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Natural constructions of some generalised Kac-Moody algebras as bosonic strings

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Natï¿½rliche Konstruktionen verallgemeinerter Kac-Moody Algebren als bosonische Strings

Zusammenfassung

Wir konstruieren die Zustandsrï¿½ume einiger bosonischer Strings als BRST-Kohomologiegruppe. Diese besitzen die Struktur einer verallgemeinerten Kac-Moody Algebra.
Dazu betrachten wir meromorphe konforme Feldtheorien von zentraler Ladung 24 mit lediglich einem Primï¿½rfeld. Dies sind mit einfachen Strï¿½men erweiterte Wess-Zumino-Witten Theorien, genauer sind es Hï¿½chstgewichtsdarstellungen von affinen Lie Algebren vom Typ $\widehat{A}_{p-1,p}$, wobei $p = 2, 3, 5$ bzw. 7 ist. Wir nehmen die Existenz der Struktur einer konformen Feldtheorie auf den zugrunde liegenden Vektorrï¿½umen an, im Fall $p = 2$ wurde sie bereits gezeigt. Es gibt Beweisideen fï¿½r die ï¿½brigen Fï¿½lle. Das Tensorprodukt von jeder dieser Theorien mit derjenigen der hyperbolischen Ebene (zentrale Ladung 2) hat zentrale Ladung 26. Daher ist der BRST-Formalismus anomaliefrei und liefert eine verallgemeinerte Kac-Moody Algebra als BRST-Kohomologiegruppe vom Grad eins. Diese Algebra interpretiert man als Zustandsraum anomaliefreier bosonischer Strings auf einem Orbifold kompaktifiziert, dessen Impulse auf einem Gitter liegen. Damit wir den BRST-Formalismus anwenden kï¿½nnen, mï¿½ssen wir die Zustandssumme, welche durch Wess-Zumino-Witten Energiespektren gegeben ist, als Summe ï¿½ber das Kompaktifizierungsgitter schreiben. Unser Hauptresultat ist eine Methode, die alle vier Fï¿½lle auf analoge Art und Weise beschreibt.

Natural constructions of some generalised Kac-Moody algebras as bosonic strings

Abstract

We construct the spaces of states of certain bosonic strings as the BRST-cohomology group. These have the structure of a generalised Kac-Moody algebra.
For this we consider meromorphic conformal field theories of central charge 24 with just one primary field. These theories are Wess-Zumino-Witten models extended by simple currents, strictly speaking they are highest weight representations of affine Lie algebras of type $\widehat{A}_{p-1,p}$, where $p = 2, 3, 5, 7$. We assume that the underlying vector spaces have the structure of a conformal field theory. Proofs are partially done and conjectured for the remaining cases. The tensor product of these theories with the conformal field theory of the hyperbolic plane has central charge 26. This allows the application of the BRST-formalism and yields a generalised Kac-Moody algebra as the degree one BRST-cohomology group. This algebra can be interpreted as the space of states of an anomaly

free bosonic string which is compactified on an orbifold whose momenta lie on a lattice. The application of the BRST-formalism requires the partition function, which is given by the Wess-Zumino-Witten energy spectra, to be rewritten as a sum over the compactification lattice. Our main result is a method describing the four cases analogously.

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Chapter 1

Introduction

The goal of this diploma thesis is to construct anomaly free bosonic strings moving on orbifolds of dimensions 18, 14, 10 and 8.

String theory is a quantum theory of a one-dimensional object, called a string. These theories became popular because string theory predicts a massless spin 2 particle which can be interpreted as the graviton. Since it also contains the gauge bosons of the other fundamental interactions one hopes that string theory could be a theory of all fundamental interactions.

As the string propagates in space-time, it sweeps out a world sheet that is the generalisation of a world line of a point particle. A key property of string theory is the invariance of the two-dimensional world sheet metric of the string under conformal transformations. Hence it can be described by a two-dimensional conformal field theory. Two dimensional conformal field theory is special, since its symmetry group is infinite and the generators of the symmetry group have the structure of a Virasoro algebra. The central element of the Virasoro algebra acts as a number and is called the central charge.

In this thesis we apply the BRST-formalism (Becchi-Rouet-Stora-Tyutin) to certain meromorphic conformal field theories of central charge 24. The BRST-cohomology group of degree one has the structure of a generalised Kac-Moody algebra and can be interpreted as the space of physical states of a bosonic string. These theories are anomaly free and the string moves in 18 (respectively 14, 10, 8) dimensional space-time.

The BRST-method is a general method for the quantisation of the fields in gauge theory. For the quantisation procedure one considers the symmetries of the Lagrangian. One requires that the Lagrangian is invariant under the symmetry group. The BRST-transformation acts on the classical Lagrangian as the gauge transformation and leaves the total Lagrangian unchanged. Then one applies the canonical quantisation procedure. By Noether's theorem any symmetry gives a conserved current and the symmetry is generated by the corresponding charge Q. The BRST-formalism requires $Q^2 = 0$, and the charge Q allows a cohomology decomposition of the space of states. It turns out that the space of physical states is the space of those states $|\psi\rangle$ which are annihilated by Q modulo the states $Q\,|\psi'\rangle$.

One can apply the BRST-formalism to string theory. It turns out that the procedure is only anomaly free if the central charge of the Virasoro algebra is 26 because the central charge of the conformal field theory of the ghosts is -26.

Otherwise $Q\frac{1}{2} \neq 0$ and the BRST-formalism breaks down. In flat Minkowski space-time central charge $c = 26$ means that the dimension of space-time is 26. In spaces with arbitrary metric the dimension is less or equal 26. We construct 18, 14, 10 and 8 dimensional string theories.

We are interested in the anomaly free construction of a bosonic string theory out of the underlying conformal field theory. Candidates for the conformal field theories are given in [S3]. All but one of the suggested theories listed there have the extended symmetry of certain Kac-Moody algebras. Conformal field theories with such a symmetry are called Wess-Zumino-Witten theories. We take four candidates listed in [S3] which are Wess-Zumino-Witten theories of type $\widehat{A}$ and construct the bosonic strings. The strings are compactified on an orbifold. Before we can apply the BRST-formalism we have to rewrite the partition function of the conformal field theories which are given by Wess-Zumino-Witten energy spectra as a sum over the compactification lattice.

From a mathematical point of view the procedure is interesting because the space of physical states has the structure of a generalised Kac-Moody algebra.

Schellekens lists meromorphic conformal field theories of central charge 24 generated by just one primary field in [S3]. We will consider the cases of the list which are highest weight representations of highest weights of affine Lie algebras of type $\widehat{A}_{p-1}$ of level p, where p is a prime number. There are two problems to be solved before applying the BRST-construction. First, the partition functions are computed in terms of Wess-Zumino-Witten characters extended by simple currents. In this formulation the compactification lattice is not obvious. This problem will be solved in this thesis. For two of the four problems under consideration this has already be done in [HSch] and [Kl]. We provide a general method, using results from the theory of modular forms, which covers all the four theories. This method can be used to construct all bosonic string theories listed in [S3]. The second problem is that only in one case it is proven that the highest weight representations suggested by Schellekens can be provided with the structure of a conformal field theory [DGM]. Proofs for the remaining cases are conjectured in [M2]. We will assume this to be true.

The main result of this thesis is that the character (partition function) of the highest weight representation of type $\widehat{A}^{48/(p^2-1)}_{p-1,p}$, $p = 2, 3, 5$ or 7 can be rewritten as a sum

$$\chi_V = \sum_{\lambda \in N'/N} f_\lambda(\tau)\, \vartheta_\lambda(\tau, \mathbf{z})$$

over the grading lattice N of genus

$$II_{2m,0}(p^{\epsilon_p(m+2)}), \qquad \epsilon_p = + \text{ for } p = 2, 5, 7 \text{ and } \epsilon_p = - \text{ for } p = 3,$$

and minimal norm 4, except for $p = 7$ it is 6. The lattice N is the unique lattice in its genus of maximal minimal norm. The ϑ_λ are theta functions of the lattice N and the coefficients f_λ which give the degeneracy of the spectrum of the string are

$$f_\lambda = \begin{cases} h(\tau) + g_0(\tau) & \text{if } \lambda = 0 \\ g_k(\tau) & \text{if } \lambda \neq 0 \text{ and } -\lambda^2/2 \equiv k/p \bmod \mathbb{Z} \end{cases} \tag{1.0.1}$$

where

$$h(\tau) = 1/(\eta(\tau)\eta(p\tau))^m = q^{-1} + m + \dots$$

and the $g_k(\tau)$ are the T-invariant parts of $h(\tau/p)$ and $\eta(\tau)$ is the Dedekind eta-function. We found three different methods for the proof of (1.0.1). The f_λ are polynomials in certain modular functions called string functions. The first method requires the knowledge of these functions. They were only known for the cases $p = 2$ and $p = 3$. We calculate them for $p = 5$. The number of distinct string functions of type $\widehat{A}_{p-1,p}$ is of order p^{p-1}, hence the problem is not tractable for $p = 7$. Thus we need a different method. We find two more methods using the theory of modular forms. The calculation of the coefficients of the string functions is laborious. For the new methods we only need to know the first coefficient. We shortly explain the most general method. We show that the f_λ, as well as $h(\tau)$ and its T-invariant parts, transform under the same Weil representation. Then we use the fact that a modular form of negative weight for a Weil representation, which is holomorphic at the cusps, must be zero. Hence, we only have to verify that the f_λ transform under a certain Weil representation and we have to calculate its expansion at the cusps. This procedure can also be applied to the remaining cases of Schellekens list.

We apply the BRST-formalism to the conformal field theory $V \otimes V_{I\!I_{1,1}} \otimes V_G$. V is the conformal field theory in [S3] with spin-1 algebra $\widehat{A}_{p-1,p}^{48/(p^2-1)}$ where $p = 2, 3, 5, 7$, $V_{I\!I_{1,1}}$ is the conformal field theory of the hyperbolic plane and V_G is the conformal field theory of the ghosts which has central charge -26. Hence the total central charge is zero and the theory is anomaly free. The space of physical states of the bosonic strings is the cohomology group of degree one. The hyperbolic extension of the grading lattice is the momentum lattice of the string. It has rank $48/(p+1)+2$ and its physical states have the structure of a generalised Kac-Moody algebra. By the above results on the character of V the number of states of a certain energy is given by the coefficients of the f_λ.

A Lie algebra can sometimes be identified by its denominator identity. So we show that the denominator identity is

$$e^\rho \prod_{\alpha \in L^+} (1 - e^\alpha)^{[h](-\alpha^2/2)} \prod_{\alpha \in L'^+} (1 - e^\alpha)^{[h](-p\alpha^2/2)}$$
$$= \sum_{w \in W} \det(w) w \left(e^\rho \prod_{n>0} (1 - e^{n\rho})^m \prod_{n>0} (1 - e^{pn\rho})^m \right).$$

One might ask the question of the physical relevance of the string theories found. The bosonic string theories found are important tools as models of theories of nature, though they only describe half of the nature, because they contain no fermions. Also, the string theories are not in four-dimensional space-time and the string is compactified on an orbifold. However, we live in four-dimensional space-time, hence we are interested in a four-dimensional theory. We saw that the space of physical states has the structure of a generalised Kac-Moody algebra. There is a candidate for a four-dimensional theory constructed by the automorphism g of the Mathieu group with eta product $\eta_g = \eta(\tau)\eta(23\tau)$. The problem is to find the underlying conformal field theory, since it is most likely generated by more than one primary field. This problem could be covered in a subsequent PhD-thesis.

We describe the chapters of this thesis.

In chapter 2 we introduce string theory and conformal field theory and its relation.

In chapter 3 we give a short overview of the results on lattices and modular forms necessary for this thesis.

The purpose of chapter 4 and 5 is to recall some facts of affine Lie algebras and its representation theory.

Chapter 6 is the heart of the diploma thesis. There we rewrite the characters of the conformal field theories, calculate the corresponding lattices and the coefficients f_λ. We give three different proofs of the identity (1.0.1).

In chapter 7 we use the results of chapter 6 to construct some generalised Kac-Moody algebras in a natural way as bosonic strings.

Chapter 8 summarises the new results obtained in this thesis.

The appendix lists transformation properties of string functions, f_λ and eta products. We also describe some proofs in more detail.

The main results of this thesis which are described in chapter 6 and 7 are summarised in the preprint [CKS] which we intend to submit soon.

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Chapter 2

Conformal field theory

In this thesis we are interested in bosonic string theories. It turns out (see section 2.1) that conformal field theories are therefore of particular interest. Thus this chapter gives an overview on conformal field theory. Further we consider its relationship to vertex algebras, in particular to those giving us bosonic string theories via the BRST-method.

Before we explain conformal field theory in general we start with an introduction of bosonic string theory, since this is the topic of our main interest. Then we give another example of conformal field theory, the Ising model.

2.1 Bosonic string theory

The simplest string theory is the bosonic string. Since it does not contain space-time fermions but contains a particle with imaginary mass, the bosonic string is not a candidate for a theory of nature. Still it is useful. There are many references, a short introduction is [S1] and a standard textbook is [P].

This section is organised as follows. We are interested in a quantum string theory. So we first look how we can describe a string classical. Then one quantises the theory and by considering all gauge symmetries there should be a theory without unphysical states. It turns out that the physical states of a bosonic string theory are the BRST-invariant states of a conformal field theory with Virasoro algebra of central charge 26.

As common practise we use the Einstein convention. First we consider a classical zero-dimensional object, a point particle. The particle moves in D flat space-time dimensions, the metric is

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1).$$

The motion of such a particle can be described by giving its position at any time. The problem is that this does not allow a covariant description of the particle. Therefore one introduces a separate variable τ parametrising its world line. The parametrisation is arbitrary and a different parametrisation of the same path is physically equivalent, i.e. any physical quantity should be invariant under the parametrisation. In order to describe the motion of the particle we want an action which is Poincarï¿½ invariant, e.g. the length of the world line

$$S_{\mathrm{PP}} = -m \int \mathrm{d}\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu}.$$

The dot denotes a τ-derivative.

The same idea can be applied to string theory. Instead of lines we consider surfaces, called world sheets, and the embedding in space-time is given by $X^\mu(\tau,\sigma)$. The purpose of σ is the description of the direction of the string and τ parametrises the time-like direction. Now the action is proportional to the surface area of the world sheet. In order to express this action in terms of $X^\mu(\tau,\sigma)$, define the induced metric $h_{ab} = \partial_a X^\mu \partial_b X_\mu$ where indices a, b run over values (τ,σ). Then the Nambu-Goto action is

$$S_{\mathrm{NG}} = -\frac{1}{2\pi\alpha'} \int_{\text{world sheet}} \mathrm{d}\tau d\sigma \sqrt{-\det\ h_{ab}}.$$

This action has the unpleasant property that it contains a square root. By introducing an independent world sheet metric $\gamma_{\alpha\beta}(\tau,\sigma)$ with Lorentzian signature $(-,+)$ we get a different but classical equivalent action, the Polyakov action (γ denotes the determinant of the world sheet metric)

$$S_P = -\frac{1}{2\pi\alpha'} \int_{\text{world sheet}} \mathrm{d}\tau \mathrm{d}\sigma \sqrt{-\gamma}\gamma^{ab} \partial_a X^\mu \partial_b X_\mu. \tag{2.1.1}$$

This action has three symmetries.

- Poincarï¿½ invariance in D dimensions
- Reparametrisation invariance in 2 dimensions. It is also sometimes called diffeomorphism invariance and in fact it is a general coordinate transformation in two dimensions:

$$\frac{\partial \sigma'^\kappa}{\partial \sigma'^\alpha} \frac{\partial \sigma'^\delta}{\partial \sigma'^\beta} \gamma'_{\kappa\delta}(\tau'(\tau,\sigma), \sigma'(\tau,\sigma)) = \gamma_{\alpha\beta}(\tau,\sigma). \tag{2.1.2}$$

- Weyl invariance. The Polyakov action is invariant under local rescalings of $\gamma_{\alpha\beta}$

$$\gamma'_{\alpha\beta} = e^{2\omega(\tau,\sigma)} \gamma_{\alpha\beta} \tag{2.1.3}$$

  for an arbitrary function $\omega(\tau,\sigma)$ without changing X^μ.

The last two symmetries are redundancies of the two-dimensional theory on the world sheet. This means that the two-dimensional action has less degrees of freedom then it seems to have. This is analogous to gauge symmetry in quantum field theory. One may fix these redundancies by a suitable gauge. We consider the conformal gauge, this means we fix the two-dimensional metric $\gamma_{\alpha\beta}$. Reparametrisation invariance allows us to put 2 components of the metric to zero. Using Weyl invariance another degree of freedom can be removed, so that finally none is left. We use this freedom to transform the metric to the flat Minkowski metric $\eta_{\alpha\beta} = \mathrm{diag}(-1,1)$. This is called conformal gauge. Two-dimensional conformal gauge is special, since only for a two-dimensional world sheet metric all degrees of freedom can be removed by reparametrisation invariance and Weyl invariance. This is a reason why string theory is easier than theories of membranes. The action is now

$$S = -\frac{1}{4\pi\alpha'}\int_{\text{world sheet}} \mathrm{d}\tau d\sigma \eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu. \tag{2.1.4}$$

This is the action of D free bosons. Varying the action with respect to X^μ yields the equation of motion in conformal gauge to be

$$\left[\frac{\partial}{\partial\tau^2} - \frac{\partial}{\partial\sigma^2}\right] X^\mu = 0 \tag{2.1.5}$$

with the general solution

$$X^\mu(\sigma,\tau) = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma). \tag{2.1.6}$$

Recall that in classical electrodynamics some symmetries are left even though the gauge was fixed. For example, if we fix the Lorentz gauge, $\partial^\mu A_\mu = 0$, then we still have not fixed the gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda$, with $\partial^\mu\partial_\mu\Lambda = 0$. The same occurs here. Consider a special coordinate transformation that changes the metric by an overall factor:

$$\frac{\partial\sigma'^\kappa}{\partial\sigma'^\alpha}\frac{\partial\sigma'^\delta}{\partial\sigma'^\beta}\gamma'_{\kappa\delta}(\tau'(\tau,\sigma),\sigma'(\tau,\sigma)) = \Lambda(\sigma,\tau)\gamma_{\alpha\beta}(\tau,\sigma). \tag{2.1.7}$$

Then one can remove $\Lambda(\sigma,\tau)$ using a Weyl transformation and the consequence is that the metric has not changed at all. The transformation of this kind are called conformal, since they preserve angles but not length. The transformations satisfying (2.1.7) form a group called the conformal group. In p space and q time dimensions it turns out to be the group $SO(p+1,q+1)$ if $p+q$ is larger than 2. If $p+q=2$ it is the group of all transformations of the form

$$\begin{aligned}\tau' &= f(\tau+\sigma) + g(\tau-\sigma)\\ \sigma' &= f(\tau+\sigma) - g(\tau-\sigma)\end{aligned}$$

with arbitrary continuous functions f and g.

Now we consider the spectrum of a string. The string has a centre of mass motion and vibration. The vibration can be decomposed in normal modes. Is the string in one of these modes then the energy of the mode can be viewed as the mass of the string.

First we consider closed strings. Then the world sheet of the string is a cylinder, i.e. we have periodic boundary conditions

$$X^\mu(0,\tau) = X^\mu(2\pi,\tau).$$

We can write X^μ in a Fourier series

$$X^\mu(\sigma,\tau) = \sum_{n\in\mathbb{Z}} e^{in\sigma} f^\mu_n(\tau).$$

(2.1.5) implies

$$\partial^2_\tau f^\mu_n(\tau) = -n^2 f^\mu_n(\tau)$$

with the solution

$$f^\mu_n(\tau) = a^\mu_n e^{in\tau} + b^\mu_n e^{-in\tau} \text{ for } n \neq 0 \text{ and } f^\mu_0(\tau) = p^\mu\tau + q^\mu.$$

So the final result with some convenient factors is

$$X^\mu(\sigma,\tau) = q^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}(\alpha_n^\mu e^{-in(\tau+\sigma)} + \bar{\alpha}_n^\mu e^{-in(\tau-\sigma)}). \tag{2.1.8}$$

X^μ should be real which implies

$$(\alpha_n^\mu)^* = \alpha_{-n}^\mu. \tag{2.1.9}$$

The discussion of the open string with boundaries 0 and π is slightly more laborious. The variation of the action (2.1.4) is

$$\delta_X S = \frac{1}{2\pi\alpha'}\int d\tau \int_0^\pi d\sigma (\delta X_\mu)\partial_a\partial^a X^\mu - \frac{1}{2\pi\alpha'}\int d\tau \delta X_\mu \partial_\sigma X^\mu \Big|_{\sigma=0}^{\sigma=\pi}. \tag{2.1.10}$$

The first term vanishing requires the equations of motion $\partial_a\partial^a X^\mu = 0$. The second term vanishes by imposing the Neumann boundary conditions $\partial_\sigma X^\mu = 0$ at the boundaries. The same procedure as above yields

$$X^\mu(\sigma,\tau) = q^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n}(\alpha_n^\mu e^{-in\tau} \cos n\sigma). \tag{2.1.11}$$

Now we quantise the theory. The Lagrangian is

$$L = -\frac{1}{4\pi\alpha'}\int_0^{\rho\pi} d\sigma \partial_a X^\mu \partial^a X_\mu$$

so that $S = \int d\tau L$. The number ρ is one for open strings and two for closed strings. Then the canonical momentum is

$$\Pi_\mu = \frac{1}{2\pi\alpha'}\partial_\tau X_\mu.$$

The substitution of Poisson brackets and commutators gives the relations of the quantised theory:

$$\begin{aligned} &[X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \quad \text{and} \\ &[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = [\Pi^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)] = 0 \end{aligned} \tag{2.1.12}$$

Then the relations for the modes are

$$\begin{aligned} &[\alpha_k^\mu, \alpha_l^\nu] = [\bar{\alpha}_k^\mu, \bar{\alpha}_l^\nu] = k\eta^{\mu\nu}\delta_{k+l,0} \\ &[\alpha_k^\mu, \bar{\alpha}_l^\nu] = 0 \quad \text{and} \quad [q^\mu, p^\nu] = i\eta^{\mu\nu}. \end{aligned} \tag{2.1.13}$$

This justifies the interpretation of q^μ and p^μ as centre-of-mass coordinate and momentum. From (2.1.9) we get

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu. \tag{2.1.14}$$

So the modes become rescaled harmonic oscillators.

Since the metric $\eta^{\mu\nu}$ is not positive definite there are states with negative norm, called tachyons. The reason for this is that we have not considered the second equation of motion obtained by the variation of the free boson action

(2.1.4) with respect to $\eta^{\alpha\beta}$. It is $T_{ab} = 0$ with the energy-momentum tensor T with components

$$T_{00} = T_{11} = \dot{X}^2 + X'^2 \quad \text{and} \quad T_{01} = T_{10} = \dot{X} \cdot X'.$$

Now let us see what this means for the quantised theory. First choose more convenient coordinates $\sigma^{\pm} = \tau \pm \sigma$. Then the derivatives are $\partial_{\pm} = 1/2(\partial_\tau \pm \partial_\sigma)$. In these coordinates the components of the energy-momentum tensor are

$$\begin{aligned} T^{++} &= \frac{1}{2}(T_{00} + T_{01}) = \partial_+ X \cdot \partial_+ X \\ T^{--} &= \frac{1}{2}(T_{00} - T_{01}) = \partial_- X \cdot \partial_- X \end{aligned}$$

The definition of the modes for closed strings is

$$\begin{aligned} L_n &= \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma e^{in\sigma} T_{++} \\ \bar{L}_n &= \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma e^{-in\sigma} T_{--} \end{aligned}$$

Substitution of the mode expansion of the modes of X yields

$$L_n = \frac{1}{2} \sum_m : \alpha_{n-m} \cdot \alpha_m : \quad \text{and} \quad \bar{L}_n = \frac{1}{2} \sum_m : \bar{\alpha}_{n-m} \cdot \bar{\alpha}_m :$$

with $\alpha_0^\mu = \bar{\alpha}_0^\mu = \sqrt{\frac{1}{2}\alpha'} p^\mu$ and : : denotes normal ordering. Similarly one has for open strings

$$L_n = \frac{1}{2\pi\alpha'} \int_0^{\pi} d\sigma e^{-in\sigma} T_{--} + e^{in\sigma} T_{++} = \frac{1}{2} \sum_m : \alpha_{n-m} \cdot \alpha_m :$$

with $\alpha_0^\mu = \sqrt{2\alpha'} p^\mu$. One calculates the commutators of the modes L_n to be

$$[L_n, L_m] = (m - n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{2.1.15}$$

c operates as a number and is called the central or conformal charge. The algebra consisting out of the modes L_n and the central charge c is called a Virasoro algebra. Often c is also called conformal anomaly.

It remains the problem of the unphysical negative norm states. There are some methods to get rid of them: the light-cone gauge, the covariant operator method, the covariant path integral method and the BRST-quantisation.

We will apply the BRST-method in this thesis. It is a more general way of quantising gauge theories than the other three methods mentioned above. In section 2.9 we explain it in a more technical way.

The total Lagrangian consists out of the classical Lagrangian, the gauge fixing Lagrangian and the ghost Lagrangian. The BRST-transformation acts on the classical Lagrangian as a gauge transformation and leaves the total Lagrangian invariant. Then the fields are quantised by turning Dirac brackets into commutators. By Noether's theorem, any symmetry yields a conserved current and a charge Q. In turn the charge Q generates the symmetry transformation.

The BRST-formalism requires the corresponding charge to be nilpotent, strictly speaking $Q^2 = 0$. The central charge of the ghost Virasoro algebra is -26. Q^2 can only be zero if the total central charge is zero, hence the BRST-formalism requires the central charge of the Virasoro algebra of the underlying conformal field theory to be 26. Otherwise the BRST-method breaks down and there are anomalies. Now one considers the physical space $\mathcal{H}$ of states. quires $Q^2 = 0$, and the charge Q allows a cohomology decomposition of the space of states. It turns out that the space of physical states is the space of those states $|\psi\rangle$ which are annihilated by Q modulo the states $Q|\psi'\rangle$. Furthermore, they have the properties:

- $b_0 |\Psi\rangle = 0$ for a certain ghost operator b_0 (without a shift by the conformal weight the ghost operator is b_1).
- $L_0 |\Psi\rangle = 0$ and L_0 determines the mass of the string.

We remark that for a string theory in flat Minkowski space central charge equals 26 means that the dimension of the space must be 26 and in a space with arbitrary metric this requires the dimension to be less or equal 26. In this thesis applying the BRST-method (see for chapter 7) we obtain bosonic string theories of dimension $18, 14, 10$ resp. 8.

In order to find a bosonic string theory we must find conformal field theories with a representation of a Virasoro algebra of central charge 26.

2.2 The Ising model

Before we explain what a conformal field theory is we give an example of their appearance in physics, the critical Ising model on the lattice (see [Ru]).

The Ising model is an example of a critical system in statistical mechanics. Statistical systems that are close to second order phase transition are characterised by long range phase transition of order parameters. They lead to singularities in the thermodynamic functions. This can be described by two-dimensional Euclidean field theory. At the critical point the correlation length diverges and hence the effective field theory becomes scale invariant. This together with the assumption that the interaction is local implies conformal invariance.

A QFT is considered to be the continuum limit of a lattice model with the sum over the states on the lattice a discrete version of the path integral.

Let L_N be a square lattice of $N \times N$ sites. The configuration of spins is a function $s : L_N \to \{\pm 1\}$ which assigns each lattice site a spin. The energy of a configuration is then

$$E[s] = -\sum_{<x,y>} s_x s_y$$

where the sum is only over neighbouring sites on L_N. The partition function is the sum over all possible states

$$Z_{L_N} = \sum_{\text{all configurations s}} e^{-\beta E[s]}$$

where $\beta > 0$ is the inverse of the temperature. In general a field $\phi(x)$ is a map $s \mapsto \phi(x)[s] \in \mathbb{C}$ which depends only on spines at sites in a certain neighbourhood of x. An example of a field is the spin field $\sigma(x)[s] = s_x$. The correlation function of spin fields on an infinite lattice is

$$\left\langle \sigma(x_1)\dots\sigma(x_m)\right\rangle_\beta = \lim_{N\to\infty}\frac{1}{Z_{L_N}}\sum_{\text{all configurations s}} s_{x_1}\dots s_{x_m}e^{-\beta E[s]}.$$

For a generic choice of the inverse temperature β, the correlators will either go to constants or decay exponentially. Also, there exists a phase transition between the ordered, low temperature phase and the disordered, high temperature phase at the critical temperature $1/\beta_c$. At this point the correlators display a power law behaviour. The continuum limit of a correlator for points $p_1,\dots,p_m$ in $\mathbb{R}^2$ is

$$\left\langle \hat\sigma(p_1),\dots,\hat\sigma(p_m)\right\rangle = \lim_{r\to\infty} r^\alpha \left\langle \sigma(rp_1),\dots,\sigma(rp_m)\right\rangle_{\beta_c}. \tag{2.2.1}$$

The constant α is the largest number such that the limit is finite. In the present case it turns out to be $\alpha = m/8$. The hat distinguishes between continuum fields and lattice fields. Note that the continuum correlators are scale invariant by definition

$$\left\langle \hat\sigma(\lambda p_1),\dots,\hat\sigma(\lambda p_m)\right\rangle = \lim_{r\to\infty} r^\alpha \left\langle \sigma(r\lambda p_1),\dots,\sigma(r\lambda p_m)\right\rangle_{\beta_c} =$$
$$\lim_{\lambda r\to\infty} r^\alpha \left\langle \sigma(r\lambda p_1),\dots,\sigma(r\lambda p_m)\right\rangle_{\beta_c} = \lambda^{-1}\left\langle \hat\sigma(p_1),\dots,\hat\sigma(p_m)\right\rangle.$$

A scale invariant theory with a local interaction is conformal invariant. We remark that the continuum limit describes the long-range behaviour of the lattice model at a critical point.

2.3 Classical conformal field theory

Conformal field theory is a tool in theoretical physics. In the last two sections we have seen its appearance in statistical mechanics and in string theory. Recently it also became important in the AdS/CFT-correspondence.

There are many introductions to this topic. For our purpose [S2] and [Ke] are suitable. A standard textbook is [DMS].

First we consider classical conformal invariance. Recall the conformal transformation in section 2.1 which were essentially general coordinate invariance

$$\frac{\partial\sigma'^\kappa}{\partial\sigma'^\alpha}\frac{\partial\sigma'^\delta}{\partial\sigma'^\beta}\gamma'_{\kappa\delta}(\tau'(\tau,\sigma),\sigma'(\tau,\sigma)) = \gamma_{\alpha\beta}(\tau,\sigma). \tag{2.3.1}$$

and Weyl invariance

$$\gamma'_{\alpha\beta} = e^{2\omega(\tau,\sigma)}\gamma_{\alpha\beta}. \tag{2.3.2}$$

We consider a flat space, i.e. a space with metric $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1,\dots,-1,1,\dots,1)$ with q-times the eigenvalue -1 and p-times the eigenvalue 1. The dimension of the space is then $d = p+q$.

The energy-momentum tensor is defined in terms of the variation of the action S under changes of the space-time metric

$$g_{\mu\nu}\to g_{\mu\nu}+\delta g_{\mu\nu}.$$

Then the definition is

$$\delta S = \frac{1}{2}\int d^dx\sqrt{g}T^{\mu\nu}\delta g_{\mu\nu}. \tag{2.3.3}$$

General coordinate invariance in flat coordinates implies

$$\partial_\nu T^{\nu\mu} = 0$$

and Weyl invariance implies the tensor to be traceless

$$T^\mu{}_\mu = 0.$$

A conformal transformation can now be defined as a coordinate transformation which acts on the metric as a Weyl transformation. Consider a general coordinate transformation $x \to x'$ such that $x^\mu = f^\mu(x'^\nu)$ with the following effect on the metric

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial f^\rho}{\partial x'^\mu}\frac{\partial f^\sigma}{\partial x'^\nu} g_{\rho\sigma}(f(x')) \propto g_{\mu\nu}(x'). \tag{2.3.4}$$

A tensor ϕ of rank n is called a conformal field if it transforms as

$$\phi'_{\mu_1,\ldots,\mu_n}(x') = \frac{\partial f^{\nu_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial f^{\nu_n}}{\partial x'^{\mu_n}} \phi'_{\nu_1,\ldots,\nu_n}(f(x')). \tag{2.3.5}$$

We want to know which transformations have the property (2.3.4). Let $x'^\mu = x^\mu + \epsilon^\mu(x)$ be an infinitesimal transformation. Its inverse is then $x^\mu = x'^\mu - \epsilon^\mu(x') + O(\epsilon^2)$. Then we have from (2.3.4) $\delta g_{\mu\nu} = -\partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu = \omega g_{\mu\nu}$ and hence

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}\partial^\kappa\epsilon_\kappa g_{\mu\nu}. \tag{2.3.6}$$

The solutions of this equation are (for $d > 2$)

- Translations: $x^\mu \to x^\mu + \alpha^\mu$
- (Lorentz) Rotations: $x^\mu \to x^\mu + \omega^\mu_\nu x^\nu$
- Scale transformations: $x^\mu \to x^\mu + \sigma x^\mu$
- Special conformal transformations: $x^\mu \to x^\mu + b^\mu x^2 - 2m^\mu b \cdot x$.

The generators of these transformation form the algebra $SO(p+1, q+1)$.

The conserved current of the conformal symmetry is

$$J_\mu(\epsilon) = T_{\mu\nu}\epsilon^\nu$$

since $\partial_\mu J_\mu(\epsilon) = (\partial^\mu T_{\mu\nu})\epsilon^\nu + T_{\mu\nu}(\partial^\mu\epsilon^\nu)$ vanishes because of (2.3.6) and because the energy momentum tensor is conserved, symmetric and traceless.

From now on we will only consider conformal field theory in two dimensions. The case of two dimensions in Euclidean space is special. We remark that a Wick rotation of the Euclidean formulation on a cylinder yields the Minkowski spacetime formulation of a field theory. The conditions for an infinitesimal conformal transformation are Cauchy-Riemann equations $\partial_1\epsilon_1 = \partial_2\epsilon_2$ and $\partial_1\epsilon_2 = -\partial_2\epsilon_1$. In complex variables $\epsilon = \epsilon_1 - i\epsilon_2, \bar\epsilon = \epsilon_1 + i\epsilon_2$ and $z = x^1 - ix^2, \bar z = x^1 + ix^2$ the condition becomes $\partial_z\bar\epsilon(z,\bar z) = \partial_{\bar z}\epsilon(z,\bar z) = 0$. Therefore, the conformal transformations can be identified with the analytic coordinate transformation

$$z \to f(z), \quad \bar z \to \bar f(\bar z), \quad f'(z) \neq 0.$$

These transformations are generated by $L_n = -z^{n+1}\frac{d}{dz}$ and the corresponding barred quantity. These operators satisfy the relations

$$[L_n, L_m] = (n-m)L_{n+m}, \quad [\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} \text{ and } [L_n, \bar{L}_m] = 0. \quad (2.3.7)$$

The two commuting parts $\{L_n\}$ and $\{\bar{L}_n\}$ of this algebra are known as the two-dimensional local conformal algebra or Witt algebra. The independence of the algebras $\{L_n\}$ and $\{\bar{L}_n\}$ justifies the use of z and $\bar{z}$ as independent coordinates.

The energy momentum tensor can be transformed into complex coordinates. The result is using that the tensor $T_{\mu\nu}$ is conserved and traceless

$$T_{z\bar{z}} = T_{\bar{z}z} = 0, \qquad T_{zz} \equiv T(z) \text{ and } T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z}) \quad (2.3.8)$$

$T(z)$ and $\bar{T}(\bar{z})$ are the holomorphic and antiholomorphic parts. The conserved current is then

$$J_\mu(\epsilon) = T_{\mu\nu}\epsilon^\nu \;\rightarrow\; J_z = T(z)\epsilon(z), \quad J_{\bar{z}} = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z}). \quad (2.3.9)$$

The components of a tensor ϕ of rank n are of the form $\phi_{z\ldots z,\bar{z}\ldots\bar{z}}(z,\bar{z})$. Under a conformal transformation this transforms into

$$\Big(\frac{\partial f(z)}{\partial z}\Big)^p \Big(\frac{\partial \bar{f}(\bar{z})}{\partial \bar{z}}\Big)^q \phi_{z\ldots z,\bar{z}\ldots\bar{z}}(f(z), \bar{f}(\bar{z})). \quad (2.3.10)$$

2.4 Quantum conformal field theory

Now we want to quantise the theory. Therefore the transformation to complex coordinates is convenient. Further we make the space direction finite by imposing periodic boundary conditions. Scale invariance allows us to set the length of one period to 2π. The Euclidean coordinates $(x^1, x^2) = (x^1, ix^0)$ can be thought of coordinates on a cylinder. The charge is defined as

$$Q = \frac{1}{2\pi}\int_0^{2\pi} dx^1 J^0 = \frac{1}{2\pi}\int_0^{2\pi} dx^1(-iJ^2).$$

Using (2.3.9) this equals

$$-\frac{1}{2\pi}\Big\{\oint dz J_z^{\rm cyl}(z,\bar{z}) - \oint d\bar{z} J_{\bar{z}}^{\rm cyl}(z,\bar{z})\Big\}.$$

The integration is along a closed contour that encircles the cylinder. The orientation is such that $\oint dz = \oint d\bar{z} = 2\pi$. For convenience we perform a conformal transformation $w = e^{iz}$. This transformation maps the cylinder on a plane with the time coordinate mapped to the radial coordinate. Under this transformation the charge becomes

$$-\frac{1}{2\pi}\Big\{\oint dw(iw)^{h-1} J_w^{\rm plane}(w,\bar{w}) + \oint dw(-i\bar{w})^{\bar{h}-1} J_{\bar{w}}^{\rm plane}(w,\bar{w})\Big\}.$$

In classical theories the ordering of fields is irrelevant. In the quantum theory they become operators, so we have to specify the ordering of products. In order to have well defined expectation values one imposes time ordering:

$$TA(t_a)B(t_b) = \begin{cases} A(t_a)B(t_b) & \text{for } t_a > t_b \\ B(t_b)A(t_a) & \text{for } t_b > t_a \end{cases}$$

After mapping from the cylinder to the plane, the Euclidean time coordinate is mapped to the radial coordinate, and time ordering becomes radial ordering

$$RA(z,\bar{z})B(w,\bar{w}) = \begin{cases} A(z,\bar{z})B(w,\bar{w}) & \text{for } |z| > |w| \\ B(w,\bar{w})A(z,\bar{z}) & \text{for } |w| > |z| \end{cases} .$$

A correlation function in field theory on the cylinder has the form

$$\langle 0| T(A_1(t_1)\dots A_n(t_n)) |0\rangle$$

where $|0\rangle$ and $\langle 0|$ are in and out states at $t = -\infty$ and $t = \infty$ respectively. After the conformal mapping, the correlation functions are

$$\langle 0| R(A_1(z_1,\bar{z}_1)\dots A_n(z_n,\bar{z}_n)) |0\rangle$$

where $|0\rangle$ and $\langle 0|$ are states at $z = 0$ and $z = \infty$ respectively. The current for an infinitesimal conformal transformation is $T(z)\epsilon(z)$ (2.3.9). The corresponding charge is

$$Q_\epsilon = \frac{1}{2\pi i}\oint dz\epsilon(z)T(z).$$

By Noether's theorem Q_ϵ should generate the conformal transformation with the global form

$$\phi(w,\bar{w}) \to \phi'(w,\bar{w}) = \Big(\frac{\partial f(w)}{\partial w}\Big)^h \phi(f(w),\bar{w}),$$

with $f(w) = w+\epsilon(w)$ and h the conformal weight of the field ϕ. The infinitesimal form of this transformation is

$$\delta_\epsilon\phi(w,\bar{w}) = h\partial_w\epsilon(w)\phi(w,\bar{w}) + \epsilon(w)\partial_w\phi(w,\bar{w}).$$

The desired commutation relation is

$$\delta_\epsilon\phi(w,\bar{w}) = [Q_\epsilon, \phi(w,\bar{w})]. \tag{2.4.1}$$

Considering radial ordering the commutator is

$$[Q_\epsilon, \phi(w,\bar{w})] = \frac{1}{2\pi i}\oint dz\epsilon(z)R(T(z)\phi(w,\bar{w})).$$

The integration makes only sense if the radially ordered product is analytic in some neighbourhood of the point w. Therefore it can be expanded in a Laurent series. The contour integral will give the desired result (2.4.1) if the Laurent series takes the form

$$R(T(z)\phi(w,\bar{w})) = \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) + \dots . \tag{2.4.2}$$

This property plus the corresponding property for the anti-holomorphic quantities defines a conformal field.

2.5 The Virasoro algebra

The operator product of the energy momentum tensor is

$$R(T(z)T(w)) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + \ldots. \tag{2.5.1}$$

If the first term were absent ($c = 0$) then $T(z)$ would be a conformal field of weight 2, which would be expected classical. Quantum effects yield this extra term, called conformal anomaly (we will call it also central charge).

Consider the transformations $z \to z' = z - z^{n+1}$, the corresponding current is $J^n(z) = T(z)z^{n+1}$ and the normalised operators are

$$L_n = \frac{1}{2\pi i}\oint dz z^{n+1} T(z).$$

The inverse of this relation is

$$T(z) = \sum_n z^{-n-2} L_n.$$

The commutators of the L_n are calculated using contour integrals, they give the commutation relations of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{2.1.15}$$

Since the Virasoro algebra arises naturally in a conformal field theory we are interested in its representations. A lowest weight representation is a representation containing a state with a smallest eigenvalue of L_0. It is reasonable that a physical theory has this property, since $L_0 + \bar{L}_0$ is the Hamiltonian, which is usually bounded from below. Let $|h\rangle$ be such a state, then $L_0 L_n |h\rangle = [L_0, L_n]|h\rangle + L_n L_0 |h\rangle = (h-n)L_n |h\rangle$, hence

$$L_n |h\rangle = 0, \quad \text{for } n \geq 1.$$

This allows us to interpret the $L_n, n \geq 1$ as annihilators. The states $L_n, n \leq -1$ generate new states called descendants.

The vacuum of the theory is defined by the condition that it respects the maximum number of symmetries. The maximal symmetry is

$$L_n |0\rangle = 0, \quad \text{for } n \geq -1.$$

This state is the vacuum and it will always be assumed that it is the unique state with this property.

The connection between lowest weight states and conformal fields is

$$|h, \bar{h}\rangle = \phi(0,0)|0\rangle \tag{2.5.2}$$

for a conformal field $\phi(z, \bar{z})$ of weights h and $\bar{h}$. This state is a lowest weight state with L_0-eigenvalue h and $\bar{L}_0$-eigenvalue $\bar{h}$. The conformal fields are also often called (Virasoro) primary fields.

2.6 Wess-Zumino-Witten theories

Often one deals with theories that have more symmetries than just the Virasoro algebra. These theories have a larger symmetry algebra which contains the Virasoro algebra as a subalgebra. These generalised algebras are often called chiral algebras, since they are generated by currents that are holomorphic or anti-holomorphic. One advantage of a larger symmetry algebra is that less primary fields are required. The conformal spin of a current is the difference between the holomorphic and anti-holomorphic conformal weights $h - \overline{h}$. In terms of the conformal spin a classification of extensions of the Virasoro algebra is

$\frac{1}{2}$ Free fermions

1 Affine Lie algebras

$\frac{3}{2}$ Superconformal algebras

2 Virasoro tensor products

> 2 W-algebras.

We are interested in conformal field theories with the extended symmetry of an affine Lie algebra.

In section 2.8 we will describe vertex algebras. A vertex algebra has the structure of the chiral part of a conformal field theory. Vertex algebras can be constructed from affine Lie algebras [FZ] and they have the additional symmetry of the affine Lie algebra. Then the conformal fields correspond to the highest weights of a highest weight representation of an affine Lie algebra in the same way as in (2.5.2) for Virasoro highest weights. In physics these conformal field theories are called Wess-Zumino-Witten theories. A nice reference is [F1] and a summary is given in [Schw].

Wess-Zumino-Witten theories are defined as those conformal field theories whose chiral symmetry algebra is generated by at least the energy-momentum tensor $T(z)$ and the currents

$$J^a(z) = \sum_m z^{-m-1} \hat{T}^a_m$$

whose modes satisfy the commutation relations of the Virasoro algebra and $[L_n, \hat{T}^a_m] = -m\hat{T}^a_{n+m}$. One also requires the Virasoro generators L_n to be of the Sugawara form, i.e.

$$L_n = \frac{1}{2(k^\vee + g^\vee)} \sum_m \bar{\kappa}(\bar{T}^a, \bar{T}^b) : \hat{T}^a_{n+m} \hat{T}^b_{-m} :,$$

where the $\hat{T}^a_{n+m}$ generate the centrally extended loop algebra $\hat{\mathfrak{g}}$ of a certain Lie algebra $\bar{\mathfrak{g}}$ (Adding a derivation to $\hat{\mathfrak{g}}$ gives the affinisation $\mathfrak{g}$ of $\bar{\mathfrak{g}}$ (4.2.2)). The level $k^\vee$ of the relevant $\mathfrak{g}$-modules, the dual Coxeter number $g^\vee$ and the killing form $\bar{\kappa}$ are introduced in chapter 4. The sum makes only sense if one introduces normal ordering : :. A possible normal ordering prescription is

$$: a_m b_n := \begin{cases} a_m b_n & \text{for } m \leq 0 \\ b_n a_m & \text{for } m > 0 \end{cases} .$$

To get a consistent theory the factor $(k^\vee + g^\vee)^{-1}$ and hence the level $k^\vee$ must be a constant. Therefore it is necessary that all representations appearing in a Wess-Zumino-Witten theory have the same level. Many quantities of interest can be studied in terms of the finite-dimensional simple Lie algebra $\bar{\mathfrak{g}}$ and of the level $k^\vee$. For example the Virasoro central charge is

$$c(\mathfrak{g}, k^\vee) = \frac{k^\vee \dim \bar{\mathfrak{g}}}{k^\vee + g^\vee}.$$

2.7 Fusion rules and simple currents

The three point function $\langle 0|\, \phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\, |0\rangle$ of conformal fields ϕ_i, ϕ_j, ϕ_k of conformal weight h_1, h_2, h_3, equals

$$C_{ijk}(z_1 - z_2)^{h_3-h_1-h_2}(z_2 - z_3)^{h_1-h_2-h_3}(z_3 - z_1)^{h_2-h_3-h_1}.$$

The operator product of two operators ϕ_i, ϕ_j can be expanded in a complete set of operators. Taking the limit $z_1 \to z_2$ one can show that the operator product expansion has the form

$$\phi_i(z, \overline{z})\phi_j(w, \overline{w}) = C_{ijk}(z - w)^{h_k-h_i-h_j}(\overline{z} - \overline{w})^{\overline{h}_k-\overline{h}_i-\overline{h}_j}\phi_k(w, \overline{w}).$$

The coefficients C_{ijk} satisfy certain selection rules called fusion rules which can be written as follows

$$\phi_i \times \phi_j = \sum_k N_{ij}^k \phi_k.$$

The value of N_{ij}^k indicates the number of distinct ways of coupling the fields.

The fusion rules are related to the modular transformation properties of the partition function of the theory. The partition function can be written as

$$P(\tau, \overline{\tau}) = \sum_{ij} M_{ij}\chi_i(\tau)\chi_j(\overline{\tau}). \tag{2.7.1}$$

Here i and j label certain highest weight states, i standing for the holomorphic part and j for the anti-holomorphic part, M_{ij} its multiplicity. The functions χ are the characters of the representations (see chapter 6). The transformation properties of χ_i under $\tau \mapsto -\frac{1}{\tau}$ is given by the symmetric matrix S

$$\chi_i(-1/\tau) = \sum_{j=0}^{N-1} S_{ij}\chi_j(\tau).$$

The fusion rule coefficients are determined by S:

$$N_{ij}{}^k = \sum_{n=0}^{N-1} \frac{S_{in}S_{jn}S^\dagger_{kn}}{S_{0n}}.$$

This is the Verlinde formula, it implies that the fusion algebra is both associative and commutative.

If a primary field ϕ_i has the following simple fusion rules

$$\phi_i \times \phi_j = \phi_{i'}$$

for all fields ϕ_j, then ϕ_i is called a simple current. Simple currents organise the fields in a conformal field theory into orbits of order dividing N. They form an abelian group called the centre of the conformal field theory. One can regard the action of the centre on the primary fields as an extension of the conformal field theory. Then the fields are given by a set of orbit representatives and the centre. In this case the centre is also called glue group.

In this thesis we are interested in simple currents of certain Wess-Zumino-Witten theories. Therefore we need the following result of [F2]: Except for the trivial simple current, a primary field ϕ_Λ is a simple current of the Wess-Zumino-Witten theory with underlying affine Lie algebra $\mathfrak{g}$ (except for $\bar{\mathfrak{g}} = E_8$ level 2) at level k if and only if Λ is a k-multiple of a cominimal fundamental weight.

2.8 Vertex algebras

A nice reference for this section is [K2] and a compact overview is [Sch3].

A chiral algebra is the algebra of all holomorphic fields of a conformal field theory. Chiral algebras and vertex algebras are essentially the same, we state the definition of [K2] of a vertex algebra. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a direct sum of two vector spaces. $\bar{0}$ and $\bar{1}$ stand for the cosets in $\mathbb{Z}/2\mathbb{Z}$ of 0 and 1. An element a in V has parity $p(a)$ in $\mathbb{Z}/2\mathbb{Z}$ if a in $V_{p(a)}$. A field is a series of the form $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ where a_n in $\mathrm{End}(V)$ and for each v in V one has $a_n v = 0$ for n sufficiently large. The vector space V together with a vacuum vector $|0\rangle$ in $V_{\bar{0}}$ and a parity preserving state-field correspondence $a \mapsto a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ (parity preserving means $p(a_n b) = p(a)+p(b)$) is a vertex algebra if the following axioms are satisfied:

The operator T on V defined by $Ta = a_{-2}\,|0\rangle$ satisfies

$$[T, a(z)] = \partial a(z);$$

(translation covariance)

$|0\rangle\,(z)a = a$ and $a(z)\,|0\rangle\,|_{z=0} = a$; (vacuum)

$(z-w)^n a(z)b(z) = (-1)^{p(a)p(b)}(z-w)^n b(z)a(z)$ holds for n sufficiently large. (locality)

A vertex algebra containing a Virasoro element ω in $V_{\bar{0}}$ satisfying

1. The operators $L_m = \omega_{m+1}$ give a representation of the Virasoro algebra of central charge c, i.e.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c;$$

2. L_0 is diagonalisable on V;

3. $T = L_{-1}$;

is called a conformal vertex algebra of central charge c.

We are interested in vertex algebras associated to representations of affine Lie algebras. Frenkel and Zhu showed in [FZ] that one can provide certain representations of affine Lie algebras with the structure of a vertex algebra.

We are also interested in vertex algebras associated to a lattice L of finite rank d, since it can be thought of as a bosonic string theory with d space-time dimensions compactified on a torus. L represents the allowed momentum vectors of the theory.

We summarise the construction of a vertex algebra to a given integral lattice L with bilinear form $(\cdot,\cdot)$. First we decompose the integral lattice $L = L_{\bar{0}} \cup L_{\bar{1}}$ with $L_{\bar{0}} = \{\alpha \in L | \alpha^2 = 0 \mod 2\}$ and $L_{\bar{1}} = \{\alpha \in L | \alpha^2 = 1 \mod 2\}$. Define $h = L \otimes_{\mathbb{Z}} \mathbb{R}$ and the bosonic Heisenberg algebra

$$\widehat{h} = h \otimes \mathbb{R}[t, t^{-1}] \oplus \mathbb{R}c$$

with central element c and commutation relation

$$[h_1(m), h_2(n)] = m\delta_{m+n,0}(h_1, h_2)c.$$

Here $h_1(m)$ denotes $h_1 \otimes t^m$. Then $\widehat{h}^- = h \otimes t^{-1}\mathbb{R}[t^{-1}]$ is an abelian subalgebra of $\widehat{h}$ and by $S(\widehat{h}^-)$ we mean the symmetric algebra of polynomials in $\widehat{h}^-$. Physically the elements of $S(\widehat{h}^-)$ can be interpreted as oscillators. Further we need a 2-cocycle $\epsilon : L \times L \to \{\pm 1\}$ with the properties:

$$\begin{aligned} \epsilon(\alpha, 0) = \epsilon(0, \alpha) &= 1 \\ \epsilon(\alpha, \beta) &= (-1)^{(\alpha,\beta)+\alpha^2\beta^2}\epsilon(\beta, \alpha) \\ \epsilon(\alpha, \beta+\gamma)\epsilon(\beta, \gamma) &= \epsilon(\alpha, \beta)\epsilon(\alpha+\beta, \gamma) \end{aligned}$$

This gives us the twisted group algebra $\mathbb{R}[L]_\epsilon$ with basis e^α, α in L and products

$$e^\alpha e^\beta = \epsilon(\alpha, \beta)e^{\alpha+\beta}.$$

Now we have the vector space

$$V = S(\widehat{h}^-) \otimes \mathbb{R}[L]_\epsilon$$

which can be provided with the structure of a vertex algebra.

V decomposes into a direct sum $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with $V_{\bar{i}} = S(\widehat{h}^-) \otimes \mathbb{R}[L_{\bar{i}}]_\epsilon$.

The Heisenberg algebra $\widehat{h}$ has a natural action on V with $h(-n)$ acting multiplicatively for $n > 0$, $h(0)e^\alpha = (h, \alpha)e^\alpha$ and $h(n)e^\alpha = 0$ for $n > 0$. Therefore we call the $h(-n), n > 0$ creation operators and the $h(n), n > 0$ annihilation operators.

A vertex algebra structure is obtained by defining:

$$\begin{aligned} e^\alpha(z) &= e^\alpha(z)^+ e^\alpha(z)^- \\ \text{with} \quad e^\alpha(z)^+ &= e^\alpha \exp\{\sum_{m>0} \alpha(-m)\frac{z^m}{m}\} \\ e^\alpha(z)^- &= z^{\alpha(0)} \exp\{-\sum_{m>0} \alpha(m)\frac{z^{-m}}{m}\} \end{aligned}$$

and with $h(z) = \sum_{n\in\mathbb{Z}} h(n)z^{-n-1}$ we set

$$h(-n-1)(z) = \partial_z^{(n)} h(z) \quad \text{for } n \geq 0.$$

The bosonic normal ordering is defined by putting all creation operators to the left of all annihilation operators. Then with

$$(h_1(-n_1-1)\cdots h_k(-n_k-1)e^{\alpha})(z) = \\ e^{\alpha}(z)^{+} : h_1(-n_1-1)(z)\cdots h_k(-n_k-1)(z) : e^{\alpha}(z)^{-}$$

extended linearly to V it becomes a vertex algebra with vacuum $1 \otimes e^0$.

2.9 The BRST-construction

The BRST-construction (Becchi-Rouet-Stora-Tyutin) is a formalism to obtain a Lie algebra $\mathfrak{g}$ out of a vertex algebra V. Since V describes the chiral states of a conformal field theory, $\mathfrak{g}$ can be interpreted as a Lie algebra of chiral physical states. The BRST-construction can be applied to a $\mathbb{Z}$ graded differential algebra $A = \bigoplus_{n\in\mathbb{Z}} A_n$ with an additional $\mathbb{Z}_2$-grading $A = A_{\bar{0}} \oplus A_{\bar{1}}$ the elements of $A_{\bar{0}}$ being the bosonic states and those of $A_{\bar{1}}$ the fermionic states plus the ghosts. Since we are interested in a bosonic string theory we don't need fermions, so $A_{\bar{1}}$ contains only ghosts. We assume that there is a fermionic operator Q satisfying $Q^2 = 0$ and $QA_p \subseteq A_{p+1}$. Then we have the following chain

$$\cdots \xrightarrow{Q} A_{p-1} \xrightarrow{Q} A_p \xrightarrow{Q} A_{p+1} \xrightarrow{Q} \cdots$$

with $\operatorname{Im} Q_{|A_{p-1}} \subseteq \operatorname{Ker} Q_{|A_p}$. This allows us to define the cohomology groups $H^p = \operatorname{Ker} Q_{|A_p} / \operatorname{Im} Q_{|A_{p-1}}$ whose elements can be interpreted as physical states. We will apply the construction explicitly in chapter 7. We are interested in an anomaly free bosonic string theory. Therefore let V be a conformal vertex algebra whose Virasoro algebra has central charge 24 and $V_{I\!I_{1,1}}$ the vertex algebra of the unique even unimodular Lorentzian lattice $I\!I_{1,1}$. Then $V \otimes V_{I\!I_{1,1}}$ represents the Fock space of a bosonic string. The ghost system is the lattice vertex algebra of the one dimensional standard lattice $\mathbb{Z}$. It has central charge -26, hence the central charge of $V \otimes V_{I\!I_{1,1}} \otimes V_{\mathbb{Z}}$ vanishes, which makes the theory anomaly free.

2.10 Meromorphic $c = 24$ conformal field theories

This thesis bases on the list of possible meromorphic conformal field theories of central charge 24 in [S3]. We already noted that such conformal field theories are of particular interest, since they allow us to construct bosonic string theories.

In [S3] conformal field theories with the simplest kind of fusion rules are considered. These are those theories with one primary field $\mathbf{1}$ and fusion rule $\mathbf{1} \times \mathbf{1} = \mathbf{1}$. The modular transformation properties of these theories are simple. In particular if the central charge is a multiple of 24 the partition function is modular invariant, and one can consider a corresponding purely chiral conformal field theory. In such a theory all correlation functions are meromorphic, therefore we call these theories meromorphic conformal field theories.

The aim of [S3] is to obtain a list of all possible meromorphic conformal field theories of central charge 24. Therefore some trace-identities and the fact that the character must be modular invariant are considered. It is shown that if the number of spin-1 currents does not equal 0, then the chiral algebra contains a spin-1 algebra with total charge 24. Hence the partition function of any such theory must be a modular invariant combination of affine Lie algebra characters. The trace identities hold only for 69 distinct combinations of affine Lie algebra highest weight representations. The modular transformation properties of the characters are well known (chapter 13 in [K1]). So one calculates exactly one modular invariant partition function for each of the 69 cases. The results are listed at the end of [S3]. In this thesis we consider conformal field theories V with spin-1 algebra $\widehat{A}^r_{p-1,p}$ where $r = 48/(p^2-1)$ and $p = 2, 3, 5$ or 7.

So far it is a problem to construct all these theories. There are 24 conformal field theories associated to the Niemeier lattices. Applying $\mathbb{Z}_2$ orbifold twists yield 15 more theories (see [DGM]). Proofs for most of the remaining cases are conjectured using $\mathbb{Z}_3$ orbifold twists or applying $\mathbb{Z}_2$ orbifold twists to the twisted theories (see [M1] and [M2]).

Chapter 3

Modular forms and lattices

In this chapter some aspects of modular forms and lattices are briefly introduced, since they are necessary to proof the main results of this thesis. Introductions to modular forms are found in [KK] and [FB]. Introductions to lattices are found in [CS] and [E]. For a complete overview we refer to [CS], except for the part about the Weil representation which can be found in [Bu].

3.1 The modular group

The modular group $SL(2,\mathbb{Z}) = \Gamma$ is the group consisting of all 2×2 - matrices with determinant 1. It acts on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im } \tau > 0\}$ via

$$\tau \mapsto M\tau = \frac{a\tau + b}{c\tau + d},$$

with $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$. $SL(2,\mathbb{Z})$ is generated by $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, which will be proven later. One defines the exact fundamental domain of Γ as

$$\mathbb{F}(\Gamma) := \{\tau \in \mathbb{H} \mid -\frac{1}{2} < \text{Re } \tau \leq \frac{1}{2},\ |\tau| \geq 1 \text{ and } |\tau| > 1 \text{ for } -\frac{1}{2} < \text{Re } \tau \leq 0\}$$

The name is justified by the following facts:
- for every τ in $\mathbb{H}$ exists an M in Γ such that $M\tau$ in $\mathbb{F}(\Gamma)$
- τ and $M\tau$ in $\mathbb{F}(\Gamma)$, M in Γ, iff
  1) $\tau = M\tau$ and $\tau = i$, $M = \pm S$ or
  2) $\tau = M\tau$ and $\tau = e^{2\pi i/6}$, $M = \pm TS$ or $\pm(TS)^2$

Now we consider a subgroup Λ of Γ. A fundamental domain of Λ is a subset $\mathcal{F}$ of $\mathbb{H}$ with the following properties:
- $\mathcal{F}$ is closed in $\mathbb{H}$.
- for every τ in $\mathbb{H}$ exists a M in Λ with $M\tau$ in $\mathcal{F}$.
- If τ and $M\tau$, M in Λ, are in the open kernel of $\mathcal{F}$, then $M = \pm Id$.

Let Λ' be the subgroup of Γ generated by Λ and $-Id$ and

$$\Gamma = \bigcup_{1\leq\nu\leq[\Gamma:\Lambda']} \Lambda' M_\nu$$

a disjoint union of Γ in right congruence classes of Λ', where the M_ν are not unique. One can show that

$$\mathbb{F}(\Lambda) = \bigcup_{1\leq\nu\leq[\Gamma:\Lambda']} M_\nu\overline{\mathbb{F}}$$

is a fundamental domain of Λ [KK]. The cusps of $\mathbb{F}(\Lambda)$ are the images of $\Lambda' M_\nu i\infty$ of $i\infty$.
The principal congruence group (mod N) is defined as follows:

$$\Gamma(N) = \{M \text{ in } \Gamma \mid M \equiv Id \bmod N\}.$$

Another important class of subgroups of Γ is

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in } \Gamma \mid c \equiv 0 \mod N \right\}. \tag{3.1.1}$$

The index of $\Gamma_0(N)$ in Γ is given by $N\prod_{p|N}(1+\frac{1}{p})$. Therefore the number of cusps of $\Gamma_0(N)$ is finite, a complete set of representatives for the cusps is given by $\frac{a}{c}$ for $c|N$, $c>0$, $0<a\leq(c,\frac{N}{c})$ and $(a,c)=1$. $\Gamma_0(p)$, for p prime, is generated by T and $\left(\begin{smallmatrix} a & b \\ p & k \end{smallmatrix}\right) \in \Gamma$, $0<k<p$, a and b such that the matrix is in Γ. This fact is proven by induction to $|c|$: For $c=0$ it is true, since $T^m = \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$. Now let $|c|>0$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\Gamma_0(p)$ arbitrary, then $\left(\begin{smallmatrix} \star & \star \\ p & k \end{smallmatrix}\right) T^m \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \star & \star \\ c' & \star \end{smallmatrix}\right)$, $c' = p(a+mc)+kc$. Choose m such that $-|c| < (a+mc) < 0$, then $|c'| = |p(a+mc)+kc| < |c|$ for one k, $0<k<p$.

3.2 Modular forms

Let f be a meromorphic function on $\mathbb{H}$, then for M in $SL(2,\mathbb{R})$ the meromorphic function on $\mathbb{H}$ $f|_k M$ is defined by

$$(f|_k M)(\tau) = (c\tau+d)^{-k} f(M\tau), \quad k \text{ in } \mathbb{Z}.$$

The function f is called modular of weight k, if f is meromorphic on $\mathbb{H}$ and $f|_k M = f$ for every M in Γ. A function f is called a modular form of weight k, if f is modular of weight k and f has at most one pole at $i\infty$, or equivalently if f has a Fourier-expansion of the form

$$f(\tau) = \sum_{m\geq m_0} \alpha_f(m)\ e^{2\pi i m\tau},$$

which for suitable $\gamma>0$ for $Im\ \tau \geq \gamma$ absolute and compact-uniform converges. The modular forms of weight k form a vector space over $\mathbb{C}$. The modular forms of weight 0 form a field. A modular form f of weight k is called integral, if f is holomorphic on $\mathbb{H}$ and if there is no pole at $i\infty$, or equivalently if f has a Fourier-expansion of the form

$$f(\tau) = \sum_{m\geq 0} \alpha_f(m)\ e^{2\pi i m\tau},$$

which for suitable $\gamma>0$ for $Im\ \tau \geq \gamma$ absolute and compact-uniform converges. The space of integral modular forms of weight 0 is $\mathbb{C}$.

Now we consider modular forms of a congruence group. A subgroup Λ of Γ is called a congruence group, if there exists a positive n, such that $\Gamma(n) \subset \Lambda$. The index in Γ of a congruence group is finite. A group homomorphism

$$\chi : \Lambda \longrightarrow \{z \in \mathbb{C} \mid |z| = 1\}$$

is called an abelian character of Λ. The trivial character, which maps every element of Λ onto 1, will be denoted by 1.
A function $f : \mathbb{H} \to \mathbb{C}$ is called an integral modular form of weight k for the congruence group Λ and the abelian character χ, if
- f is holomorphic on $\mathbb{H}$
- $f|_k L = \chi(L) f$ for every L in Λ.
- $f|_k M$ is holomorphic at $i\infty$ for every M in Γ.

The set $\mathbb{M}_k(\Lambda, \chi)$ of all integral modular forms of weight k to Λ and χ is a vector space over $\mathbb{C}$. One can show ([KK] p.172) that

$$\mathbb{M}_0(\Lambda, 1) = \mathbb{C}. \tag{3.2.1}$$

3.3 The Dedekind η-function

The Dedekind η-function is a holomorphic function $\eta : \mathbb{H} \to \mathbb{C}$ defined by

$$\eta(\tau) = e^{\pi i \tau/12} \cdot \prod_{m=1}^{\infty} (1 - e^{2\pi i m \tau})$$

The T transformation is $\eta(\tau + 1) = e^{\pi i/12} \cdot \eta(\tau)$, and the S transformation is $\eta(-1/\tau) = \sqrt{\tau/i} \cdot \eta(\tau)$([KK] p.168). Since the product $\prod_{m=1}^{\infty}(1 - e^{2\pi i m \tau})$ is absolute convergent one has $\eta(\tau) \neq 0$ for every τ in $\mathbb{H}$ (cf. [FB] p.196). The general transformation properties are the following (cf. [R] p.163):

$$\begin{aligned}
&\text{for } M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \text{ with } c > 0 \\
&\eta(M\tau) = \epsilon(M) \cdot \sqrt{c\tau + d/i} \cdot \eta(\tau) \\
&\epsilon(M) := \begin{cases} \left(\frac{d}{c}\right) i^{\frac{1-c}{2}} \; e^{\frac{\pi i}{12}(bd(1-c^2)+c(a+d))} & , c \text{ odd} \\ \left(\frac{c}{d}\right) e^{\frac{\pi i d}{4}} \; e^{\frac{\pi i}{12}(ac(1-d^2)+d(b-c))} & , d \text{ odd} \end{cases} \\
&\left(\frac{d}{c}\right) \text{ is the Legendre-Jacobi symbol}
\end{aligned} \tag{3.3.1}$$

this yields [Sch1],

$$\begin{aligned}
\eta((kS\tau + j)/m) &= \epsilon(A)\sqrt{m\tau/m'i}\; \eta((k'\tau + j')/m') \\
A &= \begin{pmatrix} j/k' & -(jj' + kk')/km \\ m/k' & -j'/k \end{pmatrix}
\end{aligned} \tag{3.3.2}$$

with j, k, m and j', k' and m' integers such that A in $SL(2, \mathbb{Z})$, $km = k'm'$ and $m/k' > 0$. Some examples are calculated in appendix C since they are necessary for our purpose.

3.4 Lattices

A lattice L is a finitely generated free $\mathbb{Z}$-module with a $\mathbb{Q}$-valued quadratic form $q : L \to \mathbb{Q}$, such that the associated bilinear form $(\cdot|\cdot)$, $((a,b) \mapsto (a|b) := q(a+b) - q(a) - q(b))$ is non degenerate. Note that $q(a) = a^2/2$. Two lattices L_1 and L_2 are isomorphic (isometric) if there exists an isomorphism of $\mathbb{Z}$-modules $\nu : L_1 \to L_2$ which preserves the bilinear form.

A lattice (L, q) is called integral if the bilinear form takes only integral values on L. Further, we say that L is an even lattice if the quadratic form q is integral. The vector space $V := L \otimes_{\mathbb{Z}} \mathbb{R}$ arises in a natural way from the lattice L. A quadratic form on L can be uniquely continued to a quadratic form of V. Then there exists a basis $\mathcal{B} = \{e_1, \ldots, e_r\}$ of V with $L = \sum_{i=1}^{r} \mathbb{Z}e_i$. $\mathcal{B}$ is called a basis of the lattice L, r the dimension of the lattice and the signature of L is the signature of the space (V, q). The matrix

$$A := A(L, \mathcal{B}) := ((e_i|e_j))_{i,j=1,\ldots,r}$$

is called the Gram matrix of the lattice L relative to the basis $\mathcal{B}$. Two lattices L and M of the same dimension r are isomorphic if and only if there exist two basis $\mathcal{B}$ and $\mathcal{C}$ and a T in $GL_r(\mathbb{Z})$ such that

$$A(M, \mathcal{C}) = T^t A(L, \mathcal{B}) T.$$

The discriminant $\mathrm{disc}(L)$ of a lattice L is the absolute value of the determinant of the Gram matrix, (note that the determinant is independent of the choice of the basis). An unimodular lattice is a lattice with discriminant 1.

The dual lattice L' of an integral lattice L is the module

$$L' = \{v \in L \otimes_{\mathbb{Z}} \mathbb{R} : (v|L) \subseteq \mathbb{Z}\}.$$

with the quadratic form q. The Gram matrix of L' is the inverse of the Gram matrix of L. Since L is integral $L \subset L'$, the factor group L'/L is called the discriminant group of L, it is of the order $\mathrm{disc}(L)$. The level of an even lattice is the smallest natural number N such that $Nq(\gamma)$ is integral for every γ in L'. For a positive definite lattice the minimal norm is the smallest norm of the nonzero lattice vectors.

Two integral quadratic forms are said to be in the same genus if they are equivalent over $\mathbb{R}$ and the p-adic integers for all prime numbers p. For our purposes it is sufficient to consider genera of even lattices whose discriminant group is $\mathbb{Z}_p^k$, where p is prime and $k \in \mathbb{Z}_{>0}$. Such genera are uniquely determined by the signature (r, s), p, k and a generalised version of the Jacobi-Legendre symbol which can take the values $+$ and $-$. It is symbolised by

$$\mathit{II}_{r,s}(p^{\pm k}).$$

Unless $k = r + s$ and $p \neq 2$, the sign can be determined by the equation (cf. [CS, p. 386, Th. 13])

$$r - s \equiv \pm 2 - 2 - (p-1)k \mod 8 \tag{3.4.1}$$

For $p = 2$ the sign is listed in table 15.5 on page 387 of [CS]. Two lattices of the same genus are not necessarily isometric, but lattices of the same isometry class are in the same genus.

It is often necessary to rescale a lattice, changing L to $\tilde{L} = kL = \{kx : x \in L\}$, k in $\mathbb{R}$. The parameters of L and $\tilde{L}$ are related as follows: $\tilde{A} = k^2 A$, $\mathrm{disc}(\tilde{L}) = k^{2n}\,\mathrm{disc}(L)$ (n is the rank of L) and minimal norm of $\tilde{L} = k^2$ minimal norm of L. We denote $\sqrt{k}L$ by $L(k)$. Obviously, $kL \cong L(k^2)$. If L is even and k in $\mathbb{Z}_{>0}$ then the map

$$L'/kL \to (\sqrt{k}L)'/(\sqrt{k}L) = 1/\sqrt{k}L'/\sqrt{k}L$$

with

$$x + kL = \sqrt{k}(x/\sqrt{k} + \sqrt{k}L) \mapsto x/\sqrt{k} + \sqrt{k}L$$

is an isomorphism of groups.

3.5 Gluing Theory

We want to describe the general n-dimensional integral lattice L that has a sublattice which is a direct orthogonal sum $L_1 \oplus L_2 \oplus \cdots \oplus L_k$ of given integral lattices $L_1, \ldots, L_k$ of total dimension n. Any vector of L can be written

$$y = y_1 + y_2 + \cdots + y_k \tag{3.5.1}$$

where each component y_i is in the subspace spanned by L_i. Since the inner product of y_i with any vector of L_i is the same as the inner product of y with that vector it is an integer and hence y_i must be in the dual lattice L_i'.

Any y_i can be altered by adding a vector of L_i, so y_i is a representative of a system of representatives for the cosets of L_i in L_i'. They are called glue vectors. The quotient group L_i'/L_i is called the glue group for L_i. So L is generated by $L_1 \oplus L_2 \oplus \cdots \oplus L_k$ and certain glue vectors y_i (3.5.1).

3.6 The Lattice A_n and the hyperbolic plane

In this thesis Euclidean lattices ($s = 0$) (especially A_n) and Lorentzian lattices ($s = 1$) are considered. An important Lorentzian lattice is the hyperbolic plane, which is the unique even unimodular two dimensional lattice $I\!I_{1,1}$ with Gram matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The lattice A_n, $n \geq 1$ is defined by

$$A_n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{Z}^{n+1} \mid x_0 + \ldots + x_n = 0\}, \tag{3.6.1}$$

with basis $\mathcal{B} = \{(-1,1,0,\ldots,0), (0,-1,1,0,\ldots,0), \ldots, (0,\ldots,-1,1)\}$ and Gram matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix}. \tag{3.6.2}$$

The minimal norm is 2, the discriminant is $n+1$, the glue group is $\mathbb{Z}/(n+1)\mathbb{Z}$ and the glue vectors are

$$[i] = \Big(\underbrace{\frac{i}{n+1},\dots,\frac{i}{n+1}}_{(i-n-1)-times},\underbrace{\frac{i-n-1}{n+1},\dots,\frac{i-n-1}{n+1}}_{i-times}\Big) \quad i = 0,\dots,n \tag{3.6.3}$$

This yields

$$A_n' = \bigcup_{i=0}^{n}([i] + A_n),$$

the discriminant of A_n' is $1/(n+1)$ and the minimal norm is $n/(n+1)$.

3.7 The Weil representation

Of primary importance for our considerations are modular forms for the Weil representation of $SL(2,\mathbb{Z})$.

Let L be an even lattice of even dimension, L'/L its discriminant group. The set of formal linear combinations $\sum_{\gamma\in L'/L} x_\gamma e^\gamma$, x_γ in $\mathbb{C}$, can be extended to a $\mathbb{C}$-algebra by defining $e^\gamma e^\delta := e^{\gamma+\delta}$. This algebra is the group ring $\mathbb{C}[L'/L]$ of the discriminant group. It has a hermitian bilinear form $(\sum_\gamma x_\gamma e^\gamma, \sum_\delta y_\gamma e^\delta) := \sum_\gamma x_\gamma \bar{y}_\gamma$.

The Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$ is defined by

$$\begin{aligned}\rho_L(T)e^\gamma &= e(-\gamma^2/2)\, e^\gamma\\ \rho_L(S)e^\gamma &= \frac{e(\mathrm{sign}(L)/8)}{\sqrt{|L'/L|}} \sum_{\beta\in L'/L} e((\gamma,\beta))\, e^\beta\,.\end{aligned}$$

where $S = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ are the standard generators of $\mathrm{SL}_2(\mathbb{Z})$.

A modular form for the Weil representation ρ_L of weight k, k in $\mathbb{Z}$, is a holomorphic map F from the upper half plane $\mathbb{H}$ to $\mathbb{C}[L'/L]$ which transforms as

$$F\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^k \rho_L\big(\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\big)F(\tau).$$

$F(\tau)$ can be written as

$$F(\tau) \sum_{\gamma\in L'/L} f_\gamma(\tau)e^\gamma.$$

The transformation properties of the components f_γ are ($e(a) = e^{2\pi i a}$):

$$\begin{aligned}f_\gamma(\tau+1) &= e(-\gamma^2/2)\, f_\gamma(\tau)\\ f_\gamma(-1/\tau) &= \frac{e(\mathrm{sign}(L)/8)}{\sqrt{|L'/L|}}\tau^k \sum_{\beta\in L'/L} e((\gamma,\beta))\, f_\beta(\tau)\end{aligned}$$

Such modular forms can be constructed by lifting scalar valued modular functions for $\Gamma_0(N)$ [Sch1]. Suppose that the level of L'/L divides N. Let f be a scalar valued modular function for $\Gamma_0(N)$ of integral weight k and character χ_L. Then

$$F(\tau) = \sum_{M\in\Gamma_0(N)\backslash\Gamma} f|_M(\tau)\,\rho_L(M^{-1})e^0$$

is a vector valued modular form for ρ_L of weight k which is invariant under the automorphisms of the discriminant group.

Now we consider the following cases. Let p be a prime such that p^2-1 divides 48, i.e. $p = 2, 3, 5$ or 7. Then there is an automorphism of the Leech lattice of cycle shape $1^m p^m$ with $m = 24/p + 1$. The fix-point lattice Λ_p is the unique lattice its genus without roots. Let

$$L = \Lambda_p \oplus II_{1,1}(p) \oplus II_{1,1}\,.$$

Then L has level p and genus

$$II_{2m+2,2}(p^{\epsilon_p(m+2)})$$

with $\epsilon_p = +, -, +, +$ for $p = 2, 3, 5, 7$. The eta product

$$h(\tau) = \frac{1}{\eta(\tau)^m\eta(p\tau)^m} = q^{-1} + m + \frac{1}{2}\dots \tag{3.7.1}$$

is a modular function for $\Gamma_0(p)$ of weight $-m$ with poles at the cusps 0 and $i\infty$. Note that for $p = 7$ the function f has character $\chi(j) = \left(\frac{j}{7}\right)$. We define functions g_k by

$$h(\tau/p) = g_0(\tau) + g_1(\tau) + \cdots + g_{p-1}(\tau) \tag{3.7.2}$$

with $g_j|_T(\tau) = e(j/p)g_j(\tau)$, i.e.

$$g_j(\tau) = \frac{1}{p}\sum_{k=0}^{p-1} e(-kj/p)h\big((\tau+k)/p\big)\,. \tag{3.7.3}$$

We can lift the function h to a modular form $F = \sum F_\gamma e^\gamma$ on the discriminant of L. Then F has components

$$F_\gamma(\tau) = f(\tau) + g_0(\tau) \qquad \text{if } \gamma = 0 \tag{3.7.4}$$

$$= g_j(\tau) \qquad \text{if } \gamma \neq 0 \text{ and } \gamma^2/2 = -j/p \mod 1\,. \tag{3.7.5}$$

The components F_γ with $\gamma^2/2 = 0 \mod 1$ are modular for $\Gamma_0(p)$ of weight $-m$ and with nontrivial quadratic character in the case $p = 7$.

This result will become important in chapter 6, where we calculate the coefficients of the partition function.

3.8 Theta Functions

The theta functions of a lattice transform under the dual Weil representation.

Let $L \subset \mathbb{R}^n$, n even, be a lattice, z in $\mathbb{R}^n$ and τ in $\mathbb{H}$ then the theta function $\vartheta_{z+L}(\tau)$ is defined as:

$$\vartheta_{z+L}(\tau) = \sum_{x\in L} e^{\pi i\tau(x+z)^2} \tag{3.8.1}$$

One can also show that the theta functions are invariant under the action of the principal congruence subgroup of N, where N is the level of L [E]

$$(\vartheta_{\rho+L}|_{\frac{n}{2}}\mathrm{A})(\tau) = \vartheta_{\rho+L}(\tau) \quad \text{for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in } \Gamma(N) \text{ and } \rho \text{ in } L'$$

and also

$$(\vartheta_L|_{\frac{n}{2}}\mathrm{A})(\tau) = \left(\frac{\Delta}{d}\right)\vartheta_L(\tau) \quad \text{for } \mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in } \Gamma_0(N)$$

holds ($\Delta = (-1)^{\frac{n}{2}}\mathrm{disc}(L)$ and $\left(\frac{\Delta}{d}\right)$ is the Jacobi-Legendre symbol).

Chapter 4

Lie algebras

In this chapter Lie algebras are considered, in particular affine Lie algebras and generalised Kac-Moody algebras. A complete introduction from a physical point of view is given in [KMPS] and [F1]. For mathematicians the standard reference is [K1]. [H] and [FH] introduce finite dimensional Lie algebras.

We first recall how Lie algebras typically appear in quantum mechanics. A quantum field theory is usually formulated in terms of the Lagrangian. Of major importance is the symmetry group of the Lagrangian, which is the group of those transformations of the fields that leaves the Lagrangian unchanged. The continuous part of this group is the Lie group G of the Lagrangian. The set of infinitesimal transformations of G form a Lie algebra, which is often viewed as a linear approximation of the group. For many purposes, e.g. if one is interested in local symmetries, the Lie algebra is more convenient than the Lie group G.

In quantum mechanics one has a Hilbert space of physical states on which the observables act as linear operators. The Hilbert space carries representations of G and its Lie algebra, a knowledge of the representation theory is of primary importance in finding a solution to a quantum mechanical problem. Symmetry properties of the problem can be related to known properties of Lie groups and Lie algebras.

4.1 Finite dimensional Lie algebras

A Lie group is a set endowed simultaneously with the compatible structure of a group and a $\mathcal{C}^\infty$ manifold. Compatible means that multiplication and inverse operation are differential maps. A Lie algebra $\mathfrak{g}$ is a vector space together with a skew-symmetric bilinear map, the Lie bracket,

$$[\ ,\ ] :\ \mathfrak{g}\ \times\ \mathfrak{g} \longrightarrow\ \mathfrak{g}$$

satisfying the Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0 \text{ for } X,Y,Z \text{ in } \mathfrak{g}.$$

Lie algebras are related to Lie groups. The tangent space on the neutral element of a Lie group G is a Lie algebra $\mathfrak{g}$. This can be seen as follows: let x

and y be smooth curves in G containing the unit element, say $x(0) = y(0) = 1$, then the tangent vectors of x and y on 1 are

$$X = \frac{\mathrm{d}}{\mathrm{d}t}x(t)|_{t=0} \quad \text{and} \quad Y = \frac{\mathrm{d}}{\mathrm{d}t}y(t)|_{t=0}.$$

The Lie algebra $\mathfrak{g}$ is the real span of all such tangent vectors. We have the scalar multiplication

$$\lambda X = \frac{\mathrm{d}}{\mathrm{d}t}x(\lambda t)|_{t=0},$$

the addition

$$X + Y = \frac{\mathrm{d}}{\mathrm{d}t}x(t)y(t)|_{t=0}$$

and the Lie bracket

$$[X, Y] = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}[x(t), y(t)]|_{t=0},$$

where $[x(t), y(t)]$ is the group commutator. Then skew-symmetry and Jacobi identity are satisfied. Of primary importance is the exponential map, $\exp : \mathfrak{g} \to G$ which maps some neighbourhood of 0 in $\mathfrak{g}$ to some neighbourhood of 1 in G. For any $X = \frac{\mathrm{d}}{\mathrm{d}t}x(t)|_{t=0}$ in $\mathfrak{g}$ we obtain the adjoint representation $\mathrm{Ad}(g)$, g in G, by conjugating x with g:

$$\mathrm{Ad}(g)(X) = \frac{\mathrm{d}}{\mathrm{d}t}gx(t)g^{-1}|_{t=0}$$

If $g = \exp(Y)$ then the adjoint representation ad of the Lie algebra $\mathfrak{g}$ is

$$(\mathrm{ad}\, Y)(X) = [y, x], \quad X, Y \in \mathfrak{g},$$

its relation to Ad is $(\mathrm{Ad}(\exp Y))(X) = (\exp(\mathrm{ad}\, Y))X$, via the Baker-Campbell-Hausdorff formula.

Denote by T the maximal abelian connected subgroup of G, and by $\mathfrak{g}_0'$ its Lie algebra. Set $\mathfrak{g}_0 = i\mathfrak{g}_0'$, then we have the modified exponential map $\exp(2\pi i(\cdot)) : \mathfrak{g}_0 \longrightarrow T, X \longrightarrow \exp(2\pi i X)$. The kernel of $\exp(2\pi i(\cdot))$ in $\mathfrak{g}_0$ is a lattice L (the bilinear form is the killing form (4.1.3)).

Suppose that G acts as unitary linear operators on some complex finite dimensional vector space V. Since T is abelian its operators can be simultaneously diagonalised (which means in physics that the corresponding observables can be measured simultaneously). Then there exists a basis $\{v_j\}$ of V such that $tv_j = c_j(t)v_j$, t in T and $c_j(t)$ in the unit circle. For X in $\mathfrak{g}_0$, we then have a map $\mu_j : \mathfrak{g}_0 \to \mathbb{R}$ with $\exp(2\pi i(X))v_j = e^{2\pi i \mu_j(X)}v_j$. In particular, for X in $L, \exp(2\pi i(X)) = 1 \in t$,thus $\mu_j : L \to \mathbb{Z}$. The μ_j are the weights of the module V, they lie in the dual lattice L'. V decomposes into a direct sum of weight subspaces

$$\begin{aligned} V &= \bigoplus_{\mu \in L'} V^\mu, \\ V^\mu &= \{v \in V | \exp(2\pi i(X))v = e^{2\pi i \mu_j(X)}v, \text{ for all X in } \mathfrak{g}_0\}, \end{aligned} \tag{4.1.1}$$

the dimension of the weight subspaces is called the multiplicity of the weight. The set of weights will be denoted by $P(V)$. Via the adjoint representation we obtain the root space decomposition $\mathfrak{g} = \bigoplus_{\mu \in L'} \mathfrak{g}^{\mu}$. Let $\Delta = P(\mathfrak{g}) \setminus \{0\}$, then the root space decomposition is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha} \tag{4.1.2}$$

with $\Delta = -\Delta$ and $\mathfrak{h} := \mathfrak{g}_0$ is a Cartan subalgebra. The elements in Δ are called roots. For finite dimensional $\mathfrak{g}$, all roots have multiplicity one, in general this is not true. The sublattice of L' generated by Δ is the root lattice Q. L'/Q is the congruence group and its elements are the congruence classes.

Up to now the lattices have no geometric structure, therefore we introduce a bilinear form, called the killing form.

$$(X|Y) = \operatorname{Tr} \operatorname{ad}(X) \operatorname{ad}(Y), \qquad X, Y \in \mathfrak{g} \tag{4.1.3}$$

which is invariant

$$([X,Y]|Z) = (X|[Y,Z]) \text{ for } X, Y, Z \in \mathfrak{g}$$

For semisimple Lie algebras it is non degenerate. L is integral, so we have $L \subseteq L'$. The isomorphism

$$\nu : \mathfrak{h} \longrightarrow \mathfrak{h}^* \qquad h \mapsto (h|\cdot) \tag{4.1.4}$$

allows the weight lattice and the root lattice to be considered as subsets of the Cartan subalgebra $\mathfrak{h}$. For each root $\alpha \in \Delta$ define the Weyl reflection

$$r_\alpha : \mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}, \quad \mu \mapsto \mu - 2\frac{(\mu|\alpha)}{(\alpha|\alpha)}\alpha.$$

The group W generated by all r_α is the Weyl group. Each generator r_α has a set of fixed points

$$P_\alpha = \{\mu \in \mathfrak{h}_{\mathbb{R}} : (\mu|\alpha) = 0\}$$

which is a hyper plane in $\mathfrak{h}_{\mathbb{R}}$. They divide $\mathfrak{h}_{\mathbb{R}}$ into disjoint open sets, which are cones. These cones are called Weyl chambers and W acts simply transitively on them. If we choose one such chamber C, and a set of roots bounding it, we obtain a basis $\alpha_1, \dots, \alpha_n$ of Q. Since $P_\alpha = P_{-\alpha}$, we may choose the basis in such a way that C is the positive cone

$$C = \{\mu \in \mathfrak{h}_{\mathbb{R}} : (\mu|\alpha_i) \geq 0 \text{ for } i = 1, \dots, n\}.$$

The roots $\alpha_1, \dots \alpha_n$ are called simple roots. Note that, since Δ is stable under the action of W, the simple roots are a set of orbit representatives in Δ of W.

Each root can be expressed as a sum of simple roots with entirely non-negative or non-positive coefficients. This gives a partition

$$\Delta = \Delta_+ \dot\cup \Delta_-$$

where Δ_+ refers to non-negative coefficient roots and Δ_- to non-positive coefficient roots. For $\alpha \in \Delta_+$ we simply write $\alpha > 0$. With $\mathfrak{n}_\pm := \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}^\alpha$ (4.1.2) implies

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-.$$

We later identify $\mathfrak{n}_\pm$ with sets of annihilation and creation operators acting on highest weight representations of $\mathfrak{g}$.

$\{\alpha_j^\vee\}$ with $\alpha_i^\vee = 2\frac{(\,\cdot\,|\alpha_i)}{(\alpha_i|\alpha_i)}$ is a basis of L, the coroot basis. Together, the α_i and $\alpha_i^\vee$ define the Cartan matrix

$$A = (a_{ij})_{i,j=1}^n := \left(2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}\right)_{i,j=1}^n .$$

Any simple Lie algebra is completely characterised by the indecomposable Cartan matrix, requiring that

(C1) $a_{ii} = 2$,

(C2) $a_{ij} \leq 0$ for $i \neq j$,

(C3) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$,

(C4) $a_{ij} \in \mathbb{Z}$ and

(C5) $\det A > 0$

It is possible to recover the Lie algebra corresponding to the Cartan matrix up to isomorphism from the Cartan matrix A:

Define the Cartan subalgebra as $\mathfrak{h} := \mathbb{C}^{2n-r}$, r the rank of A, with standard scalar product $\langle\cdot|\cdot\rangle$ and orthonormal basis $\{h_i\}$. Choose simple roots $\alpha_i, i = 1,\ldots,n$, such that $\alpha_i(h_j) = \delta_{ij}$, and simple coroots satisfying $\alpha_i(\alpha_j^\vee) = a_{ij}$.

Then one obtains the Lie algebra corresponding to the Cartan matrix via the Chevalley-Serre construction: let the Lie algebra $\mathfrak{g}$ be generated by $h \in \mathfrak{h}$ and elements $e_1,\ldots,e_n, f_1,\ldots,f_n$ such that the relations

$$\begin{aligned}
&[h,h'] = 0 \text{ for } h,h' \in \mathfrak{h},\\
&[e_i,f_j] = \delta_{ij}\alpha_i^\vee,\\
&[h,e_i] = \alpha_i(h)e_i,\\
&[h,f_i] = -\alpha_i(h)f_i \text{ for } h \in \mathfrak{h},\\
&(\operatorname{ad} e_i)^{1-a_{ij}}e_j = (\operatorname{ad} f_i)^{1-a_{ij}}f_j = 0 \text{ for } i \neq j
\end{aligned} \tag{4.1.5}$$

hold.

4.2 Kac-Moody algebras

Lie algebras arise in the description of symmetries, since many systems possess infinitely many independent symmetries, infinite dimensional Lie algebras are also important to physics. In particular Kac-Moody algebras are of importance, we already mentioned its applications in two-dimensional conformal field theory. They are also closely related to the symmetries of integrable quantum systems (often called quantum groups).

A Kac-Moody algebra is a generalisation of the simple Lie algebras. A matrix $A = (a_{ij})_{i,j=1}^n$ obeying (C1–C4) is called a generalised Cartan Matrix. Further, if all proper principal minors of a matrix are positive,

$$\det\, A_{\{i\}} > 0 \;\forall i = 0,\ldots,r \tag{4.2.1}$$

we call this matrix degenerate positive semidefinite. A Kac-Moody algebra is obtained from a generalised Cartan matrix. An affine Lie algebra corresponds to a degenerate positive semidefinite generalised Cartan matrix with determinant 0. The simple Lie algebras are those with Cartan matrix A with $\det A > 0$. Instead of (4.2.1) the requirement that there exists a diagonal matrix D such that DA is symmetric and positive semidefinite and that rank $A = r$ is also sufficient. Since the rank of an affine Cartan matrix is r, it has one right $(a_i)_{i=0}^r$ and one left $(a_i^\vee)_{i=0}^r$ eigenvector with eigenvalue zero. The a_i ($a_i^\vee$) are called Coxeter labels (dual Coxeter labels). Their sums are called Coxeter number h (dual Coxeter number $h^\vee$). This allows us to fix D in such a way that

$$(\alpha_i|\alpha_j) = \frac{a_i^\vee}{a_i} a_{ij}.$$

The affine Lie algebras can be classified completely. By deleting the zeroth column and row of an affine generalised Cartan Matrix one obtains its simple counterpart. In the non twisted case one can also do the converse, this is called affinisation. Given a simple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ and Killing form $(\cdot|\cdot)$ the affinisation of $\mathfrak{g}$ is the vector space

$$\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D \tag{4.2.2}$$

provided with (for $x, y \in \mathfrak{g}$, $n, m \in \mathbb{Z}$ and $\lambda, \mu \in \mathbb{C}$) the Lie bracket

$$[x \otimes t^m + \lambda D, y \otimes t^n + \mu D] := \\ [x, y] \otimes t^{m+n} + \lambda n t^n \otimes y - \mu m t^m \otimes x + m\delta_{m+n}\,(x|y)\,K.$$

$\mathbb{C}K$ is the centre of $\widehat{\mathfrak{g}}$ and $K = \alpha_0^\vee + \ldots + \alpha_{p-1}^\vee$ is the canonical central element. $\widehat{\mathfrak{g}}$ is isomorphic to the non twisted simple affine Lie algebra of the same type and rank as $\mathfrak{g}$. The Cartan subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$ is $\mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$. The element D acts as a derivation $t\frac{\mathrm{d}}{\mathrm{d}t}$ on $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. Setting $\lambda(K) = \lambda(D) = 0$ extends a linear functional on $\mathfrak{h}$ to $\widehat{\mathfrak{h}}$. Then $\widehat{\mathfrak{h}}^*$ can be defined as

$$\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$$

with linear functionals Λ_0 and δ defined by

$$\begin{aligned} \Lambda_0(h \oplus \mathbb{C}d) &= 0 & \Lambda_0(K) &= 1 \\ \delta(h \oplus \mathbb{C}K) &= 0 & \delta(d) &= 1\,. \end{aligned}$$

This gives a natural projection $\bar{\cdot} : \widehat{\mathfrak{h}} \to \mathfrak{h}$ with $\overline{\Lambda_0} = \overline{\delta} = 0$.

A set of simple roots $\{\alpha_1, \ldots, \alpha_l\}$ of $\mathfrak{g}$ can be extended to a set of simple roots $\{\alpha_0, \ldots, \alpha_l\}$ of $\widehat{\mathfrak{g}}$ with $\alpha_0 = \delta - \theta$ and $\theta = a_1\alpha_1 + \cdots + a_n\alpha_n$ the highest root.

4.3 Generalised Kac-Moody algebras

The fake monster Lie algebra led Borcherds to the definition of generalised Kac-Moody algebras. Therefore these algebras are sometimes called Borcherds algebras. A generalised Kac-Moody algebra is defined as follows. Let $A = (a_{ij})$ be a real quadratic matrix satisfying

$a_{ij} = a_{ji}$

$a_{ij} \leq 0$ for $i \neq j$

$2a_{ij}/a_{ii} \in \mathbb{Z}$ if $a_{ii} > 0$.

Then the universal generalised Kac-Moody algebra $\widehat{G}$ is the Lie algebra with generators $\{e_i, f_i, h_{ij}\}$ defined by the relations

$$\begin{aligned}
&[e_i, f_j] = h_{ij} \\
&[h_{ij}, e_j] = \delta_{ij} a_{ik} e_k \\
&[h_{ij}, f_j] = -\delta_{ij} a_{ik} f_k \\
&(\mathrm{ad}\, e_i)^{1-a_{ij}/a_{ii}} e_j = (\mathrm{ad}\, f_i)^{1-a_{ij}/a_{ii}} f_j = 0 \text{ if } a_{ii} > 0,\ i \neq j \\
&[e_i, e_j] = [f_i, f_j] = 0 \text{ if } a_{ij} = 0.
\end{aligned} \tag{4.3.1}$$

A generalised Kac-Moody algebra G is obtained from a universal generalised Kac-Moody algebra $\widehat{G}$ by factoring out a subspace of the centre and adding a commuting algebra of outer derivations.

Chapter 5

Highest weight modules over Kac-Moody algebras

In this chapter we state results of the representation theory of Kac-Moody algebras, in particular the notion of a highest weight representation. Recall the lowest weight representation of the Virasoro algebra on the space of conformal fields (2.5.2). We are particularly interested in the character of such a representation. One of the main tasks of this thesis is to rewrite the characters of some conformal field theories in a suitable manner. These theories are representations of affine Lie algebras and hence its characters are composed of affine Lie algebra characters.

The standard textbook about affine Lie algebras and its representation is [K1].

5.1 Highest weight representations

A representation ϕ of the Lie algebra $\mathfrak{g}$ is a homomorphism of $\mathfrak{g}$ into End V, V a vector space, i.e.

$$\phi([X,Y]) = \phi(X) \circ \phi(Y) - \phi(Y) \circ \phi(X).$$

V is often called a $\mathfrak{g}$-module, and accordingly one writes $X(v)$ instead of $\phi(X)(v)$ for $v \in V$. While for $\mathfrak{g}$ there is no definition of a product other than the Lie bracket, for the representation ϕ one has the composition of maps. With the help of this products one can define arbitrary power series. This is the concept of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ which consists of all finite formal power series in the elements of $\mathfrak{g}$ (actually the definition of the universal enveloping algebra is different but the equivalence to finite formal power series is shown with the help of the Poincaré-Birkhoff-Witt-theorem). Each representation of $\mathfrak{g}$ uniquely induces a representation of the universal enveloping algebra $U(\mathfrak{g})$.

Recall the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$$

and the corresponding decomposition of the universal enveloping algebra

$$U(\mathfrak{g}) = U(\mathfrak{n}_+) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_-).$$

A $\mathfrak{g}$-module V is called a highest weight module with highest weight Λ in $\mathfrak{h}^*$ if there exists a nonzero vector v_Λ in V, the highest weight vector, such that $\mathfrak{h}$ acts diagonally, $\mathfrak{n}_+$ annihilates v_Λ and V is the orbit of v_Λ under the action of $U(\mathfrak{n}_-)$:

$$\begin{aligned} h(v_\Lambda) &= \Lambda(h)v_\Lambda \quad \text{for } h \text{ in } \mathfrak{h}, \\ \mathfrak{n}_+(v_\Lambda) &= 0 \quad \text{and} \\ U(\mathfrak{g})(v_\Lambda) &= V \end{aligned} \tag{5.1.1}$$

There exists a unique up to isomorphism $\mathfrak{g}$-module $M(\Lambda)$ with highest weight Λ with the property that every $\mathfrak{g}$-module $M(\Lambda)$ with highest weight Λ is a quotient of $M(\Lambda)$. This module is called a Verma module. It is a free $U(\mathfrak{n}_-)$-module of rank one generated by a highest weight vector and $M(\Lambda)$ contains a unique proper maximal submodule $M'(\Lambda)$. The quotient $L(\Lambda) = M(\Lambda)/M'(\Lambda)$ is the unique irreducible highest weight module with highest weight Λ. Any highest weight module V can be decomposed into a direct sum of weight spaces

$$V_\lambda = \{v \in V | h(v) = \lambda(h)v \text{ for } h \in \mathfrak{h}\}.$$

The dimension of V_λ is called multiplicity of λ and is denoted by $\text{mult}_V \lambda$.

Having an irreducible highest weight representation $L(\Lambda)$ one defines the character ch_Λ to be the formal e^λ expression

$$\text{ch}_\Lambda = \sum_{\lambda \in P(V)} \text{mult}_\Lambda(\lambda) e^\lambda. \tag{5.1.2}$$

$P(\Lambda)$ is the set of weights of $L(\Lambda)$. The character of affine Lie algebras can be identified with modular functions by setting $e^\lambda = q = e^{2\pi i \tau}, \tau \in \mathbb{H}$. For the character of a Kac-Moody algebra there exists a so called Weyl-Kac character formula

$$\text{ch}_\Lambda = \frac{\sum_{w \in W} \det(w) e^{w(\Lambda+\rho)}}{\sum_{w \in W} \det(w) e^{w(\rho)}} \tag{5.1.3}$$

with the Weyl vector ρ which has the property $(\rho|\alpha) = -\alpha^2/2$ for all simple roots α. $\det(w)$ is one for w a product of an even number of simple Weyl reflections and minus one otherwise. For the denominator the identity

$$e^\rho \prod_{\alpha \in \Delta_+} (1 - e^\alpha)^{\text{mult}(\alpha)} = \sum_{w \in W} \det(w) w(e^\rho)$$

holds.

For generalised Kac-Moody algebras, the imaginary simple roots affect the right hand side, so that one gets a correction term S

$$e^\rho \prod_{\alpha \in \Delta_+} (1 - e^\alpha)^{\text{mult}(\alpha)} = \sum_{w \in W} \det(w) w\left(e^\rho S\right). \tag{5.1.4}$$

S takes into account all roots composed of the imaginary simple roots, that is

$$S = \sum_{\alpha \in \Delta \cup \{0\}} \epsilon(\alpha) e^\alpha \tag{5.1.5}$$

where $\epsilon(\alpha)$ is $(-1)^n$ if α is the sum of a set of n pairwise orthogonal imaginary simple roots, and 0 otherwise (cf. [B3]). Identities of this form can be used to construct generalised Kac-Moody algebras.

5.2 Integrable highest weight representations of affine Lie algebras

We are interested in integrable highest weight representations.

The weight lattice P is the set

$$P = \{\lambda \in \mathfrak{h}^* | \lambda(\alpha_i^\vee) \in \mathbb{Z}\ i = 1, \dots, n\}. \tag{5.2.1}$$

Elements from P are called integral weights, P contains the root lattice Q. The set of dominant weights P_+ is the set of those elements of P with $\lambda(\alpha_i^\vee)$ nonnegative. A highest weight representation is called integrable if all e_i and f_i are locally nilpotent on V. This condition is equivalent to

$$e_i(v_\Lambda) = 0, \qquad f_i^{\Lambda(\alpha_i^\vee)+1}(v_\Lambda) = 0 \qquad \text{for all } 1 \leq i \leq n$$

Now we consider integrable highest weight representations of affine Lie algebras. Let K be the canonical central element and define the level k of Λ in $\mathfrak{h}^*$ to be $\Lambda(K)$. Denote by P^k (respectively P_+^k) the set of (dominant) weights of level k. The dual element of the canonical central element K is the null-root δ. A weight λ is called maximal if $\lambda + \delta$ is not a weight, the set of those weights is denoted by $\max(\Lambda)$. The null-root yields a decomposition of the set of weights into weight strings

$$P(V) = \biguplus_{\lambda \in \max(\Lambda)} \{\lambda - n\delta : n \in \mathbb{Z}_{\geq 0}\}. \tag{5.2.2}$$

The aim of this section is to simplify the character, therefore we define the modular anomaly of any dominant weight Λ:

$$m_\Lambda = \frac{|\Lambda + \rho|^2}{2(h^\vee + k)} - \frac{|\rho|^2}{2h^\vee} \tag{5.2.3}$$

and introduce the normalised character

$$\chi_\Lambda = e^{-m_\Lambda \delta}\, \mathrm{ch}(V). \tag{5.2.4}$$

Further we define for any weight λ in $P(V)$ the rational number

$$m_{\Lambda,\lambda} = m_\Lambda - \frac{|\lambda|^2}{2k}$$

and the string function

$$c_\lambda^\Lambda = e^{-m_{\Lambda,\lambda}\delta} \sum_{n \in \mathbb{Z}} \mathrm{mult}_{L(\Lambda)}(\lambda - n\delta) e^{-n\delta} \tag{5.2.5}$$

(and $c_\lambda^\Lambda = 0$ if λ not in $P(V)$).

For any $\Lambda \in P_+^k$, the character of Λ can be written as ([K1] chapter 12) (the coefficients c_λ^Λ, called string functions, will be introduced in the next section)

$$\chi_\Lambda = \sum_{\lambda \in P^k/(kM + \mathbb{C}\delta)} c_\lambda^\Lambda \tilde{\theta}_\lambda.$$

M is the root lattice of $\widehat{A}_n$ ([K1] chapter 6). Introduce the orthogonal projection $\bar{\cdot}$ from $\widehat{\mathfrak{h}}^*$ to $M \otimes_{\mathbb{Z}} \mathbb{C}$ (with $\bar{\delta} = 0$). The condition $\langle \lambda, K \rangle = k$ allows us to express the sum in terms of $\overline{P}$:

$$\chi_\Lambda = \sum_{\overline{\lambda} \in \overline{P}/kM} c^\Lambda_{\overline{\lambda}} \tilde{\theta}_{\overline{\lambda}}$$

P is the weight lattice of $\widehat{A}_n$, hence

$$\chi_\Lambda = \sum_{\lambda \in M^*/kM} c^\Lambda_\lambda \theta_\lambda = \sum_{\lambda \in M^*(1/k)/M(k)} c^\Lambda_\lambda \theta_\lambda = \sum_{\lambda \in M(k)^*/M(k)} c^\Lambda_\lambda \theta_\lambda \qquad (5.2.6)$$

where

$$\theta_\lambda = \sum_{\mathbf{s} \in \lambda + M(k)} q^{\frac{1}{2}\mathbf{s}^2} e^{\sqrt{k}\mathbf{s}}.$$

θ_λ is a modular function on $\mathbb{H} \times \mathbb{C}^\ell$ by mapping $\mathbf{s} \to (\mathbf{s}, \mathbf{z})$ and setting $q = e^{2\pi i \tau}$.

5.3 String functions

Now only the explicit calculation of the string functions is missing. This is done for some examples in [KP]. In this thesis the string functions of $\widehat{A}_4$-characters are calculated in section 6.3.

A string function is holomorphic on the upper half plane and for a given module $L(\Lambda)$ there are only a finite number of distinct string functions, since

$$c^\Lambda_{w(\lambda)+m\gamma+a\delta} = c^\Lambda_\lambda \qquad \text{for } w \in W,\ \gamma \in M \text{ and } a \in \mathbb{C}.$$

The coefficients of the Fourier expansion at $i\infty$ of the string function c^Λ_λ are the weight multiplicities $\mathrm{mult}_\Lambda(\lambda - n\delta)$. They can be calculated with the Freudenthal recursion formula

$$(|\Lambda+\rho|^2 - |\lambda+\rho|^2)\,\mathrm{mult}_\Lambda(\lambda) = 2 \sum_{\alpha \in \Delta_+} \mathrm{mult}(\alpha) \sum_{k \geq 1} (\alpha|\lambda + k\alpha)\,\mathrm{mult}_\Lambda(\lambda + k\alpha). \qquad (5.3.1)$$

It is a consequence of the denominator identity ([KMPS]). Further we describe the transformation properties of the string functions (for type A, B, E) under the modular group Γ which is a result of [KP]

$$c^\Lambda_\lambda(-\frac{1}{\tau}) = |M'/kM|^{-1/2}(-i\tau)^{-l/2} \sum_{\substack{\Lambda' \in P^k_+ \mod \mathbb{C}\delta \\ \lambda' \in P^k \mod (kM' + \mathbb{C}\delta)}} S_{\Lambda,\Lambda'} e((\bar{\lambda}, \bar{\lambda'})/k)\ c^{\Lambda'}_{\lambda'}(\tau)$$

$$S_{\Lambda,\Lambda'} = i^{|\bar{\Delta}_+|} |M'/(k+h^\vee)M|^{-1/2} \sum_{w \in \bar{W}} \epsilon(w) e(-(\bar{\Lambda} + \bar{\rho}, w(\bar{\lambda}' + \bar{\rho}))/(k + h^\vee))$$

$$c^\Lambda_\lambda(\tau + 1) = e(m_{\Lambda,\lambda}) c^\Lambda_\lambda(\tau) \qquad (5.3.2)$$

where $e(a) = e^{2\pi i a}$. These are our main tools in finding the results of the following chapter. The Freudenthal formula gives us the first coefficients of the Fourier expansion of the string functions. This suggests some identities with products of the Dedekind eta-function. These identities are proven by comparing their transformation properties.

Chapter 6

Characters of some conformal field theories

In this chapter we consider some irreducible highest weight representations of the list from Schellekens [S3] and suppose that they can be provided with the structure of a conformal field theory.

The most laborious task of this diploma thesis is to rewrite the character as a sum over the discriminant group over certain lattices N

$$\chi_V = \sum_{\lambda \in N'/N} f_\lambda \, \vartheta_\lambda(\tau, z)$$

and to calculate the coefficients f_λ. This will be done in this chapter.

The important new results are indicated by a box.

6.1 The characters

In this chapter we consider the theories in [S3] with spin-1 algebra $\widehat{A}^r_{p-1,p}$ with $r = 48/(p^2-1)$ and $p = 2, 3, 5$ or 7. We will rewrite their characters in a very simple form which is convenient for our purposes and also shows that they are invariant under $\mathrm{SL}(2, \mathbb{Z})$.

Therefore we note some useful facts.

Let $p = 2, 3, 5$ or 7. The central element of the affine Kac-Moody algebra $\widehat{A}_{p-1}$ is given by $K = \alpha_0^\vee + \ldots + \alpha_{p-1}^\vee$ and the weight $\lambda = (n_0, \ldots, n_{p-1}) = n_0\alpha_0^\vee + \ldots + n_{p-1}\alpha_{p-1}^\vee$ has level $\lambda(K) = n_0 + \ldots + n_{p-1}$. The weights of $\widehat{A}_{p-1}$ of level p can be identified with the weights of A_{p-1}. We call $A'_{p-1}/A_{p-1} \cong \mathbb{Z}/p\mathbb{Z}$ congruence group and its elements congruence classes. We define the class of λ as $n_0 + 2n_1 + \ldots + pn_{p-1} \mod p$. The group of simple currents of $\widehat{A}_{p-1}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and acts on the weight $\lambda = (n_0, \ldots, n_{p-1})$ by cyclic shifting of the coefficients n_i. The class of λ is invariant under the simple currents.

The string functions of $\widehat{A}_{p-1}$ of level p are invariant under the following action of a simple current s:

$$c^\Lambda_\lambda = c^{s.\Lambda}_{s.\lambda} \tag{6.1.1}$$

This can be shown using Freudenthal's formula (5.3.1).

The discriminant of $\sqrt{p}A_{p-1}$ is isomorphic to

$$(\mathbb{Z}/p\mathbb{Z})^{p-2} \times (\mathbb{Z}/p^2\mathbb{Z})\,.$$

There is also a natural isomorphism from the discriminant to A'_{p-1}/pA_{p-1} (cf. p.29).

Let λ be in A'_{p-1}. Write $\lambda = n_1\alpha_1^\vee + \ldots + n_{p-1}\alpha_{p-1}^\vee$. We identify λ with the weight $n_0\alpha_0^\vee + n_1\alpha_1^\vee + \ldots + n_{p-1}\alpha_{p-1}^\vee$ where $n_0 = p - (n_1 + \ldots + n_{p-1})$ of $\widehat{A}_{p-1}$ of level p. Then we map $n_0\alpha_0^\vee + n_1\alpha_1^\vee + \ldots + n_{p-1}\alpha_{p-1}^\vee$ to $(\overline{n}_0, \ldots, \overline{n}_{p-2})$ with $\overline{n}_j = n_j \mod p$ and $\overline{n}_{p-2} = n_0 + 2n_1 + \ldots + pn_{p-1} \mod p^2$. This gives us a map

$$\pi : A'_{p-1} \to (\mathbb{Z}/p\mathbb{Z})^{p-2} \times (\mathbb{Z}/p^2\mathbb{Z})$$

which induces an isomorphism $A'_{p-1}/pA_{p-1} \to (\mathbb{Z}/p\mathbb{Z})^{p-2} \times (\mathbb{Z}/p^2\mathbb{Z})$.

If λ is a weight of $\widehat{A}_{p-1}$ of level p and class i then $\pi(\lambda) = (*, \ldots, *, j)$ with $j = i \mod p$.

There is a one-to-one correspondence between congruence classes and simple currents such that for any weight μ of congruence class i and the corresponding simple current s the identity

$$c^\Lambda_{\lambda+p\mu} = c^\Lambda_{s.\lambda}$$

holds for all Λ in P^k_+ and λ in P^k.Let $\lambda = (n_0, \ldots, n_{p-1}) \in P^k$.

This can be seen as follows. Let s be the simple current which shifts every entry of a weight one to the right and μ an element of congruence class 1. Then $\pi(\lambda + p\mu) = \pi(\nu)$ with $\nu = (n_0, \ldots, n_{p-3}, n_{p-2} - p, n_{p-1} + p)$. Applying the Weyl reflection $w = w_{\alpha_0} \ldots w_{\alpha_{n-1}}$ to ν gives $w(\nu) = (n_{p-1}, n_0, \ldots, n_{p-2}) = s.\lambda$ so that by the invariance of the string functions under the Weyl group $c^\Lambda_{\lambda+p\mu} = c^\Lambda_{w(\lambda+p\mu)} = c^\Lambda_{s.\lambda}$. The general case now follows by induction on the congruence class because every element of congruence class $i+1$ can be written as the sum of an element in class 1 and an element in class i.

Let V be the conformal field theory with spin-1 algebra $\widehat{A}^r_{p-1,p}$ where $r = 48/(p^2-1)$ and $p = 2, 3, 5$ or 7. Then V is the sum of irreducible highest weight modules of $\widehat{A}^r_{p-1}$ the weight of each factor $\widehat{A}_{p-1}$ having level p. The highest weights are obtained by acting with a subgroup G of the group of simple currents of $\widehat{A}^r_{p-1}$ on orbit representatives. We denote by M the set of highest weights. We consider the group G also as a linear code in $\mathbb{F}_p^r$.

We describe this in more detail (cf. [S3]).

In the case $p = 2$ the glue code G is the binary Hamming code of order 16 generated by the rows of the matrix

$$\begin{pmatrix}
1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0\\
1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0
\end{pmatrix}$$

and the orbit representatives are

- $(2,0)^{16}$.
- $8 \times (1,1)^{16}$.
- The remaining orbit representatives can be described as follows. In the dual binary Hamming code of length 16, for every codeword of weight 8, identify the 1-components with the highest weight $(1,1)$ and for the 0-components allow all combinations of $(2,0)$ and $(0,2)$ such that both of these highest weights appear an odd number of times.

In the case $p = 3$ the glue code G is the ternary zero-sum code of length 6 generated by the rows of the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the orbit representatives are

- $(3,0,0)^6$.
- $(1,1,1)^4(3,0,0)^2$ and all permutations.
- $(2,0,1)^5(0,1,2)$ and $(2,1,0)^5(0,2,1)$.
- $6 \times (1,1,1)^6$.

In the case $p = 5$ the glue code is $\mathbb{F}_5^2$ and the orbit representatives are

- $(5,0,0,0,0)^2$.
- $(2,0,1,0,2)^2$.
- $(2,0,0,2,1)(3,0,1,1,0)$ and $(3,0,1,1,0)(2,0,0,2,1)$.
- $(1,1,1,1,1)(1,0,0,1,3)$ and $(1,0,0,1,3)(1,1,1,1,1)$.
- $4 \times (1,1,1,1,1)^2$.

In the case $p = 7$ the glue code is $\mathbb{F}_7$ and the orbit representatives are

- $(7,0,0,0,0,0,0)$.
- $(2,0,0,1,3,0,1)$ and $(2,1,0,3,1,0,0)$.
- $(2,0,0,2,0,3,0)$.
- $(1,0,1,0,1,2,2)$.
- $3 \times (1,1,1,1,1,1,1)$

The elements of the code G and the dual code $G^\perp$ are in $(\mathbb{Z}/p\mathbb{Z})^r$ so they can be identified naturally with the discriminant of A^r_{p-1}. Then $G^\perp$ is the orthogonal complement of G. We denote by $A^r_{p-1} \times G$ the lattice obtained by gluing the elements of G to A^r_{p-1}.

Case-by-case the congruence classes are calculated. The result is that $M \subset A^r_{p-1} \times G^\perp$. Two elements are in the same congruence class if and only if they differ by a root lattice vector. This means $c^\Lambda_\lambda \neq 0$ only if Λ and λ are in the same congruence class. Therefore any nonzero contribution to the character of V comes from a weight λ in $A^r_{p-1} \times G^\perp$.

Now we are ready to rewrite the character.

$$\begin{aligned}
\chi_V &= \sum_{\Lambda \in M} \text{mult}\,\Lambda\ \chi_\Lambda \\
&= \sum_{\Lambda \in M} \text{mult}\,\Lambda \prod_{i=1}^{r} \sum_{\lambda \in A'_{p-1}/pA_{p-1}} c^{\Lambda_i}_\lambda(\tau)\,\theta_\lambda(\tau, z_i) \\
&= \sum_{\lambda \in Dual A^r_{p-1}/pA^r_{p-1}} \sum_{\Gamma \in M/G} \sum_{\Lambda \in G.\Gamma} \text{mult}\,\Lambda \prod_{i=1}^{r} c^{\Lambda_i}_{\lambda_i}(\tau)\,\theta_{\lambda_i}(\tau, z_i) \\
&= \sum_{\lambda \in A^r_{p-1} \times G^\perp/pA^r_{p-1}} \sum_{\Lambda \in M/G} \sum_{g \in G} (\text{mult}\,\Lambda)/|G_\Lambda| \prod_{i=1}^{r} c^{g_i.\Lambda_i}_{\lambda_i}(\tau)\,\theta_{\lambda_i}(\tau, z_i)
\end{aligned}$$

Since any nonzero contribution to the character of V comes from a weight λ in $A^r_{p-1} \times G^\perp$.

$$\chi_V = \sum_{\lambda \in A^r_{p-1} \times G^\perp/pA^r_{p-1}} \sum_{\Lambda \in M/G} \sum_{g \in G} (\text{mult}\,\Lambda)/|G_\Lambda| \prod_{i=1}^{r} c^{\Lambda_i}_{g_i.\lambda_i}(\tau)\,\theta_{\lambda_i}(\tau, z_i)$$

Using $c^\Lambda_\lambda = c^{s.\Lambda}_{s.\lambda}$ for any simple current s in G.

$$\chi_V = \sum_{\lambda \in A^r_{p-1} \times G^\perp/pA^r_{p-1}} \prod_{i=1}^{r} \theta_{\lambda_i}(\tau, z_i) \sum_{\Lambda \in M/G} \sum_{g \in G} (\text{mult}\,\Lambda)/|G_\Lambda| \prod_{i=1}^{r} c^{\Lambda_i}_{g_i.\lambda_i}(\tau)\,.$$

Using the one-to-one correspondence between congruence classes and simple currents we get

$$\begin{aligned}
\chi_V &= \sum_{\lambda\in A_{p-1}^r\times G^\perp/p(A_{p-1}^r\times G)} \;\sum_{\mu\in p(A_{p-1}^r\times G)/pA_{p-1}^r} \prod_{i=1}^r \theta_{\lambda_i+\mu_i}(\tau,z_i) \\
&\qquad \sum_{\Lambda\in M/G}\sum_{g\in G}(\operatorname{mult}\Lambda)/|G_\Lambda| \prod_{i=1}^r c^{\Lambda_i}_{g_i.\lambda_i+\mu_i}(\tau) \\
&= \sum_{\lambda\in A_{p-1}^r\times G^\perp/p(A_{p-1}^r\times G)} \;\sum_{\mu\in p(A_{p-1}^r\times G)/pA_{p-1}^r} \prod_{i=1}^r \theta_{\lambda_i+\mu_i}(\tau,z_i) \\
&\qquad \sum_{\Lambda\in M/G}\sum_{g\in G}(\operatorname{mult}\Lambda)/|G_\Lambda| \prod_{i=1}^r c^{\Lambda_i}_{g_{\mu_i}.g_i.\lambda_i}(\tau) \\
&= \sum_{\lambda\in A_{p-1}^r\times G^\perp/p(A_{p-1}^r\times G)} \;\sum_{\mu\in p(A_{p-1}^r\times G)/pA_{p-1}^r} \prod_{i=1}^r \theta_{\lambda_i+\mu_i}(\tau,z_i) \\
&\qquad \sum_{\Lambda\in M/G}\sum_{g\in G}(\operatorname{mult}\Lambda)/|G_\Lambda| \prod_{i=1}^r c^{\Lambda_i}_{g_i.\lambda_i}(\tau)\,.
\end{aligned}$$

Now

$$\begin{aligned}
&\sum_{\mu\in p(A_{p-1}^r\times G)/pA_{p-1}^r} \prod_{i=1}^r \theta_{\lambda_i+\mu_i}(\tau,z_i) = \\
&\sum_{\mu\in p(A_{p-1}^r\times G)/pA_{p-1}^r} \prod_{i=1}^r \sum_{\nu\in pA_{p-1}} e(\tau(\lambda_i+\mu_i+\nu)^2/2p + (\lambda_i+\mu_i+\nu, z_i)) = \\
&\sum_{\mu\in p(A_{p-1}^r\times G)/pA_{p-1}^r} \;\sum_{\nu\in pA_{p-1}^r} \prod_{i=1}^r e(\tau(\lambda_i+\mu_i+\nu_i)^2/2p + (\lambda_i+\mu_i+\nu_i, z_i)) = \\
&\sum_{\mu\in p(A_{p-1}^r\times G)} \prod_{i=1}^r e(\tau(\lambda_i+\mu_i)^2/2p + (\lambda_i+\mu_i, z_i)) = \\
&\sum_{\mu\in\sqrt{p}(A_{p-1}^r\times G)} \prod_{i=1}^r e(\tau(\lambda_i+\mu_i)^2/2 + \sqrt{p}(\lambda_i+\mu_i, z_i)) =: \vartheta_\lambda(\tau,z),
\end{aligned}$$

where $\vartheta_\lambda(\tau,z)$ are the theta functions of the lattice $N=\sqrt{p}(A_{p-1}^r\times G)$. Finally the character is

$$\boxed{\begin{aligned}
\chi_V &= \sum_{\lambda\in(1/\sqrt{p})(A_{p-1}^r\times G^\perp)/\sqrt{p}(A_{p-1}^r\times G)} \vartheta_\lambda(\tau,z) \\
&\qquad \sum_{\Lambda\in M/G}\sum_{g\in G}(\operatorname{mult}\Lambda)/|G_\Lambda| \prod_{i=1}^r c^{\Lambda_i}_{g_i.\sqrt{p}\lambda_i}(\tau) \\
&= \sum_{\lambda\in(1/\sqrt{p})(A_{p-1}^r\times G^\perp)/\sqrt{p}(A_{p-1}^r\times G)} f_\lambda(\tau)\,\vartheta_\lambda(\tau,z)
\end{aligned}} \tag{6.1.2}$$

with

$$\boxed{f_\lambda(\tau) = \sum_{\Lambda \in M/G} \sum_{g \in G} (\operatorname{mult}\Lambda)/|G_\Lambda| \prod_{i=1}^{r} c^{\Lambda_i}_{g_i \cdot \sqrt{p}\lambda_i}(\tau).}$$

6.2 The grading lattice

Consider the lattice $N = \sqrt{p}(A_{p-1}^r \times G)$. First observe that the dual lattice N' is $(1/\sqrt{p})(A_{p-1}^r \times G^\perp)$. In the cases $p = 5$ and $p = 7$ this is obvious, since $N = \sqrt{p}A'^r_{p-1}$ and $N' = (1/\sqrt{p})A_{p-1}^r$. For the other two cases we calculate this explicitly. First one computes that the product of two weights λ, μ in A'_{p-1} depends only on the congruence class, i.e.

$$(\lambda_i, \lambda_j) = -ij/p \mod 1$$

for λ_i in congruence class i and λ_j in congruence class j. Recall that the product of two elements $x = (i_1, \dots, i_r)$ and $y = (j_1, \dots, j_r)$ of two codes is $i_1 j_1 + \dots i_r j_r$ mod p. This means (λ, μ) in $\mathbb{Z}$ if and only if the code word corresponding to the congruence class of λ is orthogonal to the code word corresponding to the congruence class of μ. $G^\perp$ is the orthogonal complement of G, hence N' is the dual lattice of N.

We want to know the genus of the lattices N. Recall the generators of G as the rows of the matrices listed in the last section. Denote the i-th row by b_i. Then in the case $p = 2$ the code $G^\perp$ is generated by $b_1 + \dots + b_7, b_1 + b_2 + b_3, b_1 + b_4 + b_5, b_2 + b_4 + b_6, b_8 + b_9 + b_{10}$ and in the case $p = 3$ the code $G^\perp$ is generated by $b_1 + \dots + b_5$. Hence in all four cases we have $G^\perp \subset G$. But this means $pN' \subset N$. Therefore the discriminant of N is

$$\begin{aligned}
\operatorname{disc}(N) &= \operatorname{disc}(\sqrt{p}(A_{p-1}^r \times G)) \\
&= p^{p-1}\operatorname{disc}(A_{p-1}^r \times G) \\
&= p^{p-1}\operatorname{disc}(A_{p-1}^r)/|G|^2 \\
&= p^p/|G|^2 \\
&= 2^{32}/2^{22} = 2^{10} \quad \text{for } p = 2, \\
&= 3^{18}/3^{10} = 3^8 \quad \text{for } p = 3, \\
&= 5^{10}/5^4 \ = 5^6 \quad \text{for } p = 5, \\
&= 7^7/7^2 \ \ = 7^5 \quad \text{for } p = 7.
\end{aligned}$$

The signum is $+$ except for the case $p = 3$ it is $-$ ((3.4.1) and ([CS, p. 387, Th. 13]). The minimal norm of $\sqrt{p}A'_{p-1}$ is $p - 1$. Hence the minimal norm of N is $(p - 1)$ times the minimal distance of G, where the distance of a nonzero codeword $(a_1, \dots, a_p)$ is the number of nonzero entries $a_i \neq 0$. The minimal distances are 4 for $p = 2$, 2 for $p = 3$ and 1 otherwise. Hence the minimal norm is 4 except for $p = 7$ it is 6.

We conclude: the grading lattice N is of genus

$$\boxed{II_{2m,0}(p^{\epsilon_p(m+2)}),} \tag{6.2.1}$$

with $\epsilon_p = +$ for $p = 2, 5, 7$ and $\epsilon_p = -$ for $p = 3$, $m = 24/(p + 1)$ and minimal norm 4, except for $p = 7$ it is 6.

N is the unique lattice in its genus of maximal minimal norm.

6.3 The coefficients f_λ

The result of the last two sections was that the character of the conformal field theories V can be written as

$$\chi_V(\tau,z) = \sum_{\lambda\in N'/N} f_\lambda(\tau)\vartheta_\lambda(\tau,z) \tag{6.3.1}$$

over the grading lattice N of genus

$$II_{2m,0}(p^{\epsilon_p(m+2)}), \qquad \epsilon_p = + \text{ for } p=2,5,7 \text{ and } \epsilon_p = - \text{ for } p=3,$$

and minimal norm 4, except for $p=7$ where it is 6.

The coefficients f_λ have the form

$$f_\lambda(\tau) = \sum_{\Lambda\in M/G}\sum_{g\in G}(\operatorname{mult}\Lambda)/|G_\Lambda|\prod_{i=1}^{r} c^{\Lambda_i}_{g_i\cdot\sqrt{p}\lambda_i}(\tau).$$

for λ in $N' = (1/\sqrt{p})(A^r_{p-1}\times G^\perp)$. From now on we will naturally identify $(1/\sqrt{p})(A^r_{p-1}\times G^\perp)$ with $A^r_{p-1}\times G^\perp$ to get rid of the factor $\sqrt{p}$.

Recall the eta product

$$h(\tau) = \frac{1}{\eta(\tau)^m\eta(p\tau)^m} = q^{-1} + m + \frac{1}{2}\dots$$

and the T-invariant parts of $h(\tau/p)$

$$g_j(\tau) = \frac{1}{p}\sum_{k=0}^{p-1} e(-kj/p)h\big((\tau+k)/p\big)$$

introduced in section 3.7. Then the f_λ have the following form:

$$\boxed{f_\gamma = \begin{cases} h(\tau)+g_0(\tau) & \text{if } \gamma = 0\\ g_k(\tau) & \text{if } \gamma\neq 0 \text{ and } -\gamma^2/2\equiv k/p \bmod \mathbb{Z}\end{cases}} \tag{6.3.2}$$

There are three methods to prove the identities (6.3.2). We will explain them, since this has been most of the work and the first proofs led us to the final one.

In the cases of $\widehat{A}_{1,2}$ and $\widehat{A}_{2,3}$ the string functions are explicitly known. They are modular forms for the modular group $\tilde{\Gamma}(16)$ of weight $-1/2$, respectively $\Gamma(18)$ of weight -1. Hence the f_λ are modular functions for $\Gamma(\tilde{16})$ of weight -8, respectively of $\Gamma(18)$ of weight -6. The space of modular forms for a congruence group of given weight is finite dimensional, its dimension can be calculated with the theorem of Riemann and Roch, but this is difficult. We found another way to calculate an upper bound of the dimension.

Denote by $\mathbb{V}_k(\Gamma(N))$ the space of modular forms of the principal congruence group $\Gamma(N)$ with poles of order one. Consider the map

$$\begin{aligned} m_\Delta : \mathbb{V}_k(\Gamma(N)) &\longrightarrow \mathbb{V}_{k+12}(\Gamma(N))\\ f_\lambda(\tau) &\mapsto \Delta(\tau)\cdot f_\lambda(\tau)\end{aligned} \tag{6.3.3}$$

with the Delta function $\Delta(\tau) = (\sqrt{2\pi}\eta(\tau))^{12}$. The kernel of this map is trivial since $\eta(\tau)$ and hence also $\Delta(\tau)$ have nonzero values on the entire upper half plane $\mathbb{H}$. This means that the dimension of $\mathbb{V}_k(\Gamma(N))$ is at most $\mathbb{V}_{k+12}(\Gamma(N))$. The order of the poles of the $f_\lambda(\tau)$ is one, hence $\Delta(\tau) \cdot f_\lambda(\tau)$ has no poles at the cusps, which means it is holomorphic at the cusps. So the $\Delta(\tau) \cdot f_\lambda(\tau)$ are integral modular forms. Example 3 on page 26 of [G] gives a formula for the dimension of integral modular forms of given weight k

$$\dim(\mathbb{M}_k(\Gamma(N))) = \frac{(2k-1)N+6}{24} N^2 \prod_{\substack{p|N \\ p \text{ prime}}} (1 - \frac{1}{p^2}).$$

Hence we have

$$\begin{aligned} &\dim(\mathbb{V}_{-8}(\Gamma(16))) \leq 944, \\ &\dim(\mathbb{V}_{-6}(\Gamma(18))) \leq 1836 \quad \text{and} \\ &\dim(\mathbb{V}_{-4}(\Gamma(10))) \leq 468. \end{aligned}$$

So in order to prove (6.3.2) we only have to compare sufficiently many coefficients.

For the two remaining cases the string functions are not explicitly known. So we calculate them for $\widehat{A}_{4,5}$. Using the Freudenthal formula, one gets the first coefficients, comparing them with the first coefficients of the Dedekind η-function suggests the following identities. We use the short hand notation $c^\Lambda_{s.\lambda}(\tau)$ for the sum over all simple currents s. This list of string functions is one of the main new results.

$$\begin{aligned} c^{12002}_{s.31001}(\tau) &= c^{30110}_{s.12002}(\tau) = c^{10220}_{s.11111}(\tau) - c^{50000}_{s.11111}(\tau) \\ c^{12002}_{s.12002}(\tau) &= c^{30110}_{s.30110}(\tau) = c^{10220}_{s.10220}(\tau) - c^{50000}_{s.10220}(\tau) \\ c^{12002}_{s.10220}(\tau) &= c^{30110}_{s.31001}(\tau) = c^{10220}_{s.30110}(\tau) - c^{50000}_{s.30110}(\tau) \\ c^{12002}_{s.30110}(\tau) &= c^{30110}_{s.50000}(\tau) = c^{10220}_{s.31001}(\tau) - c^{50000}_{s.31001}(\tau) = c^{30110}_{s.11111}(\tau) \\ c^{12002}_{s.50000}(\tau) &= c^{30110}_{s.10220}(\tau) = c^{10220}_{s.12002}(\tau) - c^{50000}_{s.12002}(\tau) = c^{12002}_{s.11111}(\tau) \end{aligned} \tag{6.3.4}$$

$$\begin{aligned} c^{11111}_{10220}(\tau) &= c^{31001}_{s.10220}(\tau) \\ c^{11111}_{30110}(\tau) &= c^{31001}_{s.30110}(\tau) \\ 4\, c^{11111}_{11111}(\tau) + c^{11111}_{50000}(\tau) &= 4\, c^{31001}_{s.11111}(\tau) + c^{31001}_{s.50000}(\tau) \end{aligned} \tag{6.3.5}$$

$$\begin{aligned} [c^{11111}_{11111}(\tau) - c^{31001}_{s.11111}(\tau)] &= \\ \tfrac{1}{4}\, [c^{31001}_{s.50000}(\tau) - c^{11111}_{50000}(\tau)] &= \frac{\eta(5\tau)}{\eta(\tau)^5} \end{aligned} \tag{6.3.6}$$

$$\begin{aligned} &c^{50000}_{s.50000}(\tau) - c^{10220}_{s.50000}(\tau) + \tfrac{1}{2}\, [c^{30110}_{s.12002}(\tau) + c^{12002}_{s.31001}(\tau)] = \\ &c^{50000}_{s.50000}(\tau) - c^{10220}_{s.50000}(\tau) + \ [c^{10220}_{s.11111}(\tau) - c^{50000}_{s.11111}(\tau)] = \frac{1}{(\eta(\tau)\eta(5\tau))^2} \end{aligned} \tag{6.3.7}$$

$$\begin{gathered}
c^{50000}_{s.50000}(\tau) + c^{10220}_{s.50000}(\tau) + c^{11111}_{50000}(\tau) + c^{31001}_{s.50000}(\tau) - \\
c^{50000}_{s.11111}(\tau) - c^{10220}_{s.11111}(\tau) - c^{11111}_{11111}(\tau) - c^{31001}_{s.11111}(\tau) = \\
\frac{\eta(\tau)^3\eta(5\tau)}{\eta(\tau/2)^3\eta(2\tau)^3\eta(5\tau/2)\eta(10\tau)}
\end{gathered} \tag{6.3.8}$$

$$c^{11111}_{11111}(\tau) - c^{11111}_{50000}(\tau) = \frac{\eta(10\tau)\eta(2\tau)^3}{\eta(5\tau)^2\eta(\tau)^6} \tag{6.3.9}$$

$$c^{11111}_{11111}(\tau) + 4c^{11111}_{50000}(\tau) - 5c^{31001}_{s.11111}(\tau) = \frac{\eta(10\tau)^2}{\eta(5\tau)^3\eta(2\tau)^2\eta(\tau)} \tag{6.3.10}$$

$$\begin{gathered}
\tfrac{6}{5}\ [c^{31001}_{s.11111}(\tau) - c^{11111}_{11111}(\tau)] + \tfrac{1}{20}[c^{31001}_{s.50000}(\tau) - c^{11111}_{50000}(\tau)] + \\
c^{31001}_{s.31001}(\tau) - c^{11111}_{31001}(\tau) + c^{31001}_{s.12002}(\tau) - c^{11111}_{12002}(\tau) + \\
\tfrac{3}{2}[c^{31001}_{s.30110}(\tau) - c^{11111}_{30110}(\tau) + c^{31001}_{s.10220}(\tau) - c^{11111}_{10220}(\tau)] = \\
\tfrac{1}{5}\ [c^{31001}_{s.11111}(\tau) - c^{11111}_{11111}(\tau)] - \tfrac{1}{5}[c^{31001}_{s.50000}(\tau) - c^{11111}_{50000}(\tau)] + \\
c^{31001}_{s.31001}(\tau) - c^{11111}_{31001}(\tau) + c^{31001}_{s.12002}(\tau) - c^{11111}_{12002}(\tau) + \\
-[c^{31001}_{s.30110}(\tau) - c^{11111}_{30110}(\tau) + c^{31001}_{s.10220}(\tau) - c^{11111}_{10220}(\tau)] = \frac{\eta(\tau/5)}{\eta(\tau)^5}
\end{gathered} \tag{6.3.11}$$

$$\begin{aligned}
&2\ c^{12002}_{s.31001}(\tau) + 2\ c^{30110}_{s.12002}(\tau) +\ [c^{10220}_{s.11111}(\tau) - c^{50000}_{s.11111}(\tau)] + \\
&2\ c^{12002}_{s.12002}(\tau) + 2\ c^{30110}_{s.30110}(\tau) +\ [c^{10220}_{s.10220}(\tau) - c^{50000}_{s.10220}(\tau)] + \\
&2\ c^{12002}_{s.10220}(\tau) + 2\ c^{30110}_{s.31001}(\tau) +\ [c^{10220}_{s.30110}(\tau) - c^{50000}_{s.30110}(\tau)] + \\
&2\ c^{12002}_{s.30110}(\tau) + 2\ c^{30110}_{s.11111}(\tau) +\ [c^{10220}_{s.31001}(\tau) - c^{50000}_{s.31001}(\tau)] + \\
&2\ c^{30110}_{s.10220}(\tau) + 2\ c^{12002}_{s.11111}(\tau) +\ [c^{10220}_{s.12002}(\tau) - c^{50000}_{s.12002}(\tau)] = \\
&\tfrac{19}{10}\ c^{12002}_{s.31001}(\tau) + \tfrac{19}{10}\ c^{30110}_{s.12002}(\tau) + \tfrac{12}{10}\ [c^{10220}_{s.11111}(\tau) - c^{50000}_{s.11111}(\tau)] + \\
&\tfrac{14}{10}\ c^{12002}_{s.12002}(\tau) + \tfrac{24}{10}\ c^{30110}_{s.30110}(\tau) + \tfrac{12}{10}\ [c^{10220}_{s.10220}(\tau) - c^{50000}_{s.10220}(\tau)] + \\
&\tfrac{24}{10}\ c^{12002}_{s.10220}(\tau) + \tfrac{14}{10}\ c^{30110}_{s.31001}(\tau) + \tfrac{12}{10}\ [c^{10220}_{s.30110}(\tau) - c^{50000}_{s.30110}(\tau)] + \\
&\tfrac{24}{10}\ c^{12002}_{s.30110}(\tau) + \tfrac{1}{10}\ c^{30110}_{s.50000}(\tau) + \tfrac{18}{10}\ c^{30110}_{s.11111}(\tau) + \\
&\tfrac{7}{10}\ [c^{10220}_{s.31001}(\tau) - c^{50000}_{s.31001}(\tau)] + \\
&\tfrac{1}{10}\ c^{12002}_{s.50000}(\tau) + \tfrac{24}{10}\ c^{30110}_{s.10220}(\tau) + \tfrac{18}{10}\ c^{12002}_{s.11111}(\tau) + \\
&\tfrac{7}{10}\ [c^{10220}_{s.12002}(\tau) - c^{50000}_{s.12002}(\tau)] = \\
&\qquad\qquad\qquad\qquad \frac{1}{(\eta(\tau)\eta(\tau/5))^2}
\end{aligned} \tag{6.3.12}$$

$$\begin{gathered}
\sum_{\lambda\ \in\ M} [c^{31001}_{s.\lambda}(\tau) + c^{11111}_{\lambda}(\tau) + c^{50000}_{s.\lambda}(\tau) + c^{10220}_{s.\lambda}(\tau)]\ = \\
\frac{\eta(\tau)^3\eta(\tau/5)}{\eta(\tau/2)^3\eta(2\tau)^3\eta(2\tau/5)\eta(\tau/10)}
\end{gathered} \tag{6.3.13}$$

$$\sum_{\lambda \in M} [c^{31001}_{s.\lambda}(\tau) + c^{11111}_{\lambda}(\tau) - c^{50000}_{s.\lambda}(\tau) - c^{10220}_{s.\lambda}(\tau)] = \frac{\eta(\tau/10)\eta(\tau/2)^3}{\eta(\tau/5)^2\eta(\tau)^6} \quad (6.3.14)$$

$$\sum_{\lambda \in M} [c^{50000}_{s.\lambda}(\tau) + c^{10220}_{s.\lambda}(\tau) - 2c^{11111}_{\lambda}(\tau)] = \frac{\eta(\tau/10)^2}{\eta(\tau/5)^3\eta(\tau/2)^2\eta(\tau)} \quad (6.3.15)$$

$$M = \{ 11111 , 10220 , 12002 , 30110 , 31001 \}$$

The identities (6.3.11) - (6.3.15) are the S transformations of (6.3.6) - (6.3.10). (6.3.5) follows directly from (6.3.6), the string function identities in (6.3.4) hold because of (6.3.6) and the S transformation, i.e. the identities, which are not obtained by (6.3.6), one gets from the S transformation (appendix B) of the identities already obtained. So we have to prove (6.3.6) - (6.3.10). The proof is as follows: we consider the quotient of the left hand side and right hand side of an identity and call it A(τ). We show, that A(τ) is an integral modular form for a certain congruence group of weight zero and hence constant (3.2.1). Comparing the first coefficient yields the constant.

Consider (6.3.6), then

$$A(\tau) = (c^{11111}_{11111}(\tau) - c^{31001}_{s.11111})(\tau)/(\eta(5\tau)/\eta(\tau)^5).$$

$\Gamma_0(5)$ is generated by T, ST^5S, ST^3ST^2S, which are easily calculated for the string functions using the transformation matrices in appendix B. One gets:

$$\begin{aligned}
((c^{11111}_{11111}(\tau) - c^{31001}_{s.11111}(\tau))|T) &= c^{11111}_{11111}(\tau) - c^{31001}_{s.11111}(\tau) \\
((c^{11111}_{11111}(\tau) - c^{31001}_{s.11111}(\tau))|ST^5S) &= \frac{1}{(5\tau-1)^2}(c^{11111}_{11111}(\tau) - c^{31001}_{s.11111})(\tau) \\
((c^{11111}_{11111}(\tau) - c^{31001}_{s.11111}(\tau))|ST^3ST^2S) &= \frac{1}{(5\tau-2)^2}(c^{11111}_{11111}(\tau) - c^{31001}_{s.11111}(\tau))
\end{aligned}$$

The transformations of $\eta(5\tau)/\eta(\tau)^5$ are obtained using (3.3.2) and (C.0.4):

$$\begin{aligned}
(\eta(5\tau)/\eta(\tau)^5|T) &= \eta(5\tau)/\eta(\tau)^5 \\
(\eta(5\tau)/\eta(\tau)^5|ST^5S) &= (\frac{1}{\sqrt{5}\tau^2}(\eta(\tau/5)/\eta(\tau)^5)|ST^5) \\
&= (e^{-25\pi i/12}\frac{1}{\sqrt{5}(\tau+5)^2}(\eta((\tau+5)/5)/\eta(\tau)^5)|S) \\
&= \frac{1}{(5\tau-1)^2}\eta(5\tau)/\eta(\tau)^5
\end{aligned}$$

$$
\begin{aligned}
&(\eta(5\tau)/\eta(\tau)^5|ST^3ST^2S) = \\
&(\frac{1}{\sqrt{5}\tau^2}(\eta(\tau/5)/\eta(\tau)^5)|ST^3ST^2) = \\
&(e^{-10\pi i/12}\frac{1}{\sqrt{5}(\tau+2)^2}(\eta((\tau+2)/5)/\eta(\tau)^5)|ST^3S) = \\
&(e^{-10\pi i/12}\frac{1}{\sqrt{5}(-\frac{1}{\tau}+2)^2}\frac{1}{\tau^2}(\eta((\tau+2)/5)/\eta(\tau)^5)|ST^3) = \\
&(e^{-25\pi i/12}\frac{1}{\sqrt{5}(-\frac{1}{\tau+3}+2)^2}\frac{1}{(\tau+3)^2}(\eta((\tau+5)/5)/\eta(\tau)^5)|S) = \\
&(\frac{1}{\sqrt{5}(2\tau+5)^2}(\eta(\tau/5)/\eta(\tau)^5)|S) = \\
&\frac{1}{(5\tau-2)^2}\eta(5\tau)/\eta(\tau)^5
\end{aligned}
$$

We have shown that $A(\tau)$ is invariant under the action of $\Gamma_0(5)$. Since every string function and $\eta(\tau)$ are holomorphic on $\mathbb{H}$ and $\eta(\tau) \neq 0$ for every τ in $\mathbb{H}$, A(τ) is also holomorphic on $\mathbb{H}$. Comparing the first coefficients of the Fourier expansion of $c^{11111}_{11111}(\tau) - c^{31001}_{s.11111}(\tau)$ and $\eta(5\tau)/\eta(\tau)^5$ yields, that $A(\tau)$ has no pole at $i\infty$. There is neither a pole at 0, since the first coefficient of the S-transformations of $c^{11111}_{11111}(\tau) - c^{31001}_{s.11111}(\tau)$ and $\eta(5\tau)/\eta(\tau)^5$ are also equal. 0 and $i\infty$ are the only cusps of $\Gamma_0(5)$, therefore $A(\tau)$ is an integral modular form for $\Gamma_0(5)$ of weight zero with trivial character, hence $A(\tau)$ is constant. Comparing the first coefficient shows that the constant must be 1.

The other identity of (6.3.6) as well as (6.3.7) are proven exactly in the same way. (6.3.9) and (6.3.10) are a little more laborious, since the eta products and the string functions $c^{11111}_{11111}(\tau) - c^{11111}_{50000}(\tau)$ and $c^{11111}_{11111}(\tau) + 4c^{11111}_{50000}(\tau) - 5c^{31001}_{s.11111}(\tau)$ are modular forms for $\Gamma_0(10)$. $\Gamma_0(10)$ is generated by $T, ST^{-3}ST^3S$, $ST^{-2}ST^5S$ T^2S, $ST^9ST^{-1}S$ and $ST^{-1}ST^2ST^{-3}S$, the cusps of $\Gamma_0(10)$ are $1/1, 1/2, 1/5$ and $1/10$.

The proof of equation (6.3.8) is the hardest one, because right-hand side and left-hand side of (6.3.8) are not T-invariant. Consider the quotient of left-hand side and right-hand side

$$
\begin{aligned}
A(\tau) = (&c^{50000}_{s.50000}(\tau) + c^{10220}_{s.50000}(\tau) - c^{50000}_{s.11111}(\tau) - c^{10220}_{s.11111}(\tau) + \\
&c^{11111}_{50000}(\tau) + c^{31001}_{s.50000}(\tau) - c^{11111}_{11111}(\tau) - c^{31001}_{s.11111}(\tau))/ \\
&\frac{\eta(\tau)^3\eta(5\tau)}{\eta(\tau/2)^3\eta(2\tau)^3\eta(5\tau/2)\eta(10\tau)}
\end{aligned}
\tag{6.3.16}
$$

We show that $A(\tau)$ is in $\mathbb{M}_0(\Gamma_0(10) \cap \Gamma(2), 1)$. $\Gamma_0(10) \cap \Gamma(2)$ is generated by

(calculated with magma)

$$\begin{aligned}
&ST\frac{1}{2}ST\frac{1}{2}ST^4S,\\
&ST^{-7}ST^{-2}STS,\\
&TST^{-3}ST^3S,\\
&ST^{-3}ST^{-6}STS,\\
&ST^{-3}ST^3STST^2ST^6ST\frac{1}{2}S,\\
&ST^{-3}ST^3STST^4ST^2ST^2S \text{ and}\\
&ST^{-3}ST^{-4}ST^{-3}ST^{-1}ST^{-2}ST^{-6}ST^{-2}ST^{-1}.
\end{aligned}$$

Using appendix B and appendix C we calculate that $A(\tau)$ is invariant under the action of $\Gamma_0(10) \cap \Gamma(2)$. The cusps of $\Gamma_0(10) \cap \Gamma(2)$ are $0, 1/3, 1/4, 1/5, 2/5$ and $i\infty$. We check that $A(\tau)$ has no poles at these cusps. Since $A(\tau)$ is holomorphic on the upper half plane $\mathbb{H}$ it is in $\mathbb{M}_0(\Gamma_0(10) \cap \Gamma(2), 1)$ and hence constant, the constant is one. This proves (6.3.8).

Using these identities we can prove (6.3.2) for $\widehat{A}_{4,5}^2$ by comparing sufficiently many coefficients.

Having computed the S and T transformation of the string functions we immediately get the transformation of the f_λ because they are polynomials in the string functions. Also the first coefficient of the string functions gives us the first coefficient of the f_λ. So we can apply the method used above to prove the string function identities in order to prove the identities (6.3.2). We consider $A_{\lambda_0}(\tau) = (f_0(\tau) - f_{\lambda_0}(\tau))/h(\tau)$ for every isotropic element λ_0 and show that $A_{\lambda_0}(\tau)$ is an integral modular form of weight zero of trivial character, hence constant. Comparing coefficients yields the constant to be one. Applying the S transformation to all of these identities yields the remaining identities.

In the appendix we list the known string functions, the S and T matrices and the first coefficients of the f_λ and we describe the proofs in more detail.

There is a third method of proof. Regarding the transformation properties of the coefficients f_λ in the case $\widehat{A}_{6,7}$ (A.4.1), we immediately observe that the f_λ transform under the Weil representation ρ_N of the grading lattice N. Computer calculations show that this is also true for the other three cases. An immediate consequence is that the character χ_V is modular invariant, since the theta functions transform under the corresponding dual Weil representation. Another immediate consequence is that equation (6.3.2) is true. This can be seen as follows: Recall the modular form $F(\tau)$ for the Weil representation ρ_N of section 3.7

$$F(\tau) = \sum_{\gamma \in N'/N} F_\gamma(\tau) e^\gamma$$

with

$$\begin{aligned}
F_\gamma(\tau) &= h(\tau) + g_0(\tau) && \text{if } \gamma = 0\\
&= g_k(\tau) && \text{if } -\gamma^2/2 \equiv k/p \mod 1
\end{aligned}$$

with

$$h(\tau) = \frac{1}{(\eta(\tau)\eta(p\tau))^m} = q^{-1} + m + \dots$$

and the g_k the T-invariant parts of $h(\tau/p) = g_0(\tau) + g_1(\tau) + \ldots + g_{p-1}(\tau)$.

Further define

$$\tilde{F}(\tau) = \sum_{\gamma \in N'/N} f_\gamma(\tau) e^\gamma$$

and consider $F(\tau) - \tilde{F}(\tau)$. This is a modular form for $SL(2, \mathbb{Z})$ (since $F(\tau)$ and $\tilde{F}(\tau)$ are modular forms) of negative weight. We calculated the first coefficients of the f_γ and observed in the previous proof that the first coefficient of f_λ equals the first coefficient of F_λ for every λ in N'/N. Furthermore $F(\tau) - \tilde{F}(\tau)$ has no singular terms so that $F(\tau) - \tilde{F}(\tau)$ is a holomorphic modular form of negative weight which is also holomorphic at the cusp $i\infty$. A well known result of the theory of modular forms is that a modular form for $SL(2, \mathbb{Z})$ of negative weight without singularities at the cusp $i\infty$ is zero. Therefore $F(\tau) = \tilde{F}(\tau)$ and since the e^γ are linear independent (6.3.2) must be true.

Chapter 7

Construction of some bosonic string theories

This chapter corresponds to chapter 4 of [Sch3] where the physical states of a bosonic string corresponding to the fake monster algebra are constructed. Let V be the vector spaces defined in the previous chapter. We assume that these vector spaces can be provided with the structure of a conformal vertex algebra of central charge 24. This has been conjectured in [M2]. We further have to assume that V has a positive definite bilinear $(\cdot,\cdot)$ form with the property that the adjoint of the Virasoro generator L_m is L_{-m}.

7.1 The Lie algebra of physical states

Recall the grading lattices $N = II_{2m,0}(p^{\epsilon_p(m+2)})$, where $p = 2, 3, 5, 7$ and $m = 24/(p+1)$, associated to the spaces V. As in section 2.9 we define the vertex superalgebras

$$\mathbf{V} = V \otimes V_{II_{1,1}} \otimes V_\sigma .$$

V_σ is the vertex superalgebra associated to the one-dimensional lattice $\mathbb{Z}\sigma$ with $\sigma^2 = 1$. We define the ghosts $b = e^{-\sigma}$ and $c = e^{\sigma}$ and the ghost current $j^G = \sigma(-1)$. The ghost number operator is $j^G_0 = \sigma(0)$, the ghost number of b is -1 and of c it is 1. A Virasoro element of V_σ of central charge -26 is

$$\omega^G := \frac{1}{2}\sigma(-1)\sigma(-1) + \frac{3}{2}\sigma(-2).$$

Suppose that ω^M is an Virasoro element of $V \oplus V_{II_{1,1}}$ then $\omega^G + \omega^M$ is a Virasoro element of central charge 0 of $\mathbf{V}$. The BRST-current is $j^{BRST} = c_{-1}(\omega^M + \frac{1}{2}\omega^G)$ and the BRST-operator is $Q = j^{BRST}_0$. Q satisfies the relations

$$Q^2 = 0, \qquad [j^N_0, Q] = Q, \qquad \{Q, b_{n+1}\} = L_n \quad \text{and} \qquad [Q, L_n] = 0.$$

Then the vector space

$$C = \mathbf{V} \cap \operatorname{Ker} b_1 \cap \operatorname{Ker} L_0$$

is invariant under Q and graded by the ghost number

$$C = \bigoplus_{\substack{\alpha \in L' \\ m \in \mathbb{Z}}} C^m_\alpha ,$$

where $L = N \oplus I\!I_{1,1}$. Since $[j_0^N, Q] = Q$ we have the sequence

$$\cdots \xrightarrow{Q} C_\alpha^{m-1} \xrightarrow{Q} C_\alpha^m \xrightarrow{Q} C_\alpha^{m+1} \xrightarrow{Q} \cdots$$

with cohomology groups H_α^m. Let

$$H = \bigoplus_{\substack{\alpha \in L' \\ m \in \mathbb{Z}}} H_\alpha^m$$

then

$$H^1 = \bigoplus_{\alpha \in L'} H_\alpha^1$$

is the space of physical states of the compactified bosonic string. The product on C defined by

$$[u, v] = (b_0 u)_0 v$$

projects down to H. It has the property $\deg[u, v] = \deg u + \deg v - 1$ hence it also projects down to H^1, with this product H^1 becomes a Lie algebra.

$V = \oplus_{m \in \mathbb{Z}} V_m$ is $\mathbb{Z}$-graded by the eigenvalues of the Virasoro generator L_0. There is an action of the Lie algebra of type A_{p-1}^r. For $\mathbf{s}$ in the weight lattice ${A'}_{p-1}^r$, we denote by $V_n(\mathbf{s})$ the subspace of V_n on which the action of the Cartan subalgebra of ${A'}_{p-1}^r$ has weight $\mathbf{s}$. Applying the no-ghost theorem (Theorem 5.1 of [B3]) we get

$$\boxed{H_\alpha^1 \cong \begin{cases} V_{1-\alpha_{I\!I_{1,1}}^2/2}(\alpha_{N'}) & \text{if } \alpha \neq 0 \\ V_1(0) \oplus \mathbb{R}^{1,1} & \text{if } \alpha = 0 \end{cases}} \tag{7.1.1}$$

where $\alpha_{I\!I_{1,1}}$ and $\alpha_{N'}$ denote the projections to the respective lattices. With the $\mathbb{Z}$-grading of [B3] we see that H^1 is a generalised Kac-Moody algebra.

The Cartan algebra H_0^1 of H^1 has dimension $2m + 2$ by (7.1.1). The dimensions of the other pieces H_α^1, $\alpha \neq 0$, can also be expressed in terms of Fourier coefficients of f_γ with $\gamma \equiv \alpha_{N'} \bmod N$. Let $[f_\gamma](n)$ be the n-th Fourier coefficient of f_γ for n in $\mathbb{Q}$, then the dimension of H_α^1 is the Fourier coefficient corresponding to the the L_0-eigenvalue of $V_{1-\alpha_{I\!I_{1,1}}^2/2}(\alpha_{N'})e^{-\alpha_{N'}}$.

$$\dim H_\alpha^1 = [f_\gamma](1 - \alpha_{N'}^2/2 - (1 - \alpha_{I\!I_{1,1}}^2/2)) = [f_\gamma](-\alpha^2/2) \tag{7.1.2}$$

(the number 1 appears here because the energy levels are counted from -1 due to modular normalisation).

We summarise the results of this section. The BRST-operator Q with $Q\frac{1}{2} = 0$ acts on the vertex superalgebra $\mathbf{V} = V \otimes V_{I\!I_{1,1}} \otimes V_\sigma$. The cohomology group of degree one, H_α^1, is the space of physical states of a bosonic string. It carries the structure of a generalised Kac-Moody algebra and it is graded by the rational lattice $N' \oplus I\!I_{1,1}$. The no-ghost theorem implies that the graded dimensions are $\dim H_\alpha^1 = 2m + 2$ if $\alpha = 0$ and $\dim H_\alpha^1 = [f_\gamma](-\alpha^2/2)$ if $\alpha \neq 0$ (with $\gamma \equiv \alpha_{N'} \bmod N$).

7.2 Multiplicities and the denominator identity

It remains to determine the denominator identity for H^1. For the case $p = 2$ this is done in [HSch] and for the case $p = 3$ it is done in [Kl]. The arguments for the other two cases are exactly the same.

Let Λ be the Leech lattice and Λ_p the sublattice fixed by an automorphism of cycle shape $1^m p^m$. For $p=2$ this is the Barnes–Wall lattice, for $p=3$ this is the Coxeter–Todd lattice, for $p=5$ this is the Maass lattice and for $p=7$ it is the Barnes–Craig lattice. The lattices $L = N \oplus I\!I_{1,1}$ and $\Lambda_p \oplus I\!I_{1,1}(p)$ are of the same genus. Since there is only one class in this genus the lattices are isomorphic by Corollary 22 in chapter 15 of [CS].

Consider the lattice $L = \Lambda_p \oplus I\!I_{1,1}(p)$ with elements $(\mathbf{s}, m, n)$, $\mathbf{s}$ in Λ_p, m, n in $\mathbb{Z}$ and norm $(\mathbf{s}, m, n)^2 = \mathbf{s}^2 - 2pmn$. The lattice L has a Weyl vector ρ of norm 0 since the lattices L are the unique lattices of its genus and it is known that these lattices contain such a vector. We choose $\rho = (0, 0, 1/p)$. Recall that a Weyl vector has the property $(\rho, \alpha) = -\alpha^2/2$ for all simple roots α. Now for any root $\alpha = (\mathbf{s}, m, n)$ is $(\rho, \alpha) = -m$. Then the simple roots of the reflection group of L' are the norm $2/p$ vectors of the form $(\mathbf{s}, 1/p, (s^2 - 2/p)/2), \mathbf{s} \in N'$ and the norm 2 vectors $(\mathbf{s}, 1, (s^2-2)/(2p))$ in N, $(\mathbf{s} \in N$ with $2p|(s^2-2))$. Now fix a Weyl vector ρ and the Weyl chamber containing ρ. The roots $n\rho$, $n \in \mathbb{N}_{>0}$ are imaginary. In a Lorentzian space the inner product of two imaginary roots in the same cone is zero only if both vectors are proportional to the same norm zero vector. Hence writing $n\rho$ as a sum of simple roots with positive coefficients, the only summands appearing in the sum are positive multiples of ρ. All the $n\rho$ are disconnected, this implies that they are all simple roots. Their multiplicities are given in (7.1.2). If $p|n$ the multiplicity is $2m$ and m otherwise (where $m = 24/(p+1)$ as always in the last chapter). We have already found all the simple roots. This can be verified as follows. Let $\mathbf{k}$ be the generalised Kac-Moody algebra with root lattice L', Cartan subalgebra $L' \otimes \mathbb{R}$ and simple roots as stated above. The denominator identity of $\mathbf{k}$ is calculated in Theorem 3.2 of [Sch1]

$$\boxed{\begin{aligned} &e^{\rho} \prod_{\alpha \in L^+} (1-e^{\alpha})^{[h](-\alpha^2/2)} \prod_{\alpha \in L'^+} (1-e^{\alpha})^{[h](-p\alpha^2/2)} \\ &\qquad = \sum_{w \in W} \det(w) w \left(e^{\rho} \prod_{n>0} (1-e^{n\rho})^m \prod_{n>0} (1-e^{pn\rho})^m \right). \end{aligned}} \tag{7.2.1}$$

W is the reflection group generated by the norm $2/p$ vectors of L' and the norm 2 vectors of $L \subset L'$. Using (7.1.2) and the definition of the f_γ we see that H^1 and $\mathbf{k}$ have the same root multiplicities. We fixed the Cartan subalgebra and the fundamental Weyl chamber, therefore the product in the denominator identity determines the simple roots of H^1. H^1 and $\mathbf{k}$ have the same simple roots and are thus isomorphic.

Chapter 8

Summary

We summarise the new results obtained in this thesis. They are also summarised in the preprint [CKS] which we intend to submit soon.

In this thesis we consider the theories in [S3] with spin-1 algebra $\widehat{A}^r_{p-1,p}$ with $r = 48/(p^2-1)$ and $p = 2, 3, 5$ or 7. We rewrite the character as

$$\chi_V = \sum_{\lambda \in N'/N} f_\lambda(\tau)\, \vartheta_\lambda(\tau, \mathbf{z})$$

over the grading lattice N of genus

$$II_{2m,0}(p^{\epsilon_p(m+2)}), \quad m = 24/(p+1), \ \ \epsilon_p = + \text{ for } p = 2,5,7 \text{ and } \epsilon_p = - \text{ for } p = 3,$$

and minimal norm 4, except for $p = 7$ it is 6. The lattices are the unique lattices of maximal minimal norm in its genus. The ϑ_λ are theta functions of the lattice N and the coefficients f_λ, which give the degeneracy of the spectrum of the string, are of the form

$$f_\lambda = \begin{cases} h(\tau) + g_0(\tau) & \text{if } \lambda = 0 \\ g_k(\tau) & \text{if } \lambda \neq 0 \text{ and } -\lambda^2/2 \equiv k/p \bmod \mathbb{Z} \end{cases} . \tag{8.0.1}$$

The $h(\tau)$ are modular for $\Gamma_0(p)$ of weight $-m = -24/(p+1)$, they are expressed in terms of the Dedekind eta-function:

$$h(\tau) = (\eta(\tau)\eta(p\tau))^{-m} = q^{-1} + m + \ldots, \qquad q = e^{2\pi i \tau}$$

The g_k are the T-invariant parts of $h(\tau/p) = g_0(\tau) + \ldots + g_{p-1}(\tau)$ with T-eigenvalue $e^{-2\pi i k/p}$.

The most laborious part of this thesis is the proof of (8.0.1). We find three methods of proof. In the cases $p = 2$ and 3 the string functions are known. The $f_\lambda(\tau)$, $h(\tau)$ and its T-invariant parts are modular forms for certain principal congruence groups $\Gamma(N)$ of some weight k. The vector space of modular forms for a principal congruence groups $\Gamma(N)$ of weight k is finite dimensional. So (8.0.1) can be verified by comparing sufficiently many coefficients. For the remaining cases there were not any string function identities known so far. Therefore, we calculate the string function identities of $\widehat{A}_{4,5}$ ((6.3.4)-(6.3.15)). The proof of the string function identities uses the fact that an integral modular form for a congruence group of weight zero and trivial character is constant. In order to prove an identity we take the quotient of the left-hand side and the right-hand side and show that this is an integral modular form for a congruence group of weight zero and trivial character. Hence it must be constant, the first coefficient of the Fourier expansion yields the constant to be one. This method requires the knowledge of the transformation properties of the string functions under the modular group. Since the $f_\lambda(\tau)$ are polynomials in the string functions we could also calculate the transformation properties of the $f_\lambda(\tau)$ and apply the same method of proof as in the case of the string functions. This is the second method of proof. Regarding the transformation properties of the $f_\lambda(\tau)$ of $\widehat{A}_{6,7}$ we observe that they transform under the Weil representation of type ρ_N, where N is the corresponding grading lattice. Computer calculations show that this is also true for the other cases. The fact that the $f_\lambda(\tau)$ transform under a certain Weil representation gives us an appealing third proof. We consider the modular form $F(\tau)$ for the Weil representation ρ_N of section 3.7

$$\begin{aligned} F(\tau) &= \sum_{\gamma\in N'/N} F_\gamma(\tau) e^\gamma \\ &\text{with} \\ F_\gamma(\tau) &= h(\tau) + g_0(\tau) \quad \text{if} \;\; \gamma = 0 \\ &= g_k(\tau) \qquad\qquad \text{if} \;\; -\gamma^2/2 \equiv k/p \mod 1 \end{aligned}$$

Further we define

$$\tilde{F}(\tau) = \sum_{\gamma\in N'/N} f_\gamma(\tau) e^\gamma$$

and consider $F(\tau) - \tilde{F}(\tau)$. This is a modular form of negative weight. We calculate the first coefficients of the f_γ and observed in the previous proof that the first coefficient of f_λ equals the first coefficient of F_λ for every λ in N'/N. Furthermore $F(\tau) - \tilde{F}(\tau)$ has no singular terms so that $F(\tau) - \tilde{F}(\tau)$ is a holomorphic modular form of negative weight which is also holomorphic at the cusp $i\infty$. A well known result of the theory of modular forms is that a modular form for $SL(2,\mathbb{Z})$ of negative weight without singularities at the cusp $i\infty$ is zero. Therefore $F(\tau) = \tilde{F}(\tau)$ and since the e^γ are linear independent (8.0.1) must be true.

Hence we finally obtain a general method which should allow the treatment of all cases of Schellekens list.

We use all these results for the construction of the space of physical states of some bosonic strings. Therefore we assume that the vector space V defined as a direct sum over the set M of highest weights given by the list of [S3] of irreducible highest weight representations of type $\widehat{A}^r_{p-1,p}$ with $r = 48/(p^2-1)$ and $p = 2, 3, 5$ or 7 has the structure of a vertex algebra of central charge 24 whose Virasoro generators satisfy $L_n = L^\dagger_{-n}$ with respect to a positive definite bilinear form. The assumption is very likely, in the case $p = 2$ it is already proven [DGM] and in most of the remaining cases proofs are conjectured [M2]. Let $V_{I\!I_{1,1}}$ be the vertex algebra of the unique even unimodular Lorentzian lattice in two dimensions $I\!I_{1,1}$. Then the tensor product with the vertex algebra V has central charge 26, so we can apply the BRST-formalism. The space of physical states $\mathfrak{g}$ is the BRST-cohomology group of degree one

$$\mathfrak{g} := H^1_{BRST}(V \otimes V_{I\!I_{1,1}}).$$

$\mathfrak{g}$ describes the states of a $(48/(p+1)+2)$-dimensional bosonic string compactified on an orbifold. It has the structure of a generalised Kac-Moody algebra. Its denominator identity is

$$e^{\rho} \prod_{\alpha \in L^+} (1-e^{\alpha})^{[h](-\alpha^2/2)} \prod_{\alpha \in L'^+} (1-e^{\alpha})^{[h](-p\alpha^2/2)}$$
$$= \sum_{w \in W} \det(w) w \left(e^{\rho} \prod_{n>0} (1-e^{n\rho})^m \prod_{n>0} (1-e^{pn\rho})^m \right),$$

where $L' = N' \oplus I\!I_{1,1}$ is the strings momentum lattice and $[h](n)$ is the n-th Fourier coefficient of $h(\tau)$.

Finally, we describe the relation between the grading lattice N, the eta product $h(\tau)$ and a generalised Kac-Moody algebra (see also [Sch2]). The Leech lattice Λ is the unique self-dual even 24-dimensional lattice. The Mathieu group M_{23} acts on the Leech lattice. Let g be an element of square-free order n such that $\sigma_1(n)|24$ ($\sigma_1(n) = \sum_{d|n} d$). As an automorphism of the Leech lattice, g has a characteristic polynomial $\prod_{d|n}(x^d-1)^{24/\sigma_1(n)}$. The corresponding eta product is $\eta_g = \prod_{d|n} \eta(d\tau)^{24/\sigma_1(n)}$. The fix point lattice Λ^g is the unique lattice in its genus without roots. We can lift $1/\eta_g$ to a vector valued modular form F_g on the lattice $L = \Lambda^g \oplus I\!I_{1,1} \oplus I\!I_{1,1}(n)$ and apply the singular theta correspondence to F_g to obtain an automorphic form Ψ_g of singular weight. This can be summarised by the diagram

$$g \to 1/\eta_g \to F_g \to \Psi_g.$$

The expansion of Ψ_g in any cusp is

$$e^{\rho} \prod_{\alpha \in L^+} (1-e^{\alpha})^{[h](-\alpha^2/2)} \prod_{\alpha \in L'^+} (1-e^{\alpha})^{[h](-p\alpha^2/2)}$$
$$= \sum_{w \in W} \det(w) w \left(e^{\rho} \prod_{n>0} (1-e^{n\rho})^m \prod_{n>0} (1-e^{pn\rho})^m \right).$$

This is the denominator identity of a generalised Kac-Moody algebra.

In our cases the fix-point lattice Λ^g is related to the grading lattice in such a way that $L = N \oplus I\!I_{1,1}$ and $\Lambda^g \oplus I\!I_{1,1}(p)$ are isomorphic and the eta product associated to the cycle shape is $1/h(\tau)$.

Elements of the Mathieu group yield ten distinct generalised Kac-Moody algebras. We have constructed four of them. A fifth candidate is the number 8 of Schellekens list. This is a highest weight representation of type $\widehat{A}_{5,6}\widehat{C}_{2,3}\widehat{A}_{1,2}$. We expect the eta product to be $h(\tau) = 1/(\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$ and the grading lattice N to be the unique lattice of genus $I\!I_{8,0}(2_{I\!I}^{+6}3^{-6})$ and maximal minimal norm. The method of proof should be exactly the same, since it is not necessary for our procedure that the highest weight representation is of type $\widehat{A}$.

It remains the question about the physical relevance of the procedure. The theories are bosonic string theories, i.e. there are no fermions. Furthermore our space of states is not four dimensional, but larger (the dimensions obtained are $18, 14, 10$ and 8). The space of states of a four-dimensional bosonic string theory might be given by the generalised Kac-Moody algebra of the fix-point lattice $I\!I_{2,0}(23^{+1})$ with eta product $\eta(\tau)\eta(23\tau)$. So far, there exists no candidate of a corresponding vertex algebra, but there are promising ideas of construction.

Appendix A

String functions

In this chapter we describe the second method of proof in more detail, in particular we state the transformation properties of the string functions and the coefficients f_λ. The transformation properties of the string functions under the modular group are (5.3.2):

$$c^\Lambda_\lambda(-1/\tau) = |M'/kM|^{-1/2}(-i\tau)^{-l/2} \sum_{\substack{\Lambda' \in P^k_+ \mod \mathbb{C}\delta \\ \lambda' \in P^k \mod (kM'+\mathbb{C}\delta)}} S_{\Lambda,\Lambda'} e((\bar\lambda, \bar\lambda')/k)\; c^{\Lambda'}_{\lambda'}(\tau)$$

$$S_{\Lambda,\Lambda'} = i^{|\bar\Delta_+|}|M'/(k+h^\vee)M|^{-1/2} \sum_{w \in \bar W} \epsilon(w) e(-(\bar\Lambda + \bar\rho, w(\bar\lambda' + \bar\rho))/(k+h^\vee))$$

$$c^\Lambda_\lambda(\tau+1) = e(m_{\Lambda,\lambda}) c^\Lambda_\lambda(\tau).$$

This allows us to calculate the transformation properties by computer. We will use the following notation. Let $B =< c_1(\tau), \ldots, c_n(\tau) >$ be a basis of string functions, then we define the S- and T-matrices by

$$c_i(\tau)|S = (-i\tau)^{-l/2} \sum_j S_{ij} c_j(\tau)$$

$$c_i(\tau)|T = \sum_j T_{ij} c_j(\tau).$$

The definition of the S- and T-matrices of the f_λ is completely analogous. They are obtained from the matrices of the string functions, since the f_λ are polynomials in the string functions.

A.1 $\widehat{A}^{16}_{1,2}$

The string functions of $\widehat{A}_{1,2}$ and $\widehat{A}_{2,3}$ are determined in [KP]. They are for $\widehat{A}_{1,2}$

$$c^{20}_{20}(\tau) = c_0(\tau) = \tilde{c}_0(\tau) + \tilde{c}_1(\tau),$$

$$c^{20}_{02}(\tau) = c_1(\tau) = \tilde{c}_0(\tau) - \tilde{c}_1(\tau),$$

$$c^{11}_{11}(\tau) = c_2(\tau) = \frac{\eta(2\tau)}{\eta(\tau)^2}$$

with

$$\tilde{c_0}(\tau) = \frac{\eta(\tau/2)}{\eta(\tau)^2} \quad \text{and} \quad \tilde{c_1}(\tau) = \frac{1}{2}\frac{\eta(\tau)}{\eta(2\tau)\eta(\tau/2)}.$$

We choose the following two bases of string functions. This has technical reasons and the results are conciser.

$$\tilde{B} =< \tilde{c_0}(\tau), \tilde{c_1}(\tau), c_2(\tau) >$$
$$B =< c_0(\tau), c_1(\tau), c_2(\tau) >$$

The S- and T-matrices are ($\xi_{16} = e^{2\pi i/16}$ a 16-th root of unity):

$$\tilde{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \qquad \tilde{T} = \begin{pmatrix} 0 & \xi_{16}^{-1} & 0 \\ \xi_{16}^{-1} & 0 & \\ 0 & 0 & 1 \end{pmatrix}, \tag{A.1.1}$$

$$S = \frac{1}{2}\begin{pmatrix} 1 & 1 & 2\sqrt{2} \\ 1 & 1 & -2\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} \xi_{16}^{-1} & 0 & 0 \\ 0 & \xi_{16}^{-9} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.1.2}$$

This yields the identity

$$2^4\tilde{c_0}(\tau)^8 = 2^4\tilde{c_1}(\tau)^8 + c_2(\tau)^8,$$

which is true because the difference $2^4\tilde{c_0}(\tau)^8 - 2^4\tilde{c_1}(\tau)^8 + c_2(\tau)^8$ is invariant under $SL(2,\mathbb{Z})$ (using A.1.1 and A.1.2) and it has no singularity at $i\infty$ (comparing coefficients). Hence it is zero.

Now, we consider the $f_\gamma(\tau)$. [HSch] shows, that they can be written in the form:

$$f_\gamma(\tau) = \begin{cases} f_0(\tau) & \text{if } \gamma = 0 \\ f_1(\tau) & \text{if } \gamma \neq 0 \text{ and } \gamma\frac{1}{2}/2 = 0 \bmod 1 \\ f_2(\tau) & \text{if } \gamma\frac{1}{2}/2 = 1/2 \bmod 1 \end{cases} \tag{A.1.3}$$

with

$$\begin{aligned} f_0(\tau) := {} & c_0(\tau)^{16} + c_1(\tau)^{16} + 140(c_0(\tau)^4 c_1(\tau)^{12} + c_0(\tau)^{12} c_1(\tau)^4) + \\ & 448(c_0(\tau)^6 c_1(\tau)^{10} + c_0(\tau)^{10} c_1(\tau)^6) + 870 c_0(\tau)^8 c_1(\tau)^8 \\ = {} & 2^{11}(\tilde{c_0}(\tau)^{16} + \tilde{c_1}(\tau)^{16}) + 15 \cdot 2^{12}\tilde{c_0}(\tau)^8\tilde{c_1}(\tau)^8 \\ f_1(\tau) := {} & 2^7(\tilde{c_0}(\tau)^8 - \tilde{c_1}(\tau)^8)c_2(\tau)^8 = 2^3 c_2(\tau)^{16} \\ = {} & 2^{11}(\tilde{c_0}(\tau)^{16} + \tilde{c_1}(\tau)^{16}) - 2^{12}\tilde{c_0}(\tau)^8\tilde{c_1}(\tau)^8 \\ f_2(\tau) := {} & 2^7(\tilde{c_0}(\tau)^8 + \tilde{c_1}(\tau)^8)c_2^8 \\ = {} & 2^{11}(\tilde{c_0}(\tau)^{16} - \tilde{c_1}(\tau)^{16}) \end{aligned}$$

Using the matrices A.1.1 and A.1.2, we obtain the transformation properties of the $f_\gamma(\tau)$:

$$S = \frac{1}{2^5}\begin{pmatrix} 1 & 1 & 1 \\ 527 & 15 & -17 \\ 496 & -16 & 16 \end{pmatrix}$$
$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{A.1.4}$$

Now we are ready to prove (6.3.2). Using (C.0.1) we observe, that $h(\tau)$ is a modular form of weight -8 and trivial character of the congruence group $\Gamma_0(2)$, which is generated by T and ST^2S. The same holds for the $f_\gamma(\tau)$ with norm 0, since T^2 =Id. The cusps of $\Gamma_0(2)$ are 0 and $i\infty$. The Fourier expansions of $h(\tau)$ and $(f_0(\tau) - f_1(\tau))$ at $i\infty$ are

$$h(\tau) = q^{-1} + 8 + \dots$$
$$f_0(\tau) - f_1(\tau) = q^{-1} + 8 + \dots$$

and at 0

$$h(-1/\tau) = 16q^{-1/2} + 128 + \dots$$
$$f_0(-1/\tau) - f_1(-1/\tau) = 16q^{-1/2} + 128 + \dots .$$

Hence $A(\tau) = (f_0(\tau) - f_1(\tau))/h(\tau)$ has no poles at the cusps 0 and $i\infty$. Furthermore $A(\tau)$ is holomorphic on the upper half plane $\mathbb{H}$, since this is true for the string functions and for the eta-function and $\eta(\tau) \neq 0 \ \forall\ \tau \in \mathbb{H}$. This means $(f_0(\tau) - f_1(\tau))/h(\tau)$ is in $\mathbb{M}_0(\Gamma_0(2), 1)$, hence constant, and the constant is one. The S transformation of $f_0(\tau) - f_1(\tau)$ (A.1.4) and $h(\tau)$ is

$$h(-1/\tau) = 2^4(g_0(\tau) + g_1(\tau)) \text{ and}$$
$$f_0(-1/\tau) - f_1(-1/\tau) = 2^4(f_1(\tau) + f_2(\tau)).$$

This completes the proof of (6.3.2).

A.2 $\widehat{A}_{2,3}^6$

The string functions in the notation of [Kl] are

$$
\begin{aligned}
c_{12}(\tau) &:= \frac{1}{\eta(6\tau)\eta(18\tau)} = q^{-1} + q^5 + 2q^{11} + 4q^{17} + \dots, \\
c_{12345}(\tau) &:= \frac{\eta(3\tau)^3\eta(2\tau)^2}{\eta(6\tau)^6\eta(\tau)} = q^{-1} + 1 - 2q^2 - 3q^3 + \dots, \\
c_5(\tau) &:= \frac{\eta(2\tau)^3\eta(3\tau)^2}{\eta(\tau)^6\eta(6\tau)} = 1 + 6q + 24q^2 + 78q^3 + \dots, \\
c_{678}(\tau) &:= \frac{1}{\eta(18\tau)\eta(6\tau)}, \\
c_{234}(\tau) &:= c_{12345}(\tau) - c_{12}(\tau) - c_5(6\tau), \\
c_2(\tau) &:= \frac{1}{18}\sum_{n=0}^{5} \zeta_6^n c_{234}\big(\frac{\tau+n}{6}\big) = q^{5/6}(1 + 6q + 20q^2 + 61q^3 + \dots), \\
c_1(\tau) &:= c_{12}(\tau/6) + c_2(\tau) = q^{-1/6}(1 + 2q + 8q^2 + 24q^3 + \dots), \\
c_3(\tau) &:= \frac{1}{18}\sum_{n=0}^{5} (-1)^{n+1} c_{234}\big(\frac{\tau+n}{6}\big) = q^{1/2}(1 + 4q + 15q^2 + 44q^3 + \dots), \\
c_4(\tau) &:= -\frac{1}{6}\sum_{n=0}^{5} \zeta_3^{2n} c_{234}\big(\frac{\tau+n}{6}\big) = q^{1/3}(2 + 10q + 36q^2 + 112q^3 + \dots), \\
c_6(\tau) &:= \frac{1}{18}\sum_{n=0}^{17} \zeta_{18}^{n} c_{678}\big(\frac{\tau+n}{18}\big) = q^{-1/18}(1 + 4q + 16q^2 + 59q^3 + \dots), \\
c_7(\tau) &:= \frac{1}{18}\sum_{n=0}^{17} \zeta_{18}^{13n} c_{678}\big(\frac{\tau+n}{18}\big) = q^{5/18}(1 + 6q + 22q^2 + 70q^3 + \dots), \\
c_8(\tau) &:= \frac{1}{18}\sum_{n=0}^{17} \zeta_{18}^{7n} c_{678}\big(\frac{\tau+n}{18}\big) = q^{11/18}(2 + 9q + 33q^2 + 98q^3 + \dots)
\end{aligned}
\tag{A.2.1}
$$

where $\xi_\ell := \exp(2\pi i/\ell)$. The point of these roots of unity is simply to extract every ℓ-th Fourier coefficient. The string functions $c_1(\tau), c_2(\tau) \dots, c_8(\tau)$ occur now according to Table A.1 (class 0) and Table A.2 (class 1). The class 2 string functions can be obtained from Table A.2 via the diagram automorphism. These results are obtained from [KP] and [Kl].

Two bases of string functions are

$$
\begin{aligned}
\mathcal{B} &=< c_1(\tau), c_2(\tau), c_3(\tau), c_4(\tau), c_5(\tau), c_6(\tau), c_7(\tau), c_8(\tau) > \\
\tilde{\mathcal{B}} &=< c_1(\tau), c_2(\tau), c_3(\tau), c_4(\tau), c_5(\tau), \tilde{c_6}(\tau), \tilde{c_7}(\tau), \tilde{c_8}(\tau) >,
\end{aligned}
$$

where $\tilde{c_6}(\tau) := \frac{1}{\sqrt{3}}(c_6(\tau)+c_7(\tau)+c_8(\tau))$, $\tilde{c_7}(\tau) := \frac{1}{\sqrt{3}}(c_6(\tau)+\xi_3 c_7(\tau)+\xi_3^2 c_8(\tau))$, $\tilde{c_8}(\tau) := \frac{1}{\sqrt{3}}(c_6(\tau) + \xi_3^2 c_7(\tau) + \xi_3 c_8(\tau))$. The $c_i(\tau)$ are expressed in terms of η-products

		Λ			
		(0,0,3)	(0,3,0)	(3,0,0)	(1,1,1)
λ	(0,0,3)	c_1	c_2	c_2	c_4
	$(0,3,0)$	c_2	c_1	c_2	c_4
	(3,0,0)	c_2	c_2	c_1	c_4
	(1,1,1)	c_3	c_3	c_3	c_5

Table A.1: String functions for class 0

		Λ		
		(0,1,2)	(1,2,0)	(2,0,1)
λ	(0,1,2)	c_6	c_8	c_7
	(1,2,0)	c_7	c_6	c_8
	(2,0,1)	c_8	c_7	c_6

Table A.2: String functions for class 1

in (A.2.1). The S- and T-matrices are ($\xi_l = e^{2\pi i/l}$ a l-th root of unity):

$$\tilde{S} = \frac{1}{3^{\frac{3}{2}}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \sqrt{3} & 0 & 0 \\ 1 & 1 & 1 & 3 & 3 & -\sqrt{3} & 0 & 0 \\ 3 & 3 & -\frac{3}{2} & 9 & -\frac{9}{2} & 0 & 0 & 0 \\ \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} & 0 & 0 & 0 \\ 3 & 3 & -\frac{3}{2} & -3 & \frac{3}{2} & 0 & 0 & 0 \\ 2\cdot 3^{\frac{3}{2}} & -3^{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3^{\frac{3}{2}}\xi_{36} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3^{\frac{3}{2}}\xi_{36}^{-1} \end{pmatrix} \qquad \text{(A.2.2)}$$

$$T = \begin{pmatrix} \xi_6^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_6^5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_{18}^{17} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \xi_{18}^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi_{18}^{11} \end{pmatrix} \qquad \text{(A.2.3)}$$

Considering all known symmetries, i.e. the $f_\gamma(\tau)$ are invariant under the Weyl group, the diagram automorphism and automorphisms of the glue group, then all pairwise distinct coefficients $f_\gamma(\tau)$ are:

$$
\begin{aligned}
f_{0,0} &= c_1^6 + 30c_1^4c_2^2 + 40c_1^3c_2^3 + 90c_1^2c_2^4 + 60c_1c_2^5 + 22c_2^6 + 15c_1^2c_4^4 + \\
&\quad 60c_2^2c_4^4 + 60c_1c_2c_4^4 + 6c_4^6, \\
f_{0,1} &= 6c_1^5c_2 + 15c_1^4c_2^2 + 60c_1^3c_2^3 + 75c_1^2c_2^4 + 66c_1c_2^5 + 21c_2^6 + \\
&\quad 60c_1c_2c_4^4 + 15c_1^2c_4^4 + 60c_2^2c_4^4 + 6c_4^6, \\
f_{1,1} &= c_1^5c_3 + 10c_1^4c_2c_3 + 40c_1^3c_2^2c_3 + 80c_1^2c_2^3c_3 \\
&\quad + 10c_1^2c_4^3c_5 + 80c_1c_2^4c_3 + 40c_1c_2c_4^3c_5 + 15c_1c_3c_4^4 \\
&\quad + 32c_2^5c_3 + 40c_2^2c_4^3c_5 + 30c_2c_3c_4^4 + 6c_4^5c_5, \\
f_{2,1} &= 3c_1^4c_3^2 + 24c_1^3c_2c_3^2 + 72c_1^2c_2^2c_3^2 + 6c_1^2c_4^2c_5^2 \\
&\quad + 96c_1c_2^3c_3^2 + 24c_1c_2c_4^2c_5^2 + 24c_1c_3c_4^3c_5 + 48c_2^4c_3^2 \\
&\quad + 24c_2^2c_4^2c_5^2 + 48c_2c_3c_4^3c_5 + 9c_3^2c_4^4 + 6c_4^4c_5^2, \\
f_{0,2} &= 9c_1^3c_3^3 + 54c_1^2c_2c_3^3 + 3c_1^2c_4c_5^3 \\
&\quad + 108c_1c_2^2c_3^3 + 12c_1c_2c_4c_5^3 + 27c_1c_3c_4^2c_5^2 \\
&\quad + 72c_2^3c_3^3 + 12c_2^2c_4c_5^3 + 54c_2c_3c_4^2c_5^2 + 27c_3^2c_4^3c_5 + 6c_4^3c_5^3, \\
f_{1,2} &= 27c_1^2c_3^4 + c_1^2c_5^4 + 108c_1c_2c_3^4 + 4c_1c_2c_5^4 + 24c_1c_3c_4c_5^3 \\
&\quad + 108c_2^2c_3^4 + 4c_2^2c_5^4 + 48c_2c_3c_4c_5^3 + 54c_3^2c_4^2c_5^2 + 6c_4^2c_5^4, \\
f_{2,2} &= 81c_1c_3^5 + 15c_1c_3c_5^4 + 162c_2c_3^5 \\
&\quad + 30c_2c_3c_5^4 + 90c_3^2c_4c_5^3 + 6c_4c_5^5, \\
f_{0,3} &= 243c_3^6 + 135c_3^2c_5^4 + 6c_5^6, \\
f_{2,3} &= 6c_6^5c_8 + 15c_6^4c_7^2 + 60c_6^3c_7c_8^2 + 60c_6^2c_7^3c_8 + 15c_6^2c_8^4 \\
&\quad + 6c_6c_7^5 + 60c_6c_7^2c_8^3 + 15c_7^4c_8^2 + 6c_7c_8^5, \\
f_{1,3} &= c_6^6 + 30c_6^4c_7c_8 + 20c_6^3c_7^3 + 20c_6^3c_8^3 + 90c_6^2c_7^2c_8^2 \\
&\quad + 30c_6c_7^4c_8 + 30c_6c_7c_8^4 + c_7^6 + 20c_7^3c_8^3 + c_8^6, \\
f_{0,4} &= 6c_6^5c_7 + 15c_6^4c_8^2 + 60c_6^3c_7^2c_8 + 15c_6^2c_7^4 + 60c_6^2c_7c_8^3 \\
&\quad + 60c_6c_7^3c_8^2 + 6c_6c_8^5 + 6c_7^5c_8 + 15c_7^2c_8^4.
\end{aligned}
\tag{A.2.4}
$$

$f_{0,0}(\tau)$ corresponds to $\gamma = 0$.

We choose $\mathcal{B} =< f_{0,0}(\tau),\ f_{0,1}(\tau),\ f_{0,2}(\tau),\ f_{0,3}(\tau),\ f_{0,4}(\tau),\ f_{1,1}(\tau),\ f_{1,2}(\tau),$ $f_{1,3}(\tau),\ f_{2,1}(\tau),\ f_{2,2}(\tau),\ f_{2,3}(\tau) >$ as a basis, then the T matrix is

$$
\begin{pmatrix}
1 &&&&&&&&&& \\
& 1 &&&&&&&&& \\
&& 1 &&&&&&&& \\
&&& 1 &&&&&&& \\
&&&& 1 &&&&&& \\
&&&&& \xi_3 &&&&& \\
&&&&&& \xi_3 &&&& \\
&&&&&&& \xi_3 &&& \\
&&&&&&&& \xi_3^{-1} && \\
&&&&&&&&& \xi_3^{-1} & \\
&&&&&&&&&& \xi_3^{-1}
\end{pmatrix}
\tag{A.2.5}
$$

We consider the $f_\gamma(\tau)$, which are listed in (A.2.4). The S-transformation for

$$\{f_{0,0}(\tau)-f_{0,1}(\tau), f_{0,0}(\tau)-f_{0,2}(\tau), f_{0,0}(\tau)-f_{0,3}(\tau), f_{0,0}(\tau)-f_{0,4}(\tau)\} \longrightarrow$$
$$\{f_{0,0}(\tau), f_{0,1}(\tau), f_{0,2}(\tau), f_{0,3}(\tau), f_{0,4}(\tau), f_{1,1}(\tau), f_{1,2}(\tau), f_{1,3}(\tau), f_{2,1}(\tau), f_{2,2}(\tau), f_{2,3}(\tau)\}$$

is (using (A.2.2))

$$S = \frac{1}{3^3}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 171 & 180 & 160 \\ 0 & 72 & 63 & 64 \\ 729 & 486 & 486 & 504 \\ 0 & 9 & 18 & 12 \\ 0 & 234 & 225 & 120 \\ 729 & 486 & 486 & 477 \\ 0 & 63 & 45 & 60 \\ 0 & 180 & 198 & 192 \\ 729 & 486 & 486 & 477 \end{pmatrix}. \tag{A.2.6}$$

Now we are ready to prove (6.3.2). Using (C.0.2) we observe, that $h(\tau)$ is a modular form of weight -6 and trivial character of the congruence group $\Gamma_0(3)$, which is generated by T and ST^3S. The same holds for the $f_\gamma(\tau)$ with norm 0, since $T^3 = \mathrm{Id}$ (A.2.5). Comparing the cusps of $h(\tau)$ and $f_\gamma(\tau)$ yields $(f_{0,0}(\tau)-f_{0,n}(\tau))/h(\tau), n = 1, ..., 4$ is in $\mathbb{M}_0(\Gamma_0(3), 1)$, hence constant, and the constant is one. The S transformation of $f_{0,0}(\tau)-f_{0,n}(\tau)$ (A.2.6) completes the proof.

A.3 $\widehat{A}^2_{4,5}$

Using (C.0.4) we observe, that $h(\tau)$ is a modular form of weight -4 and trivial character for the congruence group $\Gamma_0(5)$. The same holds for the $f_\gamma(\tau)$ with norm 0. Comparing the cusps of $h(\tau)$ and $f_\gamma(\tau)$ yields (A.3.1) $(f_\gamma(\tau)-f_{\gamma'}(\tau))/h(\tau)$ ($\gamma = 0, \gamma' \neq 0, {\gamma'}^2/2 = 0 \mod 1$) is in $\mathbb{M}_0(\Gamma_0(5), 1)$, hence constant, and the constant is one. The S transformation of $f_\gamma(\tau)$-$f_{\gamma'}(\tau)$ (A.3.2) gives us again (6.3.2).

$$\begin{aligned} &f_{(50000,50000)}(\tau) - f_{(11111,11111)}(\tau) = \\ &f_{(50000,50000)}(\tau) - f_{(50000,11111)}(\tau) = \\ &f_{(50000,50000)}(\tau) - f_{(30110,10220)}(\tau) = \\ &f_{(50000,50000)}(\tau) - f_{(31001,12002)}(\tau) = \frac{1}{(\eta(\tau)\eta(5\tau))^4} \end{aligned} \tag{A.3.1}$$

$$S = \frac{1}{5^2}\left(\begin{array}{cccccccccc} 350 & 115 & 10 & 150 & 190 & 250 & 175 & 10 & 290 & 10 \\ 300 & 120 & 5 & 200 & 220 & 250 & 150 & 5 & 270 & 5 \\ 345 & 112 & 8 & 160 & 192 & 240 & 185 & 8 & 292 & 13 \\ 360 & 108 & 12 & 145 & 198 & 240 & 180 & 7 & 288 & 12 \end{array}\right.$$
$$\left.\begin{array}{cccccccccc} 250 & 75 & 250 & 290 & 10 & 75 & 175 & 190 & 250 & 10 \\ 250 & 100 & 250 & 270 & 5 & 100 & 150 & 220 & 250 & 5 \\ 240 & 80 & 240 & 292 & 13 & 80 & 185 & 192 & 240 & 8 \\ 240 & 85 & 240 & 288 & 12 & 85 & 180 & 198 & 240 & 7 \end{array}\right)$$

S-transformation of $\mathcal{A} \longrightarrow \mathcal{C}$,where (A.3.2)

$$\mathcal{A} = \{f_{(50000,50000)} - f_{(11111,11111)}, f_{(50000,50000)} - f_{(50000,11111)}, f_{(50000,50000)} - f_{(30110,10220)}, f_{(50000,50000)} - f_{(31001,12002)}\}$$

$$\begin{aligned}\mathcal{C} = \{&f_{(10220,30110)}, f_{(11111,11111)}, f_{(11111,50000)}, f_{(12002,31001)},\\ &f_{(11111,12002)}, f_{(31001,10220)}, f_{(30110,30110)}, f_{(12002,50000)},\\ &f_{(10220,11111)}, f_{(10220,50000)}, f_{(31001,30110)}, f_{(12002,12002)},\\ &f_{(10220,12002)}, f_{(11111,30110)}, f_{(50000,30110)}, f_{(31001,31001)},\\ &f_{(10220,10220)}, f_{(31001,11111)}, f_{(30110,12002)}, f_{(31001,50000)}\ \}\end{aligned}$$

A.4 $\widehat{A}_{6,7}$

In this case it is more difficult to compare sufficiently many coefficients, since the string functions of $\widehat{A}_{6,7}$ are not known and obtaining the weight multiplicities via the Freudenthal formula is laborious. But S- and T-matrices are easily calculated. Since every λ in M is a weight of congruence class 0 (cf. section 6.1), we only have to consider $f_\gamma(\tau)$ for γ in class 0. Further we know that the f_λ are invariant under the affine Weyl group and diagram automorphisms. Hence we only have to consider a set P of pairwise distinct f_λ and define

$$R = \{\bar{\gamma} \in N'/N | f_{\bar{\gamma}} \in P\}.$$

Then the transformations are:

$$\begin{aligned} &S = (S_{\bar{\gamma},\bar{\delta}})_{\bar{\gamma},\bar{\delta}\in R} \text{ where } S_{\bar{\gamma},\bar{\delta}} = \frac{-i}{7^5} \sum_{\substack{\delta\in\bar{\delta}\\ \delta\in N'/N}} e((\bar{\gamma}|\delta)/7) \\ &T = (e(-(\bar{\gamma}|\bar{\gamma})/2)\delta_{\bar{\gamma},\bar{\delta}})_{\bar{\gamma},\bar{\delta}\in R} \end{aligned} \qquad \text{(A.4.1)}$$

Hence we can write the transformation of the $f_\gamma(\tau)$ as

$$\begin{aligned} f_\gamma(\tau+1) &= e(-(\gamma|\gamma)/2 f_\gamma(\tau) \\ f_\gamma(-1/\tau) &= \tau^{-m}\sum_{\bar\delta\in R} S_{\bar\gamma,\bar\delta} f_{\bar\delta}(\tau) \\ &= \tau^{-m}\sum_{\bar\delta\in R}\frac{-i}{7^5}\sum_{\substack{\delta\in\bar\delta\\ \delta\in N'/N}} e((\bar\gamma|\delta)/7) f_{\bar\delta}(\tau) \\ &= \frac{e(\mathrm{sign}(N)/8)}{\sqrt{|N'/N|}}\tau^{-m}\sum_{\delta\in N'/N} e((\gamma,\delta))\, f_\delta(\tau) \end{aligned} \tag{A.4.2}$$

This is exactly the way like elements of a Weil representation of type ρ_N, N the grading lattice of genus $II_{6,0}(7^{+5})$, transform (cf. section 3.7). This fact simplifies the proof of (6.3.2).

(A.4.3) is a list of the f_γ with the corresponding norm and first coefficients calculated with the Freudenthal formula.

$$\begin{aligned} \text{Norm 0 mod 7}: &\ f_{1000321} = 3+\dots,\ f_{0102031} = 3+\dots, \\ &\ f_{1111111} = 3+\dots,\ f_{7000000} = q^{-1}+\dots \\ \text{Norm 1 mod 7}: &\ f_{3020020} = 9q^{\frac{1}{7}}+\dots,\ f_{1301200} = 9q^{\frac{1}{7}}+\dots, \\ &\ f_{2101012} = 9q^{\frac{1}{7}}+\dots,\ f_{5100001} = 9q^{\frac{1}{7}}+\dots \\ \text{Norm 2 mod 7}: &\ f_{3011110} = 22q^{\frac{2}{7}}+\dots,\ f_{2202001} = 22q^{\frac{2}{7}}+\dots, \\ &\ f_{5010010} = 22q^{\frac{2}{7}}+\dots,\ f_{1300003} = 22q^{\frac{2}{7}}+\dots \\ \text{Norm 3 mod 7}: &\ f_{1005001} = 51q^{\frac{3}{7}}+\dots,\ f_{0104200} = 51q^{\frac{3}{7}}+\dots, \\ &\ f_{1210012} = 51q^{\frac{3}{7}}+\dots,\ f_{2011102} = 51q^{\frac{3}{7}}+\dots \\ \text{Norm 4 mod 7}: &\ f_{1400101} = 108q^{\frac{4}{7}}+\dots,\ f_{1030030} = 108q^{\frac{4}{7}}+\dots, \\ &\ f_{1021201} = 108q^{\frac{4}{7}}+\dots,\ f_{3200002} = 108q^{\frac{4}{7}}+\dots \\ \text{Norm 5 mod 7}: &\ f_{3110011} = 221q^{\frac{5}{7}}+\dots,\ f_{4102000} = 221q^{\frac{5}{7}}+\dots, \\ &\ f_{1201021} = 221q^{\frac{5}{7}}+\dots,\ f_{3002200} = 221q^{\frac{5}{7}}+\dots \\ \text{Norm 6 mod 7}: &\ f_{0400012} = q^{-\frac{1}{7}}+\dots,\ f_{1003300} = q^{-\frac{1}{7}}+\dots, \\ &\ f_{1120021} = q^{-\frac{1}{7}}+\dots,\ f_{3101101} = q^{-\frac{1}{7}}+\dots \end{aligned} \tag{A.4.3}$$

Recall the modular form $F(\tau)$ for the Weil representation ρ_N of section 3.7

$$F(\tau) = \sum_{\gamma\in N'/N} F_\gamma(\tau) e^\gamma$$

with

$$\begin{aligned} F_\gamma(\tau) &= h(\tau)+g_0(\tau) \quad \text{if } \ \gamma = 0 \\ &= g_k(\tau) \qquad\qquad \text{if } \ -\gamma^2/2 \equiv k/p \mod 1 \end{aligned}$$

with

$$h(\tau) := \frac{1}{(\eta(\tau)\eta(7\tau))^3} = q^{-1}+3+9q+22q^2+51q^3+108q^4+221q^5+\dots \tag{A.4.4}$$

and the g_k the T-invariant parts of $h(\tau/p) = g_0(\tau) + g_1(\tau) + \ldots + g_{p-1}(\tau)$.

Further define

$$\tilde{F}(\tau) := \sum_{\gamma \in N'/N} f_\gamma(\tau) e^\gamma$$

and consider $F(\tau) - \tilde{F}(\tau)$. This is a modular form for $SL(2,\mathbb{Z})$ (since $F(\tau)$ and $\tilde{F}(\tau)$ are modular forms) of negative weight. Regarding (A.4.3) and (A.4.4) we observe that the first coefficient of f_λ equals the first coefficient of F_λ for every λ in N'/N. Furthermore $F(\tau) - \tilde{F}(\tau)$ has no singular terms so that $F(\tau) - \tilde{F}(\tau)$ is a holomorphic modular form of negative weight which is also holomorphic at the cusp $i\infty$. A modular form for $SL(2,\mathbb{Z})$ of negative weight without singularities at the cusp $i\infty$ is zero. Therefore $F(\tau) = \tilde{F}(\tau)$ and since the e^γ are linear independent (6.3.2) must be true.

Appendix B

S- and T-matrices of $\widehat{A}_{4,5}$

We note the S- and T-matrices of the string functions of class 0 of type $\widehat{A}_{4,5}$. They are the main tool in proving the string function identities listed in section 6.3.

We use the convenient basis $\mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2$

$$\begin{aligned}\mathcal{B}_1 := \{\ & c^{11111}_{30110} - c^{31001}_{s.30110}\ ,\ c^{11111}_{10220} - c^{31001}_{s.10220}\ ,\ c^{11111}_{12002} - c^{31001}_{s.12002},\\ & c^{11111}_{31001} - c^{31001}_{s.31001}\ ,\ c^{11111}_{11111} - c^{31001}_{s.11111}\ ,\ c^{11111}_{50000} - c^{31001}_{s.50000},\\ & c^{11111}_{30110} + c^{31001}_{s.30110}\ ,\ c^{11111}_{10220} + c^{31001}_{s.10220}\ ,\ c^{11111}_{12002} + c^{31001}_{s.12002},\\ & c^{11111}_{31001} + c^{31001}_{s.31001}\ ,\ c^{11111}_{11111} + c^{31001}_{s.11111}\ ,\ c^{11111}_{50000} + c^{31001}_{s.50000},\\ & c^{50000}_{s.11111} + c^{10220}_{s.11111}\ ,\ c^{50000}_{s.10220} + c^{10220}_{s.10220}\ ,\ c^{50000}_{s.30110} + c^{10220}_{s.30110},\\ & c^{50000}_{s.31001} + c^{10220}_{s.31001}\ ,\ c^{50000}_{s.12002} + c^{10220}_{s.12002}\ ,\ c^{50000}_{s.50000} + c^{10220}_{s.50000}\}\end{aligned} \tag{B.0.1}$$

$$\begin{aligned}\mathcal{B}_2 := \{& c^{12002}_{s.31001}\ ,\ c^{12002}_{s.12002}\ ,\ c^{12002}_{s.10220}\ ,\ c^{12002}_{s.30110}\ ,\ c^{12002}_{s.50000},\\ & c^{50000}_{s.11111} - c^{10220}_{s.11111}\ ,\ c^{50000}_{s.10220} - c^{10220}_{s.10220}\ ,\ c^{50000}_{s.30110} - c^{10220}_{s.30110},\\ & c^{50000}_{s.31001} - c^{10220}_{s.31001}\ ,\ c^{50000}_{s.12002} - c^{10220}_{s.12002}\ ,\ c^{30110}_{s.11111}\ ,\ c^{12002}_{s.11111},\\ & c^{30110}_{s.12002}\ ,\ c^{30110}_{s.30110}\ ,\ c^{30110}_{s.31001}\ ,\ c^{30110}_{s.50000}\ ,\ c^{30110}_{s.10220}\ ,\\ & c^{50000}_{s.50000} - c^{10220}_{s.50000}\},\end{aligned}$$

since the representation of the string functions of $A_{4,5}$ of SL(2,$\mathbb{Z}$) decomposes into the two irreducible representations corresponding to the basis $\mathcal{B}_1$, respectively $\mathcal{B}_2$. Denote by $\xi := e^{2\pi i/10}$ a 10-th root of unity and $a = (\xi + \xi^{-1})/2$,

$b = (\xi^2 + \xi^{-2})/2$ its real parts.

$$S_1 = \frac{1}{5^{\frac{3}{2}}} \left(\begin{matrix}
10b & -10a & 0 & 0 & 5 & 30 & -10b & 10a & 0 \\
-10a & 10b & 0 & 0 & 5 & 30 & 10a & -10b & 0 \\
0 & 0 & 10a & -10b & -5 & 20 & 0 & 0 & -10a \\
0 & 0 & -10b & 10a & -5 & 20 & 0 & 0 & 10b \\
4 & 4 & -6 & -6 & -1 & 24 & -4 & -4 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -5b & 5a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5a & -5b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5a \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5b \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/2 & -1/2 & -1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & -5a & 5b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 5b & -5a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5b \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5a \\
0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 1/2
\end{matrix} \right.$$

$$\left. \begin{matrix}
0 & -5 & -30 & 5 & -10a & 10b & 0 & 0 & 30 \\
0 & -5 & -30 & 5 & 10b & -10a & 0 & 0 & 30 \\
10b & 5 & -20 & -5 & 0 & 0 & -10b & 10a & 20 \\
-10a & 5 & -20 & -5 & 0 & 0 & 10a & -10b & 20 \\
6 & 1 & -24 & -1 & 4 & 4 & -6 & -6 & 24 \\
-1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & -5/2 & -15 & 15/2 & -15a & 15b & 0 & 0 & 45 \\
0 & -5/2 & -15 & 15/2 & 15b & -15a & 0 & 0 & 45 \\
5b & 5/2 & -10 & -15/2 & 0 & 0 & -15b & 15a & 30 \\
-5a & 5/2 & -10 & -15/2 & 0 & 0 & 15a & -15b & 30 \\
3 & 1/2 & -12 & -1.5 & 6 & 6 & -9 & -9 & 36 \\
-1/2 & -1/2 & -1/2 & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 & 3/2 \\
-3 & -1/2 & 12 & -1/2 & 2 & 2 & -3 & -3 & 12 \\
0 & 5/2 & 15 & 5/2 & 5b & -5a & 0 & 0 & 15 \\
0 & 5/2 & 15 & 5/2 & -5a & 5b & 0 & 0 & 15 \\
5a & -5/2 & 10 & -5/2 & 0 & 0 & 5a & -5b & 10 \\
-5b & -5/2 & 10 & -5/2 & 0 & 0 & -5b & 5a & 10 \\
1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2
\end{matrix} \right) \tag{B.0.2}$$

$$
B_1 = \begin{pmatrix} \xi^6 & & & & & \\ & \xi^4 & & & & \\ & & \xi^2 & & & \\ & & & \xi^8 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}
$$

$$
B_2 = \begin{pmatrix} -1 & & & & \\ & \xi^9 & & & \\ & & \xi^1 & & \\ & & & \xi^3 & \\ & & & & \xi^7 \end{pmatrix} \tag{B.0.3}
$$

$$
T_1 = \begin{pmatrix} B_1 & & & \\ & B_1 & & \\ & & B_2 & \\ & & & -1 \end{pmatrix}
$$

$$
S_2 = \frac{1}{5^2} \left(\begin{array}{rrrrrrrrr}
5+5b & -5/2 & 0 & 0 & 20a & -5 & 0 & 0 & 10a \\
-5/2 & 5+5b & 0 & 0 & 20a & -5 & 0 & 0 & -10b \\
0 & 0 & 5/2 & -5-5b & 30a & 5 & 10b & -10a & 0 \\
0 & 0 & -5-5b & 5/2 & 30a & 5 & -10a & 10b & 0 \\
a & a & a & a & a & 1 & 1 & 1 & 1 \\
-6 & -6 & 4 & 4 & 24 & 1 & -4 & -4 & 6 \\
0 & 0 & 10b & -10a & 30 & -5 & -10b & 10a & 0 \\
0 & 0 & -10a & 10b & 30 & -5 & 10a & -10b & 0 \\
10a & -10b & 0 & 0 & 20 & 5 & 0 & 0 & -10a \\
-10b & 10a & 0 & 0 & 20 & 5 & 0 & 0 & 10b \\
6a & 6a & -4b & -4b & -24b & -1 & 4 & 4 & -6 \\
-6a & -6a & 4a & 4a & 24a & -1 & 4 & 4 & -6 \\
5-5a & -5/2 & 0 & 0 & -20b & -5 & 0 & 0 & -10b \\
0 & 0 & 5/2 & 5a-5 & -30b & 5 & -10a & 10b & 0 \\
-5/2 & 5-5a & 0 & 0 & -20b & -5 & 0 & 0 & 10a \\
-b & -b & -b & -b & -b & 1 & 1 & 1 & 1 \\
0 & 0 & 5a-5 & 5/2 & -30b & 5 & 10b & -10a & 0 \\
1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1
\end{array} \right.
$$

$$
\left.\begin{array}{rrrrrrrrr}
-10b & 5b & -5a & 5-5a & 0 & -5/2 & -20b & 0 & 20 \\
10a & 5b & -5a & -5/2 & 0 & 5-5a & -20b & 0 & 20 \\
0 & -5b & 5a & 0 & 5/2 & 0 & -30b & 5a-5 & 30 \\
0 & -5b & 5a & 0 & 5a-a & 0 & -30b & 5/2 & 30 \\
1 & -b & a & -b & -b & -b & -b & -b & 1 \\
6 & -1 & -1 & -6 & 4 & -6 & 24 & 4 & -24 \\
0 & 5 & 5 & 0 & -10a & 0 & 30 & 10b & -30 \\
0 & 5 & 5 & 0 & 10b & 0 & 30 & -10a & -30 \\
10b & -5 & -5 & -10b & 0 & 10a & 20 & 0 & -20 \\
-10a & -5 & -5 & 10a & 0 & -10b & 20 & 0 & -20 \\
-6 & -a & b & -6a & 4a & -6a & 24a & 4a & 24 \\
-6 & b & -a & 6a & -4b & 6a & -24b & -4b & 24 \\
10a & -5a & 5b & 5+5b & 0 & -5/2 & 20a & 0 & 20 \\
0 & 5a & -5b & 0 & 5/2 & 0 & 30a & 5b-5 & 30 \\
-10b & -5a & 5b & -5/2 & 0 & 5+5b & 20a & 0 & 20 \\
1 & a & -b & a & a & a & a & a & 1 \\
0 & 5a & -5b & 0 & -5-5b & 0 & 30a & 5/2 & 30 \\
-1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1
\end{array}\right) \tag{B.0.4}
$$

$$
B_2 := \begin{pmatrix} -1 & & & & \\ & \xi^9 & & & \\ & & \xi^1 & & \\ & & & \xi^3 & \\ & & & & \xi^7 \end{pmatrix}
$$

$$
T_2 := \begin{pmatrix} B_2 & & & & \\ & B_2 & & & \\ & & \xi^3 & & \\ & & & \xi^7 & \\ & & & & B_2 \end{pmatrix} \tag{B.0.5}
$$

Appendix C

S-transformation of $\eta((k\tau + j)/m)$

Formula (3.3.2) allows us to calculate the S-transformation of $\eta((k\tau + j)/m)$. For our proofs we have to verify that certain eta products transform exactly in the same way under certain congruence groups as the corresponding string functions (resp. f_λ).

We list all the transformations necessary for our purpose.

$$\eta((\tau+1)/2)|S = \sqrt{\tau/i}\ \eta((\tau+1)/2) \tag{C.0.1}$$

$$\eta((\tau+1(/3)|S = e^{11\pi i/12}\sqrt{\tau/i}\ \eta((\tau+2)/3) \tag{C.0.2}$$

$$\begin{aligned} \eta((\tau+1)/4)|S &= e^{-2\pi i/12}\sqrt{\tau/i}\ \eta((\tau+3)/4) \\ \eta((\tau+2)/4)|S &= \sqrt{2\tau/i}\ \eta((2\tau+1)/2) \end{aligned} \tag{C.0.3}$$

$$\begin{aligned} \eta((\tau+1)/5)|S &= e^{-3\pi i/12}\sqrt{\tau/i}\ \eta((\tau+4)/5) \\ \eta((\tau+2)/5)|S &= \sqrt{\tau/i}\ \eta((\tau+2)/5) \\ \eta((\tau+3)/5)|S &= \sqrt{\tau/i}\ \eta((\tau+3)/5) \end{aligned} \tag{C.0.4}$$

$$\begin{aligned} \eta((\tau+1)/7)|S &= e^{7\pi i/12}\sqrt{\tau/i}\ \eta((\tau+6)/7) \\ \eta((\tau+2)/7)|S &= e^{-11\pi i/12}\sqrt{\tau/i}\ \eta((\tau+3)/7) \\ \eta((\tau+4)/7)|S &= e^{-11\pi i/12}\sqrt{\tau/i}\ \eta((\tau+5)/7) \end{aligned} \tag{C.0.5}$$

$$
\begin{aligned}
\eta((\tau+1)/10)|S &= -\sqrt{\tau/i}\ \eta((\tau+9)/10) \\
\eta((\tau+2)/10)|S &= e^{-3\pi i/12}\sqrt{2\tau/i}\ \eta((2\tau+4)/5) \\
\eta((\tau+3)/10)|S &= \sqrt{\tau/i}\ \eta((\tau+3)/10) \\
\eta((\tau+4)/10)|S &= -\sqrt{2\tau/i}\ \eta((2\tau+2)/5) \\
\eta((\tau+5)/10)|S &= \sqrt{5\tau/i}\ \eta((5\tau+1)/2) \\
\eta((\tau+6)/10)|S &= \sqrt{2\tau/i}\ \eta((2\tau+3)/5) \\
\eta((\tau+7)/10)|S &= e^{-\pi i/12}\sqrt{\tau/i}\ \eta((\tau+7)/10) \\
\eta((\tau+8)/10)|S &= e^{3\pi i/12}\sqrt{2\tau/i}\ \eta((2\tau+1)/5)
\end{aligned}
\tag{C.0.6}
$$

$$
\begin{aligned}
\eta((\tau+1)/20)|S &= e^{-2\pi i/12}\sqrt{\tau/i}\ \eta((\tau+19)/20) \\
\eta((\tau+2)/20)|S &= e^{8\pi i/12}\sqrt{2\tau/i}\ \eta((2\tau+9)/10) \\
\eta((\tau+3)/20)|S &= e^{-5\pi i/12}\sqrt{\tau/i}\ \eta((\tau+13)/20) \\
\eta((\tau+4)/20)|S &= e^{-8\pi i/12}\sqrt{4\tau/i}\ \eta((4\tau+4)/5) \\
\eta((\tau+5)/20)|S &= e^{-2\pi i/12}\sqrt{5\tau/i}\ \eta((5\tau+3)/4) \\
\eta((\tau+6)/20)|S &= \sqrt{2\tau i}\ \eta((2\tau+3)/10) \\
\eta((\tau+7)/20)|S &= e^{-5\pi i/12}\sqrt{\tau/i}\ \eta((\tau+17)/20) \\
\eta((\tau+8)/20)|S &= \sqrt{4\tau i}\ \eta((4\tau+2)/5) \\
\eta((\tau+9)/20)|S &= e^{2\pi i/12}\sqrt{\tau/i}\ \eta((\tau+11)/20) \\
\eta((\tau+10)/20)|S &= \sqrt{10\tau/i}\ \eta((10\tau+1)/2) \\
\eta((\tau+12)/20)|S &= \sqrt{4\tau/i}\ \eta((4\tau+3)/5) \\
\eta((\tau+14)/20)|S &= \sqrt{2\tau/i}\ \eta((2\tau+7)/10) \\
\eta((\tau+15)/20)|S &= e^{2\pi i/12}\sqrt{5\tau/i}\ \eta((5\tau+1)/4) \\
\eta((\tau+16)/20)|S &= e^{3\pi i/12}\sqrt{4\tau/i}\ \eta((4\tau+1)/5) \\
\eta((\tau+18)/20)|S &= e^{8\pi i/12}\sqrt{2\tau/i}\ \eta((2\tau+1)/10)
\end{aligned}
\tag{C.0.7}
$$

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Erklärung:

Ich versichere, daß ich diese Arbeit selbständig verfaßt und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den . . . .

Unterschrift