}{2} \rfloor - i})^{\lfloor \frac{n}{2} \rfloor} (1 - x_j q^{\sigma(r-1) + \lfloor \frac{n}{2} \rfloor + i})^{\lfloor \frac{n}{2} \rfloor} \right\} \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_j)^{\lfloor \frac{n}{2} \rfloor}
\end{aligned}$$

$$\begin{aligned}
& \prod_{\alpha \in \varepsilon_1} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) \propto \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i=1}^n (1 - x_j^{-1} q^{\sigma r + \frac{n+1}{2} - i}) (1 - x_j q^{\sigma(r-1) - \frac{n+1}{2} + i}) \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left\{ \prod_{i=0}^{\frac{n-1}{2}} (1 - x_j^{-1} q^{\sigma(r-1) + i}) (1 - x_j q^{\sigma r - i}) \right. \\
& \quad \left. \cdot \prod_{i=1}^{\frac{n-1}{2}} (1 - x_j^{-1} q^{\sigma r - i}) (1 - x_j q^{\sigma(r-1) + i}) \right\} \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_j)
\end{aligned}$$

$$\prod_{\alpha \in \varepsilon_2} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) \propto \prod_{r=1}^{\infty} \prod_{i=1}^{\frac{n-1}{2}} (1 - q^{\sigma r + \frac{n+1}{2} - i}) (1 - q^{\sigma(r-1) - \frac{n+1}{2} + i})$$

$$\begin{aligned}
& \propto \prod_{r=1}^{\infty} \prod_{i=1}^{\frac{n-1}{2}} (1 - q^{\sigma r - i}) (1 - q^{\sigma(r-1) + i}) \\
& = \prod_{r=1}^{\infty} \prod_{i=1}^{\frac{n-1}{2}} (1 - q^{\sigma(r-1) + i}) \\
& \propto \frac{\eta(\tau)}{\eta(\sigma\tau)}
\end{aligned}$$

$$\begin{aligned}
\prod_{\alpha \in \gamma_1^+} \kappa_{11}(\tau, \alpha(z)) & \propto \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_{j_1} + c_{j_2})^n \\
\prod_{\alpha \in \gamma_2^+} \kappa_{11}(\tau, \alpha(z)) & \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, 2c_j)^{\lfloor \frac{n}{2} \rfloor} \\
\prod_{\alpha \in \gamma_3^+} \kappa_{11}(\tau, \alpha(z)) & \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_j)^{\lfloor \frac{n}{2} \rfloor} \\
\prod_{\alpha \in \gamma_1^-} \kappa_{11}(\tau, \alpha(z)) & \propto \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_{j_1} - c_{j_2})^n \\
\prod_{\alpha \in \gamma_2^-} \kappa_{11}(\tau, \alpha(z)) & \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_j)^{\lfloor \frac{n}{2} \rfloor} \\
\prod_{\alpha \in \phi} \kappa_{11}(\tau, \alpha(z)) & \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_j)
\end{aligned}$$

$$\begin{aligned}
\prod_{\alpha \in \delta_4^-} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) & \propto \prod_{r=1}^{\infty} \prod_{1 \leq i_1 < i_2 \leq \lfloor \frac{n}{2} \rfloor} (1 - q^{\sigma r - i_1 + i_2}) (1 - q^{\sigma(r-1) + i_1 - i_2}) \\
& \propto \prod_{r=1}^{\infty} \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor - 1} (1 - q^{\sigma r - i})^{\lfloor \frac{n}{2} \rfloor - i} (1 - q^{\sigma(r-1) + i})^{\lfloor \frac{n}{2} \rfloor - i}
\end{aligned}$$

For the evaluation of the remaining products two cases are distinguished: Assuming that $\lfloor \frac{n}{2} \rfloor = 2d$ or, equivalently, $n = 4d + 1$ yields

$$\begin{aligned}
\prod_{\alpha \in \delta_4^+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) & \propto \prod_{r=1}^{\infty} \left\{ \prod_{i=1}^d (1 - q^{\sigma r - 2i})^i (1 - q^{\sigma(r-1) + 2i})^i \right. \\
& \quad \cdot \prod_{i=2}^d (1 - q^{\sigma r - (2i-1)})^{i-1} (1 - q^{\sigma(r-1) + 2i-1})^{i-1} \\
& \quad \cdot \prod_{i=1}^{d-1} (1 - q^{\sigma r - 2d - (2i-1)})^{d-i} (1 - q^{\sigma(r-1) + 2d + 2i-1})^{d-i} \left. \right\}
\end{aligned}$$

$$\cdot \prod_{i=1}^{d-1} (1 - q^{\sigma r - 2d - 2i})^{d-i} (1 - q^{\sigma(r-1) + 2d + 2i})^{d-i} \Big\}$$

whereas assuming that $\lfloor \frac{n}{2} \rfloor = 2d + 1$ or, equivalently, $n = 4d + 3$ yields

$$\begin{aligned} \prod_{\alpha \in \delta_4^+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) &\propto \prod_{r=1}^{\infty} \left\{ \prod_{i=1}^d (1 - q^{\sigma r - 2i})^i (1 - q^{\sigma(r-1) + 2i})^i \right. \\ &\quad \cdot \prod_{i=2}^d (1 - q^{\sigma r - (2i-1)})^{i-1} (1 - q^{\sigma(r-1) + 2i-1})^{i-1} \\ &\quad \cdot \prod_{i=1}^d (1 - q^{\sigma r - 2d - (2i-1)})^{d+1-i} (1 - q^{\sigma(r-1) + 2d + 2i-1})^{d+1-i} \\ &\quad \left. \cdot \prod_{i=1}^d (1 - q^{\sigma r - 2d - 2i})^{d+1-i} (1 - q^{\sigma(r-1) + 2d + 2i})^{d+1-i} \right\}. \end{aligned}$$

Counting the exponentials in the individual products for any odd n shows that the products simplify to

$$\prod_{\alpha \in \delta_4^+ \cup \delta_4^-} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) \propto \left(\frac{\eta(\tau)}{\eta(\sigma\tau)} \right)^{\lfloor \frac{n}{2} \rfloor - 1}.$$

Collecting all above products it follows that the character of the \mathcal{W} -algebra reads

$$ch_{\mathcal{W}_k(\mathfrak{so}_{mn}, \iota(f))}(\tau, z) \propto 1.$$

B.3 Type C

B.3.1 Principal embedding into \mathfrak{so}_n

Let $\mathfrak{g} = \mathfrak{sp}_{2mn}$ be of type C_l . Again, the set of positive roots is given by

$$\Delta_+ = \{\varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq l\} \cup \{2\varepsilon_i | 1 \leq i \leq l\}.$$

In order to simplify computations the set of positive roots is partitioned

$$\Delta_+ = \delta_1^+ \cup \delta_1^- \cup \delta_2^+ \cup \delta_2^- \cup \varepsilon$$

with the subsets being defined as follows:

$$\delta_1^\pm = \left\{ \varepsilon_{nj_1+i_1} \pm \varepsilon_{nj_2+i_2} \mid 1 \leq i_1, i_2 \leq n \wedge 0 \leq j_1 < j_2 < m \right\}$$

$$\begin{aligned}\delta_2^\pm &= \left\{ \varepsilon_{nj+i_1} \pm \varepsilon_{nj+i_2} \mid 1 \leq i_1 < i_2 \leq n \wedge 0 \leq j < m \right\} \\ \varepsilon &= \left\{ 2\varepsilon_{nj+i} \mid 1 \leq i \leq n \wedge 0 \leq j < m \right\}\end{aligned}$$

Taking n to be odd, the element $\iota(x)$ can be defined as in (B.0.1) with $(r, s) = (2m, n)$. Let $z \in \mathfrak{h}^{\iota(f)}$ and write

$$z = \text{diag}(c_0 I_n, \dots, c_{m-1} I_n, -c_{m-1} I_n, \dots, -c_0 I_n).$$

Observe that

$$\begin{aligned}\varepsilon_{nj+i}(\iota(x)) &= \frac{n+1}{2} - i & \text{for } i = 1, \dots, n \\ \varepsilon_{nj+i}(z) &= c_j & \text{for } j = 1, \dots, m-1\end{aligned}$$

which determines the sets $\Delta_{\frac{1}{2}} = \emptyset$ and $\Delta_+^0 = \gamma_1^+ \cup \gamma_2^+ \cup \gamma_1^- \cup \phi$ where the subsets are defined as follows:

$$\begin{aligned}\gamma_1^+ &= \left\{ \varepsilon_{nj_1+i} + \varepsilon_{nj_2+n+1-i} \mid 1 \leq i \leq n \wedge 0 \leq j_1 < j_2 < m \right\} \\ \gamma_2^+ &= \left\{ \varepsilon_{nj+i} + \varepsilon_{nj+n+1-i} \mid 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \wedge 0 \leq j < m \right\} \\ \gamma_1^- &= \left\{ \varepsilon_{nj_1+i} - \varepsilon_{nj_2+i} \mid 1 \leq i \leq n \wedge 0 \leq j_1 < j_2 < m \right\} \\ \phi &= \left\{ 2\varepsilon_{nj+\frac{n+1}{2}} \mid 0 \leq j < m \right\}\end{aligned}$$

In order to determine the character of $\mathcal{W}_k(\mathfrak{g}, \iota(f))$ we first consider all the relevant products over the given subsets of the positive roots. Observe that all but two of them have already been determined in B.2. Compared to B.2, similar subsets of Δ_+ and Δ_+^0 are suggestively given the same symbols with the only difference being that $m \mapsto 2m$. In particular, we have

$$\frac{\prod_{\alpha \in \Delta_+ \setminus \varepsilon} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x)))}{\prod_{\alpha \in \Delta_+^0 \setminus \phi} \kappa_{11}(\tau, \alpha(z))} \propto \left(\frac{\eta(\tau)}{\eta(\sigma\tau)} \right)^{nm}$$

Taking $n = 2d + 1$ the remaining products are

$$\prod_{\alpha \in \phi} \kappa_{11}(\tau, \alpha(z)) \propto \prod_{0 \leq j < m} \kappa_{11}(\tau, 2c_j)$$

and

$$\prod_{\alpha \in \varepsilon} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) \propto$$

$$\begin{aligned}
& \propto \prod_{0 \leq j < m} \prod_{r=1}^{\infty} \prod_{i=1}^n (1 - x_j^{-2} q^{\sigma r + n + 1 - 2i}) (1 - x_j^2 q^{\sigma(r-1) - n - 1 + 2i}) \\
& \propto \prod_{0 \leq j < m} \prod_{r=1}^{\infty} \left\{ (1 - x_j^{-2} q^{\sigma(r+1) + 1 - 4d - 2}) (1 - x_j^2 q^{\sigma(r-2) - 1 + 4d + 2}) \right. \\
& \quad \cdot \prod_{i=1}^d \left((1 - x_j^{-2} q^{\sigma(r+1) + 1 - 2i}) (1 - x_j^{-2} q^{\sigma(r-2) - 1 + 2i}) \right. \\
& \quad \quad \cdot (1 - x_j^{-2} q^{\sigma(r+1) + 1 - 2d - 2i}) (1 - x_j^{-2} q^{\sigma(r-2) - 1 + 2d + 2i}) \left. \left. \right) \right\} \\
& \propto \prod_{0 \leq j < m} \prod_{r=1}^{\infty} \left\{ (1 - x_j^{-2} q^{\sigma(r-1) + 1}) (1 - x_j^2 q^{\sigma r - 1}) \right. \\
& \quad \cdot \prod_{i=1}^d \left((1 - x_j^{-2} q^{\sigma r + 1 - 2i}) (1 - x_j^{-2} q^{\sigma(r-1) - 1 + 2i}) \right. \\
& \quad \quad \cdot (1 - x_j^{-2} q^{\sigma r + 2 - 2i}) (1 - x_j^{-2} q^{\sigma(r-1) - 2 + 2i}) \left. \left. \right) \right\} \\
& \propto \prod_{0 \leq j < m} \kappa_{11}(\tau, 2c_j).
\end{aligned}$$

Collecting all products shows that the character is trivial.

$$ch_{\mathcal{W}_k(\mathfrak{sp}_{2mn}, \iota(f))}(\tau, z) \propto 1$$

B.3.2 Principal embedding into \mathfrak{sp}_{2n}

Let $\mathfrak{g} = \mathfrak{sp}_{2mn}$ be of type C_l . Recall that the set of positive roots is given by

$$\Delta_+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq l\} \cup \{2\varepsilon_i \mid 1 \leq i \leq l\}.$$

In order to simplify computations the set of positive roots is partitioned

$$\Delta_+ = \begin{cases} \delta & \text{if } 2|m \\ \delta \cup \varepsilon & \text{otherwise.} \end{cases}$$

where

$$\begin{aligned}
\delta &= \delta_1^+ \cup \delta_1^- \cup \delta_2^+ \cup \delta_2^- \cup \delta_+ \\
\varepsilon &= \varepsilon_1^+ \cup \varepsilon_1^- \cup \varepsilon_2^+ \cup \varepsilon_2^- \cup \varepsilon_+
\end{aligned}$$

with the subsets being defined as follows:

$$\delta_1^\pm = \left\{ \varepsilon_{2nj_1+i_1} \pm \varepsilon_{2nj_2+i_2} \mid 1 \leq i_1, i_2 \leq 2n \wedge 0 \leq j_1 < j_2 < \left\lfloor \frac{m}{2} \right\rfloor \right\}$$

$$\begin{aligned}
\delta_2^\pm &= \left\{ \varepsilon_{2nj+i_1} \pm \varepsilon_{2nj+i_2} \mid 1 \leq i_1 < i_2 \leq 2n \wedge 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\delta_+ &= \left\{ 2\varepsilon_{2nj+i} \mid 1 \leq i \leq 2n \wedge 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\varepsilon_1^\pm &= \left\{ \varepsilon_{2nj+i_1} \pm \varepsilon_{2n\lfloor \frac{m}{2} \rfloor + i_2} \mid 1 \leq i_1 \leq 2n \wedge 1 \leq i_2 \leq n \wedge 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\varepsilon_2^\pm &= \left\{ \varepsilon_{2n\lfloor \frac{m}{2} \rfloor + i_1} \pm \varepsilon_{2n\lfloor \frac{m}{2} \rfloor + i_2} \mid 1 \leq i_1 < i_2 \leq n \right\} \\
\varepsilon_+ &= \left\{ 2\varepsilon_{2n\lfloor \frac{m}{2} \rfloor + i} \mid 1 \leq i \leq n \right\}.
\end{aligned}$$

The element $\iota(x)$ may be set to equal (B.0.1) with $(r, s) = (m, 2n)$. Let $z \in \mathfrak{h}^{\iota(f)}$ and write

$$z = \begin{cases} \text{diag}(c_0 I_{2n}, \dots, c_{\frac{m-2}{2}} I_{2n}, -c_{\frac{m-2}{2}} I_{2n}, \dots, -c_0 I_{2n}) & \text{if } 2|m \\ \text{diag}(c_0 I_{2n}, \dots, c_{\frac{m-3}{2}} I_{2n}, 0 I_{2n}, -c_{\frac{m-3}{2}} I_{2n}, \dots, -c_0 I_{2n}) & \text{otherwise.} \end{cases}$$

Observe that

$$\begin{aligned}
\varepsilon_{2nj+i}(\iota(x)) &= n + \frac{1}{2} - i & \text{for } i = 1, \dots, 2n \\
\varepsilon_{2nj+i}(z) &= \begin{cases} c_j & \text{for } 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \\ 0 & \text{for } j = \left\lfloor \frac{m}{2} \right\rfloor \end{cases}
\end{aligned}$$

which determines the set

$$\Delta_+^0 = \begin{cases} \gamma & \text{if } 2|m \\ \gamma \cup \phi & \text{otherwise.} \end{cases}$$

with

$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$$

$$\phi = \phi_1 \cup \phi_2$$

where the subsets are defined as follows:

$$\begin{aligned}
\gamma_1 &= \left\{ \varepsilon_{2nj_1+i} - \varepsilon_{2nj_2+i} \mid 1 \leq i \leq 2n \wedge 0 \leq j_1 < j_2 < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\gamma_2 &= \left\{ \varepsilon_{2nj_1+i} + \varepsilon_{2nj_2+2n+1-i} \mid 1 \leq i \leq 2n \wedge 0 \leq j_1 < j_2 < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\gamma_3 &= \left\{ \varepsilon_{2nj+i} + \varepsilon_{2nj+2n+1-i} \mid 1 \leq i \leq n \wedge 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\phi_1 &= \left\{ \varepsilon_{2nj+i} - \varepsilon_{2n\lfloor \frac{m}{2} \rfloor + i} \mid 1 \leq i \leq n \wedge 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \right\} \\
\phi_2 &= \left\{ \varepsilon_{2nj+2n+1-i} + \varepsilon_{2n\lfloor \frac{m}{2} \rfloor + i} \mid 1 \leq i \leq n \wedge 0 \leq j < \left\lfloor \frac{m}{2} \right\rfloor \right\}.
\end{aligned}$$

Furthermore, it follows that $\Delta_{\frac{1}{2}} = \emptyset$.

We now proceed with computing the character of $\mathcal{W}_k(\mathfrak{sp}_{2mn}, \iota(f))$ at boundary admissible level using 6.1.3. The partitioning of the sets Δ_+ and Δ_+^0 as given above will be used to compute the products that appear in the character formula. Before computing the character we first determine the products over the subsets of Δ_+ and Δ_+^0 as given above. Let $x_j = \exp(2\pi i c_j)$.

$$\begin{aligned}
& \prod_{\alpha \in \delta_1^-} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) \propto \\
& \propto \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i_1, i_2=1}^{2n} \left(1 - \frac{x_{j_1}}{x_{j_2}} q^{\sigma r - i_1 + i_2}\right) \left(1 - \frac{x_{j_2}}{x_{j_1}} q^{\sigma(r-1) + i_1 - i_2}\right) \\
& = \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left(\prod_{i=0}^{2n-1} \left(1 - \frac{x_{j_1}}{x_{j_2}} q^{\sigma r - i}\right)^{2n-i} \left(1 - \frac{x_{j_2}}{x_{j_1}} q^{\sigma(r-1) + i}\right)^{2n-i} \right. \\
& \quad \cdot \left. \prod_{i=1}^{2n-1} \left(1 - \frac{x_{j_1}}{x_{j_2}} q^{\sigma r + i}\right)^{2n-i} \left(1 - \frac{x_{j_2}}{x_{j_1}} q^{\sigma(r-1) - i}\right)^{2n-i} \right) \\
& \propto \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left(\left(1 - \frac{x_{j_1}}{x_{j_2}} q^{\sigma r}\right) \left(1 - \frac{x_{j_2}}{x_{j_1}} q^{\sigma(r-1)}\right) \right. \\
& \quad \cdot \left. \prod_{i=0}^{2n} \left(1 - \frac{x_{j_1}}{x_{j_2}} q^{\sigma r - i}\right)^{2n-1} \left(1 - \frac{x_{j_2}}{x_{j_1}} q^{\sigma(r-1) + i}\right)^{2n-1} \right) \\
& \propto \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_{j_1} - c_{j_2})^{2n-1} \kappa_{11}(\sigma\tau, c_{j_1} - c_{j_2})
\end{aligned}$$

$$\begin{aligned}
& \prod_{\alpha \in \delta_1^+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) \propto \\
& \propto \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i_1, i_2=1}^{2n} \left(1 - x_{j_1} x_{j_2} q^{\sigma r - i_1 - i_2}\right) \left(1 - (x_{j_1} x_{j_2})^{-1} q^{\sigma(r-1) + i_1 + i_2}\right) \\
& = \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left(\prod_{i=1}^{2n+1} \left(1 - x_{j_1} x_{j_2} q^{\sigma r - i}\right)^{i-1} \left(1 - (x_{j_1} x_{j_2})^{-1} q^{\sigma(r-1) + i}\right)^{i-1} \right. \\
& \quad \cdot \left. \prod_{i=1}^{2n-1} \left(1 - x_{j_1} x_{j_2} q^{\sigma(r-1) - i}\right)^{2n-i} \left(1 - (x_{j_1} x_{j_2})^{-1} q^{\sigma r + i}\right)^{2n-i} \right) \\
& = \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left(\left(1 - x_{j_1} x_{j_2} q^{\sigma r}\right)^{2n} \left(1 - (x_{j_1} x_{j_2})^{-1} q^{\sigma(r-1)}\right)^{2n} \right)
\end{aligned}$$

$$\begin{aligned}
& \cdot \prod_{i=1}^{2n} \left(1 - x_{j_1} x_{j_2} q^{\sigma r - i}\right)^{2n-1} \left(1 - (x_{j_1} x_{j_2})^{-1} q^{\sigma(r-1)+i}\right)^{2n-1} \\
\propto & \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_{j_1} + c_{j_2})^{2n-1} \kappa_{11}(\sigma\tau, c_{j_1} + c_{j_2}) \\
\prod_{\alpha \in \delta_2^-} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) & \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{1 \leq i_1 < i_2 \leq 2n} \left(1 - q^{\sigma r - i_1 + i_2}\right) \left(1 - q^{\sigma(r-1) + i_1 - i_2}\right) \\
& = \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i=1}^{2n-1} \left(1 - q^{\sigma r + i}\right)^{2n-i} \left(1 - q^{\sigma(r-1) - i}\right)^{2n-1} \\
& = \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i=1}^{2n} \left(1 - q^{\sigma(r+1) - i}\right)^{i-1} \left(1 - q^{\sigma(r-2) + i}\right)^{i-1} \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i=1}^{2n} \left(1 - q^{\sigma r - i}\right)^{i-1} \left(1 - q^{\sigma(r-1) + i}\right)^{i-1} \\
& = \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left(\frac{1 - q^r}{1 - q^{\sigma r}}\right)^{2n-1} \\
& = \left(\frac{\eta(\tau)}{\eta(\sigma\tau)}\right)^{(2n-1)\lfloor \frac{m}{2} \rfloor}
\end{aligned}$$

$$\begin{aligned}
\prod_{\alpha \in \delta_2^+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) & \propto \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{1 \leq i_1 < i_2 \leq 2n} \left(1 - x_j^2 q^{\sigma r - i_1 - i_2}\right) \left(1 - x_j^{-2} q^{\sigma(r-1) + i_1 + i_2}\right) \\
& = \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left(\prod_{i=1}^n \left(1 - x_j^2 q^{\sigma r - 2i - 1}\right)^i \left(1 - x_j^{-2} q^{\sigma(r-1) + 2i + 1}\right)^i\right. \\
& \quad \cdot \prod_{i=2}^n \left(1 - x_j^2 q^{\sigma r - 2i}\right)^{i-1} \left(1 - x_j^{-2} q^{\sigma(r-1) + 2i}\right)^{i-1} \\
& \quad \cdot \prod_{i=n+1}^{2n-1} \left(1 - x_j^2 q^{\sigma r - 2i}\right)^{2n-i} \left(1 - x_j^{-2} q^{\sigma(r-1) + 2i}\right)^{2n-i} \\
& \quad \left. \cdot \prod_{i=n+1}^{2n-1} \left(1 - x_j^2 q^{\sigma r - 2i - 1}\right)^{2n-i} \left(1 - x_j^{-2} q^{\sigma(r-1) + 2i + 1}\right)^{2n-i}\right) \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left(\left(1 - x_j^2 q^{\sigma(r-1)}\right)^n \left(1 - x_j^{-2} q^{\sigma r}\right)^n\right)
\end{aligned}$$

$$\begin{aligned}
& \cdot \prod_{i=0}^{n-1} \left(1 - x_j^2 q^{\sigma r - 2i - 1}\right)^{n-1} \left(1 - x_j^{-2} q^{\sigma(r-1) + 2i + 1}\right)^{n-1} \\
& \cdot \prod_{i=1}^n \left(1 - x_j^2 q^{\sigma r - 2i}\right)^{n-1} \left(1 - x_j^{-2} q^{\sigma(r-1) + 2i}\right)^{n-1} \\
\propto & \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left(\left(1 - x_j^2 q^{\sigma r}\right)^n \left(1 - x_j^{-2} q^{\sigma(r-1)}\right)^n \right. \\
& \left. \cdot \prod_{i=1}^{2n} \left(1 - x_j^2 q^{\sigma r - i}\right)^{n-1} \left(1 - x_j^{-2} q^{\sigma(r-1) + i}\right)^{n-1} \right) \\
\propto & \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, 2c_j)^{n-1} \kappa_{11}(\sigma\tau, 2c_j)
\end{aligned}$$

$$\begin{aligned}
\prod_{\alpha \in \delta_+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) & \propto \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i=1}^{2n} \left(1 - x_j^2 q^{\sigma r + 2n + 1 - 2i}\right) \left(1 - x_j^{-2} q^{\sigma(r-1) - 2n - 1 + 2i}\right) \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left(\prod_{i=1}^n \left(1 - x_j^2 q^{\sigma r - 2i}\right) \left(1 - x_j^{-2} q^{\sigma(r-1) + 2i}\right) \right. \\
& \quad \left. \cdot \prod_{i=n+1}^{2n} \left(1 - x_j^2 q^{\sigma r - 2i}\right) \left(1 - x_j^{-2} q^{\sigma(r-1) + 2i}\right) \right) \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \frac{\kappa_{11}(\tau, 2c_j)}{\kappa_{11}(\sigma\tau, 2c_j)}
\end{aligned}$$

$$\begin{aligned}
\prod_{\alpha \in \varepsilon_1^-} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) & \propto \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i_1=1}^{2n} \prod_{i_2=1}^n \left(1 - x_j q^{\sigma r - i_1 + i_2}\right) \left(1 - x_j^{-1} q^{\sigma(r-1) + i_1 - i_2}\right) \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left\{ \prod_{i=1}^n \left(\left(1 - x_j q^{\sigma r - i}\right)^{n-i} \left(1 - x_j^{-1} q^{\sigma(r-1) + i}\right)^{n-i} \right. \right. \\
& \quad \left. \cdot \left(1 - x_j q^{\sigma r - n - i}\right)^{n-i} \left(1 - x_j^{-1} q^{\sigma(r-1) + n + i}\right)^{n-i} \right) \\
& \quad \left. \cdot \prod_{i=0}^n \left(\left(1 - x_j q^{\sigma r + i}\right)^{n-i} \left(1 - x_j^{-1} q^{\sigma(r-1) - i}\right)^{n-i} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& \cdot \left. \left(1 - x_j q^{\sigma r - n + i}\right)^{n-i} \left(1 - x_j^{-1} q^{\sigma(r-1) + n - i}\right)^{n-i} \right\} \\
\propto & \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left\{ \prod_{i=1}^n \left(\left(1 - x_j q^{\sigma r - i}\right)^{n-i} \left(1 - x_j^{-1} q^{\sigma(r-1) + i}\right)^{n-i} \right. \right. \\
& \cdot \left. \left(1 - x_j q^{\sigma r - n - i}\right)^{n-i} \left(1 - x_j^{-1} q^{\sigma(r-1) + n + i}\right)^{n-i} \right) \\
& \cdot \prod_{i=1}^{n+1} \left(1 - x_j q^{\sigma r - n - i}\right)^{i-1} \left(1 - x_j^{-1} q^{\sigma(r-1) + n + i}\right)^{i-1} \\
& \cdot \left. \prod_{i=0}^n \left(1 - x_j q^{\sigma r - i}\right)^i \left(1 - x_j^{-1} q^{\sigma(r-1) + i}\right)^i \right\} \\
\propto & \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \left(\kappa_{11}(\tau, c_j)^n \prod_{i=1}^n \left(1 - x_j q^{\sigma r - n - i}\right)^{-1} \left(1 - x_j^{-1} q^{\sigma(r-1) + n + i}\right)^{-1} \right)
\end{aligned}$$

$$\begin{aligned}
& \prod_{\alpha \in \varepsilon_1^+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) \propto \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i_1=1}^{2n} \prod_{i_2=1}^n \left(1 - x_j q^{\sigma(r+1) - i_1 - i_2}\right) \left(1 - x_j^{-1} q^{\sigma(r-2) + i_1 + i_2}\right) \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \prod_{i=1}^n \left\{ \left(1 - x_j q^{\sigma(r+1) - i}\right)^{i-1} \left(1 - x_j^{-1} q^{\sigma(r-2) + i}\right)^{i-1} \right. \\
& \quad \cdot \left(1 - x_j q^{\sigma(r+1) - n - i}\right)^n \left(1 - x_j^{-1} q^{\sigma(r-2) + n + i}\right)^n \\
& \quad \cdot \left. \left(1 - x_j q^{\sigma(r+1) - 2n - i}\right)^{n+1-i} \left(1 - x_j^{-1} q^{\sigma(r-2) + 2n + i}\right)^{n+1-i} \right\} \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \prod_{r=1}^{\infty} \left(1 - x_j q^{\sigma r}\right)^n \left(1 - x_j^{-1} q^{\sigma(r-1)}\right)^n \\
& \quad \cdot \prod_{i=1}^n \left\{ \left(1 - x_j q^{\sigma r - i}\right)^{n-1} \left(1 - x_j^{-1} q^{\sigma(r-1) + i}\right)^{n-1} \right. \\
& \quad \cdot \left. \left(1 - x_j q^{\sigma r - n - i}\right)^n \left(1 - x_j^{-1} q^{\sigma(r-1) + n + i}\right)^n \right\} \\
& \propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \left\{ \kappa_{11}(\tau, c_j)^n \prod_{i=1}^n \left(1 - x_j q^{\sigma r - i}\right)^{-1} \left(1 - x_j^{-1} q^{\sigma(r-1) + i}\right)^{-1} \right\}
\end{aligned}$$

Observe that combining the last two products simplifies to

$$\begin{aligned} \prod_{\alpha \in \varepsilon_1^- \cup \varepsilon_1^+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) &\propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_j)^{2n-1} \kappa_{11}(\sigma\tau, c_j). \\ \prod_{\alpha \in \varepsilon_2^-} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) &\propto \prod_{r=1}^{\infty} \prod_{1 \leq i_1 < i_2 \leq n} (1 - q^{\sigma r - i_1 + i_2}) (1 - q^{\sigma(r-1) + i_1 - i_2}) \\ &\propto \prod_{r=1}^{\infty} \prod_{i=1}^{n-1} (1 - q^{\sigma r - n - 1 - i})^i (1 - q^{\sigma(r-1) + n + 1 + i})^i \end{aligned}$$

For the product over the subset $\varepsilon_2^+ \subset \Delta_+$ we distinguish two cases: For $n = 2d$ the product is

$$\begin{aligned} \prod_{\alpha \in \varepsilon_2^+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) &\propto \prod_{r=1}^{\infty} \prod_{1 \leq i_1 < i_2 \leq n} (1 - q^{\sigma(r+1) - i_1 - i_2}) (1 - q^{\sigma(r-2) + i_1 + i_2}) \\ &\propto \prod_{r=1}^{\infty} \prod_{i=1}^d \left\{ \begin{aligned} &(1 - q^{\sigma r - (2i+1)})^i (1 - q^{\sigma(r-1) + 2i+1})^i \\ &\cdot (1 - q^{\sigma r - 2i})^{i-1} (1 - q^{\sigma(r-1) + 2i})^{i-1} \\ &\cdot (1 - q^{\sigma r - 2d - (2i+1)})^{d-i} (1 - q^{\sigma(r-1) + 2d + 2i+1})^{d-i} \\ &\cdot (1 - q^{\sigma r - 2d - 2i})^{d-i} (1 - q^{\sigma(r-1) + 2d + 2i})^{d-i} \end{aligned} \right\} \end{aligned}$$

whereas for $n = 2d + 1$ it equals

$$\begin{aligned} \prod_{\alpha \in \varepsilon_2^+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) &\propto \prod_{r=1}^{\infty} \prod_{1 \leq i_1 < i_2 \leq n} (1 - q^{\sigma(r+1) - i_1 - i_2}) (1 - q^{\sigma(r-2) + i_1 + i_2}) \\ &\propto \prod_{r=1}^{\infty} \prod_{i=1}^d \left\{ \begin{aligned} &(1 - q^{\sigma r - (2i+1)})^i (1 - q^{\sigma(r-1) + 2i+1})^i \\ &\cdot (1 - q^{\sigma r - 2i})^{i-1} (1 - q^{\sigma(r-1) + 2i})^{i-1} \\ &\cdot (1 - q^{\sigma r - 2d - (2i+1)})^{d+1-i} (1 - q^{\sigma(r-1) + 2d + 2i+1})^{d+1-i} \\ &\cdot (1 - q^{\sigma r - 2d - 2i})^{d+1-i} (1 - q^{\sigma(r-1) + 2d + 2i})^{d+1-i} \end{aligned} \right\}. \end{aligned}$$

In either case, the product over the subset $\varepsilon_2^- \cup \varepsilon_2^+ \subset \Delta_+$ can be seen to simplify by counting the multiplicity of the exponentials that appear in all terms.

$$\prod_{\alpha \in \varepsilon_2^- \cup \varepsilon_2^+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) \propto \left(\frac{\eta(\tau)}{\eta(\sigma\tau)} \right)^{n-1}$$

All remaining products are listed below.

$$\begin{aligned}
\prod_{\alpha \in \varepsilon_+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) &\propto \prod_{r=1}^{\infty} \prod_{i=1}^n \left(1 - q^{\sigma(r+1)-2i}\right) \left(1 - q^{\sigma(r-2)+2i}\right) \\
&\propto \prod_{r=1}^{\infty} \prod_{i=1}^n \left(1 - q^{\sigma r-2i}\right) \left(1 - q^{\sigma(r-1)+2i}\right) \\
&= \prod_{r=1}^{\infty} \prod_{i=1}^n \left(1 - q^{\sigma(r-1)+2i-1}\right) \left(1 - q^{\sigma(r-1)+2i}\right) \\
&\propto \frac{\eta(\tau)}{\eta(\sigma\tau)}
\end{aligned}$$

$$\begin{aligned}
\prod_{\alpha \in \gamma_1} \kappa_{11}(\tau, \alpha(z)) &\propto \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_{j_1} - c_{j_2})^{2n} \\
\prod_{\alpha \in \gamma_2} \kappa_{11}(\tau, \alpha(z)) &\propto \prod_{0 \leq j_1 < j_2 < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_{j_1} + c_{j_2})^{2n} \\
\prod_{\alpha \in \gamma_3} \kappa_{11}(\tau, \alpha(z)) &\propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, 2c_j)^n \\
\prod_{\alpha \in \phi_1} \kappa_{11}(\tau, \alpha(z)) &\propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_j)^n \\
\prod_{\alpha \in \phi_2} \kappa_{11}(\tau, \alpha(z)) &\propto \prod_{0 \leq j < \lfloor \frac{m}{2} \rfloor} \kappa_{11}(\tau, c_j)^n
\end{aligned}$$

The two cases to distinguish now are if m is even or odd. Assuming that $m = 2d$ one obtains

$$ch_{\mathcal{W}_k(\mathfrak{sp}_{2mn}, \iota(f))}(\tau, z) \propto \left(\frac{\eta(\sigma\tau)}{\eta(\tau)}\right)^d \prod_{0 \leq j_1 < j_2 < d} \frac{\kappa_{11}(\sigma\tau, c_{j_1} - c_{j_2}) \kappa_{11}(\sigma\tau, c_{j_1} + c_{j_2})}{\kappa_{11}(\tau, c_{j_1} - c_{j_2}) \kappa_{11}(\tau, c_{j_1} + c_{j_2})}$$

whereas for $m = 2d + 1$ the character is

$$\begin{aligned}
ch_{\mathcal{W}_k(\mathfrak{sp}_{2mn}, \iota(f))}(\tau, z) &\propto \left(\frac{\eta(\sigma\tau)}{\eta(\tau)}\right)^d \prod_{0 \leq j_1 < j_2 < d} \frac{\kappa_{11}(\sigma\tau, c_{j_1} - c_{j_2}) \kappa_{11}(\sigma\tau, c_{j_1} + c_{j_2})}{\kappa_{11}(\tau, c_{j_1} - c_{j_2}) \kappa_{11}(\tau, c_{j_1} + c_{j_2})} \\
&\quad \cdot \prod_{0 \leq j < d} \frac{\kappa_{11}(\sigma\tau, c_j)}{\kappa_{11}(\tau, c_j)}
\end{aligned}$$

Observe that the right hand side is proportional to the character of $\mathcal{V}_\ell(\mathfrak{so}_m)$ at boundary principal admissible level. The difference in characters reflects the difference of the root systems of type B and type D .

B.4 Type D

Let \mathfrak{so}_N be of type D_l . Recall that the set of positive roots is given by

$$\Delta_+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq l\}.$$

In the case at hand we take $N = 4nm$ and choose a partition of the set of positive roots Δ_+ with the subsets being defined as follows:

$$\begin{aligned} \delta_1^\pm &= \left\{ \varepsilon_{2nj_1+i_1} \pm \varepsilon_{2nj_2+i_2} \mid 1 \leq i_1, i_2 \leq 2n \wedge 0 \leq j_1 < j_2 < m \right\} \\ \delta_2^\pm &= \left\{ \varepsilon_{2nj+i_1} \pm \varepsilon_{2nj+i_2} \mid 1 \leq i_1 < i_2 \leq 2n \wedge 0 \leq j < m \right\} \end{aligned}$$

Set $\iota(x)$ as in (B.0.1) with $(r, s) = (2m, 2n)$. Let $z \in \mathfrak{h}^{\iota(f)}$ and write

$$z = \text{diag}(c_0 I_{2n}, \dots, c_{m-1} I_{2n}, -c_{m-1} I_{2n}, \dots, -c_0 I_{2n}).$$

Observe that

$$\begin{aligned} \varepsilon_{2nj+i}(\iota(x)) &= n + \frac{1}{2} - i & \text{for } i = 1, \dots, 2n \\ \varepsilon_{2nj+i}(z) &= c_j & \text{for } j = 0, \dots, m-1 \end{aligned}$$

which determines the sets $\Delta_+^0 = \gamma_1 \cup \gamma_2 \cup \gamma_3$ and $\Delta_{\frac{1}{2}} = \emptyset$ where the subsets are defined as follows:

$$\begin{aligned} \gamma_1 &= \left\{ \varepsilon_{2nj_1+i} - \varepsilon_{2nj_2+i} \mid 1 \leq i \leq 2n \wedge 0 \leq j_1 < j_2 < m \right\} \\ \gamma_2 &= \left\{ \varepsilon_{2nj_1+i} + \varepsilon_{2nj_2+2n+1-i} \mid 1 \leq i \leq 2n \wedge 0 \leq j_1 < j_2 < m \right\} \\ \gamma_3 &= \left\{ \varepsilon_{2nj+i} + \varepsilon_{2nj+2n+1-i} \mid 1 \leq i \leq n \wedge 0 \leq j < m \right\} \end{aligned}$$

Before determining the character of the \mathcal{W} -algebra we compute the products over the given subsets of the positive roots. Observe that all of them have been determined in B.3.2 where the relevant subsets are suggestively denoted by the same symbols. The products are as follows:

$$\begin{aligned} \prod_{\alpha \in \delta_1^+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) &\propto \prod_{0 \leq j_1 < j_2 < m} \kappa_{11}(\tau, c_{j_1} - c_{j_2})^{2n-1} \kappa(\sigma\tau, c_{j_1} - c_{j_2}) \\ \prod_{\alpha \in \delta_1^-} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) &\propto \prod_{0 \leq j_1 < j_2 < m} \kappa_{11}(\tau, c_{j_1} + c_{j_2})^{2n-1} \kappa(\sigma\tau, c_{j_1} + c_{j_2}) \end{aligned}$$

$$\begin{aligned}
\prod_{\alpha \in \delta_2^+} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) &\propto \left(\frac{\eta(\tau)}{\eta(\sigma\tau)} \right)^{(2n-1)m} \\
\prod_{\alpha \in \delta_2^-} \kappa_{11}(\sigma\tau, \alpha(z - \tau\iota(x))) &\propto \prod_{0 \leq j < m} \kappa_{11}(\tau, 2c_j)^{n-1} \kappa(\sigma\tau, 2c_j) \\
\prod_{\alpha \in \gamma_1} \kappa_{11}(\tau, \alpha(z)) &\propto \prod_{0 \leq j_1 < j_2 < m} \kappa_{11}(\tau, c_{j_1} - c_{j_2})^{2n} \\
\prod_{\alpha \in \gamma_2} \kappa_{11}(\tau, \alpha(z)) &\propto \prod_{0 \leq j_1 < j_2 < m} \kappa_{11}(\tau, c_{j_1} + c_{j_2})^{2n} \\
\prod_{\alpha \in \gamma_2} \kappa_{11}(\tau, \alpha(z)) &\propto \prod_{0 \leq j < m} \kappa_{11}(\tau, 2c_j)^n
\end{aligned}$$

The character reads

$$\begin{aligned}
ch_{\mathcal{W}_k(\mathfrak{so}_{4mn}, \iota(f))}(\tau, z) &\propto \left(\frac{\eta(\sigma\tau)}{\eta(\tau)} \right)^m \prod_{0 \leq j < m} \frac{\kappa_{11}(\sigma\tau, 2c_j)}{\kappa_{11}(\tau, 2c_j)} \\
&\cdot \prod_{0 \leq j_1 < j_2 < m} \frac{\kappa_{11}(\sigma\tau, c_{j_1} - c_{j_2}) \kappa_{11}(\sigma\tau, c_{j_1} + c_{j_2})}{\kappa_{11}(\tau, c_{j_1} - c_{j_2}) \kappa_{11}(\tau, c_{j_1} + c_{j_2})}.
\end{aligned}$$