Existence of Heteroclinic Solutions for Non-Autonomous Second Order Problems

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Abstract

We proof, using the minimization of functionals, the existence of heteroclinic solutions connecting equilibria -1 and 1 for a few number of non-autonomous second order problems like

$$\ddot{x}(t) = g(t)V'(x(t)).$$

We consider for the function g the following cases: constant, periodic, asymptotically periodic or coercive. Besides that, we treat a special condition for the problem depending on a parameter.

Introduction

Considering equations such as

$$\ddot{x} = f(t, x), \tag{1}$$

an interesting question to be treated is if there are trajectories connecting equilibria of the equation: by **equilibria** we understand constant solutions of (1). Such trajectories are called **heteroclinics** when it connects two distinct equilibria of the equation. Hence, if p and q are equilibria of (1), x is a heteroclinic trajectory connecting p and q when $\lim_{t\to-\infty} x(t) = p$ and $\lim_{t\to\infty} x(t) = q$.

Here we are interested in heteroclinics solutions for the particular case $f(t, x) = g(t)V'(x(t))$, and we can write the problem as follows:

$$\ddot{x}(t) = g(t)V'(x(t)),\ \forall t \in \mathbb{R}, \tag{2}$$

$$x(t) \to -1 \text{ if } t \to -\infty \quad \text{and} \quad x(t) \to 1 \text{ if } t \to \infty, \tag{3}$$

where $V : \mathbb{R} \to \mathbb{R}$ is a function satisfying properties

- (V_1) $V \in C^2(\mathbb{R})$;
- (V_2) $V(t) \geq 0, \forall t \in \mathbb{R}$ and $V(-1) = V(1) = 0$;
- (V_3) $V(t) > 0, \forall t \in (-1, 1)$;
- (V_4) $V''(-1), V''(1) > 0$,

and $g : \mathbb{R} \to \mathbb{R}$ is a continuos function that belongs to one of the following classes:

Class 1: g is identically a *positive constant*;

Class 2: g is a continuos *periodic* function with

$$\inf_{t\in\mathbb{R}} g(t) = g_0 > 0;$$

Class 3: g is *asymptotically periodic*, i. e., there is a continuos periodic function $g_P : \mathbb{R} \to \mathbb{R}$ such that

$$|g(t) - g_P(t)| \to 0 \quad \text{when} \quad |t| \to \infty$$

and

$$0 < \inf_{t\in\mathbb{R}} g(t) \leq g(t) < g_P(t),\ \forall t \in \mathbb{R};$$

Class 4: g is *coercive*, i. e.,

$$0 < \inf_{t\in\mathbb{R}} g(t) \quad \text{and} \quad g(t) \to \infty \quad \text{when } |t| \to \infty;$$

Class 5: $g \in L^\infty(\mathbb{R})$ and

$$\liminf_{|t|\to\infty} g(t) = g_\infty > \inf_{t\in\mathbb{R}} g(t) = g(0) > 0.$$

First of all, notice that -1 and 1 are equilibria of equation (2) since $V'(-1) = V'(1) = 0$. Then it makes sense to look for heteroclinic trajectories connecting them. The main theorem is stated as follows:

Theorem 1. *Let V with conditions $(V_1) - (V_4)$. Then problem (2)-(3) has a solution $U \in H^1_{loc}(\mathbb{R}) \cap C^2(\mathbb{R})$. Moreover, $U(t) \in (-1, 1)$ for all $t \in \mathbb{R}$.*

The Functional Associated

We will use variational methods in order to guarantee the existence of heteroclinic solutions for the problem. To be more precise, associated to the equation we consider the functional $J : H^1_{loc}(\mathbb{R}) \to [0, \infty]$ given by

$$J(x) = \int_{-\infty}^{\infty} \left(\frac{1}{2}\dot{x}(t)^2 + g(t)V(x(t)) \right) dt, \tag{4}$$

and our main goal is to minimize it in the set

$$W = \{x \in H^1_{loc}(\mathbb{R}) \mid x + 1 \in H^1(-\infty, 0) \text{ and } x - 1 \in H^1(0, \infty)\},$$

obtaining in that way solutions for the problem (2)-(3).

Note that besides W is not a Banach space, we can proof in a natural way that J is differentiable and J' is given by

$$J'(x) \cdot v = \int_{-\infty}^{\infty} \left[\dot{x}(t)\dot{v}(t) + g(t)V'(x(t))v(t)\right] dt,\ \forall x \in W \text{ e } v \in H^1(\mathbb{R})$$

and it is one of the keys to conclude that minimizers are solutions (critical points of J are solutions).

In this direction, denoting $B = \inf\{J(x) \mid x \in W\} > 0$, we have:

Lemma 2. *If $x \in W$ is such that $J(x) = B$, then x is solution of problem (2)-(3) and, besides that, $x \in C^2(\mathbb{R})$ with $x(t) \in (-1, 1)$ for all $t \in \mathbb{R}$.*

In order to obtain the result in Theorem 1 we have the following lemmas (which hold in case function g is bounded: g in classes 1, 2, 3 or 5):

Lemma 3. *Let $x \in H^1_{loc}(\mathbb{R})$ such that $J(x) < \infty$. Then*

$$x(t) \to -1 \ \text{ or } \ x(t) \to 1 \text{ when } t \to -\infty$$

and

$$x(t) \to -1 \ \text{ or } \ x(t) \to 1 \text{ when } t \to \infty.$$

Moreover

$$x + 1 \in H^1(-\infty, 0) \ \text{ or } \ x - 1 \in H^1(-\infty, 0)$$

and

$$x + 1 \in H^1(0, \infty) \ \text{ or } \ x - 1 \in H^1(0, \infty).$$

Lemma 4. *If $A > 0$ and $(x_n) \subset H^1_{loc}(\mathbb{R})$ satisfy $J(x_n) \leq A$ for all $n \in \mathbb{N}$, then there exists subsequence of (x_n), still denoted by (x_n), and a function $x \in H^1_{loc}(\mathbb{R})$ such that for all $T > 0$,*

$$\begin{array}{l} x_n \to x \text{ uniformly in } [-T, T] \ \text{ and} \\ x_n \rightharpoonup x \text{ in } H^1(-T, T) \end{array}.$$

General Idea for the Main Theorem

For all the cases of g we follow almost the same way: and that is to work with minimizers sequences for J in the set W. For that, let $(u_n) \subset W$ be a minimizer sequence for J, i. e., $J(u_n) \to B$ when $n \to \infty$. First of all, we can take for each $n \in \mathbb{N}$, $-1 \leq u_n(t) \leq 1$, for all $t \in \mathbb{R}$.

Now, for each particular case of g, we obtain for $\varepsilon_0 > 0$ (small and conveniently chosen for geometry and properties of V) a new minimizer sequence $(U_n) \subset W$, and sequences $(s_n), (t_n) \subset \mathbb{R}$, with $s_n < t_n$, such that

$$\begin{array}{c} J(U_n) \to B \text{ when } n \to \infty, \\ U_n(t) \in [-1, -1 + \varepsilon_0],\ \forall t \in (-\infty, s_n], \\ U_n(t) \in [1 - \varepsilon_0, 1],\ \forall t \in [t_n, \infty), \\ U_n(t) \in [-1 + \varepsilon_0, 1 - \varepsilon_0],\ \forall t \in [s_n, t_n], \\ U_n(s_n) = -1 + \varepsilon_0 \ \text{ e } \ U_n(t_n) = 1 - \varepsilon_0, \\ (t_n - s_n)_{n\in\mathbb{N}} \text{ is bounded in } \mathbb{R}. \end{array} \tag{5}$$

For $A = \sup_{n\in\mathbb{N}} J(U_n)$ we obtain by Lemma 4 that there exists subsequence of (U_n), still denoted by (U_n), and a function $U \in H^1_{loc}(\mathbb{R})$ such that for all $T > 0$,

$$\begin{array}{l} U_n \to U \text{ uniformly in } [-T, T] \ \text{ and} \\ U_n \rightharpoonup U \text{ in } H^1(-T, T) \end{array}. \tag{6}$$

We can note that $J(U) \leq B$. Hence to finish we just have to proof that $U \in W$.

The point now is guarantee that the sequence (s_n) is bounded in $\mathbb{R}$ (what implies (t_n) bounded too) and in that way obtain limits for subsequences of (s_n) and (t_n), apply the uniformly convergence described soon, the estimatives for U_n in (5) and conclude with Lemma 3. Furthermore, $U \in W$.

But to have the certain that sequence (s_n) is indeed bounded, it depends a lot of the class that function g belongs. In the first case, when we have a positive constant, the functional represented in (4) is invariant under translations, and this is the fundamental property that allows us to choose $s_n = 0$ for all $n \in \mathbb{N}$. In the case that g is a τ-periodic function, the functional is only invariant under certain translations. But we can choose $s_n \in [0, \tau]$ for all $n \in \mathbb{N}$, and hence (s_n) is bounded.

Now let g be an asymptotically periodic function and let g_P be periodic as in the hypothesis. For g_P we consider

$$J_P(x) = \int_{-\infty}^{\infty} \left(\frac{1}{2}\dot{x}(t)^2 + g_P(t)V(x(t)) \right) dt \quad \text{and} \quad B_P = \inf_{x\in W} J_P(x).$$

By hypothesis we note $B < B_P$ and hence there is $\delta > 0$ such that $B + \delta < B_P$. In order to proof that (s_n) is bounded, we proceed by contradiction and work with estimatives for $|g - g_P|$; also estimatives for V exploring its geometry and the choice of ε_0; and finally facts in (5). After those calculations we are able to find $B + \delta \geq B_P$, which is an absurd.

If g is coercive, things change a bit because all the preliminaries results were shown for g bounded (but of course that is not the case now). To solve that and can proceed as before using the results already done (such as Lemmas 2, 3 and 4) we introduce the sets:

$$H^1_g(\mathbb{R}) = \left\{ v \in H^1(\mathbb{R}) \mid \int_{-\infty}^{\infty} g(t)v(t)^2\, dt < \infty \right\}$$

endowed with the norm

$$\|v\|_{H^1_g(\mathbb{R})} = \left(\int_{-\infty}^{\infty} \dot{v}(t)^2\, dt + \int_{-\infty}^{\infty} g(t)v(t)^2\, dt \right)^{1/2}$$

and

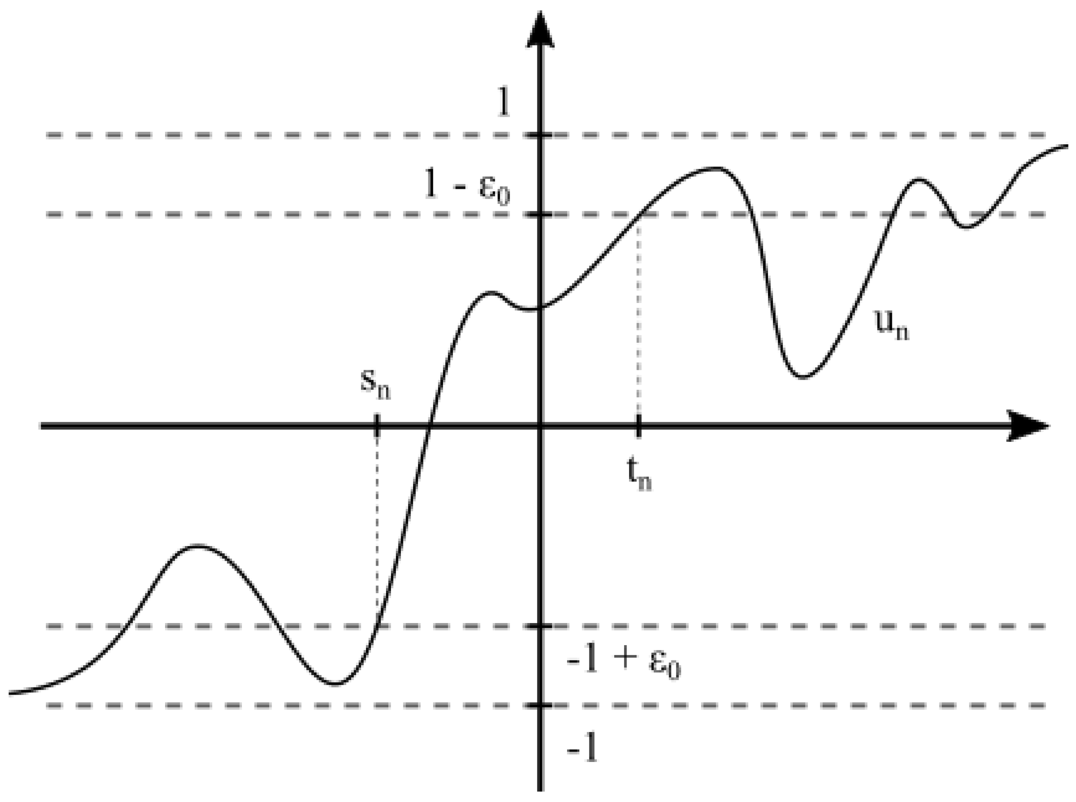
$$W_g = \left\{ x \in H^1_{loc}(\mathbb{R}) \mid x + 1 \in H^1_g(-\infty, 0) \text{ and } x - 1 \in H^1_g(0, \infty) \right\}.$$

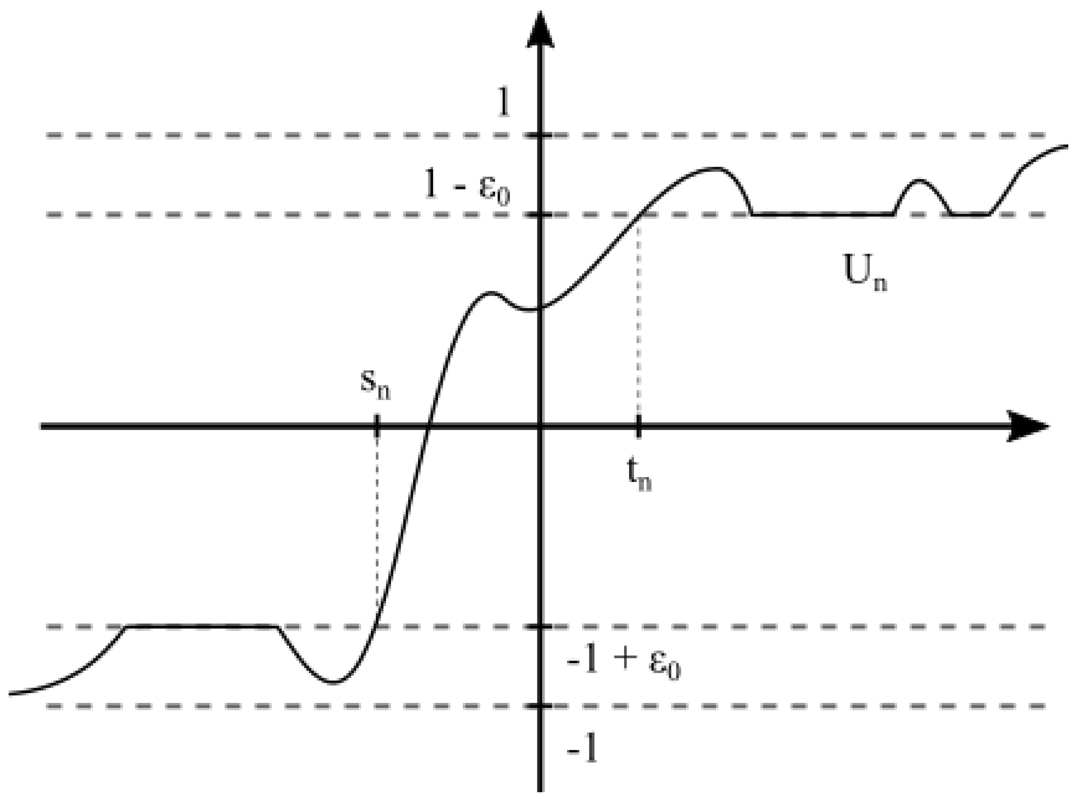
We have the continuos embeddings

$$H^1_g(\mathbb{R}) \hookrightarrow H^1(\mathbb{R}),\ H^1_g(-\infty, 0) \hookrightarrow H^1(-\infty, 0) \ \text{ and } \ H^1_g(0, \infty) \hookrightarrow H^1(0, \infty)$$

and the previous results can be formulated in terms of these new point of view.

Example of one of the modifications to obtain a new minimizer sequence.





References

[1] ALVES, C. O. *Existence of Heteroclinic Solution for a Class of Non-Autonomous Second-Order Equation*. Nonlinear Differential Equations and Applications. v. 22, pp. 1195-1212, 2015.

[2] BONHEURE, D.; SANCHEZ, L. *Heteroclinic Orbits for Some Classes of Second and Fourth Order Differenial Equation*. In: Canadã, A.; Drabek, P.; Fonda A. (eds.) Handbook of Differential Equations III, Chapter 2. Elsevier: Amsterdam, 2006.

[3] EKELAND, I. *On the Variational Principle*. Journal of Mathematical Analysis and Applications. v. 47, pp. 324-353, 1974.

[4] GAVIOLI, A. *On the Existence of Heteroclinic Trajectories for Asymptotically Autonomous Equations*. Topological Methods in Nonlinear Analysis. v. 34, pp. 251-266, 2009.

[5] SPRADLIN, G.S. *Heteroclinic Solutions to an Asymptotically Autonomous Second-Order Equation*. Electronic Journal of Differential Equations. n. 137, pp. 1-14, 2010.

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