Construction of a Fractional Schrödinger Equation and Some First Predictions on Quantum Mechanical Problems

Demian Nahuel Goos[1,2,3]

[1] Departamento de Matemática, FCEIA, Universidad Nacional de Rosario, Argentina
[2] CONICET, FCEIA, Universidad Nacional de Rosario
[3] Instituto de Estudios Nucleares y Radiaciones Ionizantes, FCEIA, Universidad Nacional de Rosario

Introduction

Quantum Mechanics is a powerful theory, which revolutionized the understanding of phenomena at small scales together with the study of the energy levels in these systems. Its consistency has experimentally been proven in a huge range of applications. Nevertheless, there are still different problems which are not sufficiently well described by these postulates.
Within this theory, the Schrödinger equation plays a fundamental role, since it describes the changes over time of a physical system in which quantum effects are significant. A simple way to construct this equation is by combining the momentum $\hat{p}$ and energy $\hat{E}$ operators and their general interpretations.
If we now consider fractional generalizations of these operators,

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \longrightarrow \hat{p}_\alpha = -i\hbar^\alpha\frac{\partial^\alpha}{\partial x^\alpha} \qquad \hat{E} = i\hbar\frac{\partial}{\partial t} \longrightarrow \hat{E}_\alpha = i\hbar^\alpha\frac{\partial^\alpha}{\partial t^\alpha}$$

which can be obtained by replacing the regular plane wave $\Psi(x,t) = e^{i(kx-\omega t)}$ by a fractional approximation,
$\Psi(x,t) = E_{\alpha,1}(ik^\alpha x^\alpha)E_{\alpha,1}(-i\omega^\alpha t^\alpha)$, where

$$E_{\alpha,\beta}(s) = \sum_{j=0}^{\infty}\frac{s^j}{\Gamma(\alpha j+\beta)},$$

the resulting Fractional Schrödinger Equation (FSE) is given by

$$i\hbar^\alpha\frac{\partial^\alpha}{\partial t^\alpha}\Psi(x,t) = \left[-\frac{\hbar^{2\alpha}}{2\mu^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha} + V(x)\right]\Psi(x,t).$$

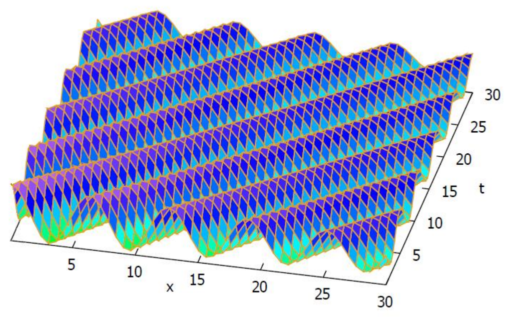


Fig. 1: Fractional wave $\Psi(x,t) = E_{\alpha,1}(ik^\alpha x^\alpha)E_{\alpha,1}(-i\omega^\alpha t^\alpha)$

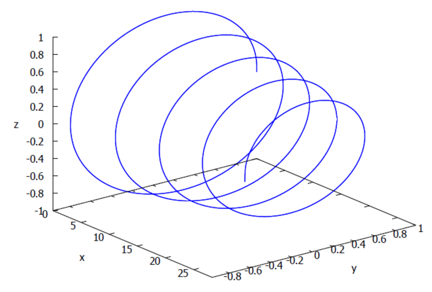


Fig. 2: Modulus of the fractional wave $\Psi(x,t)$

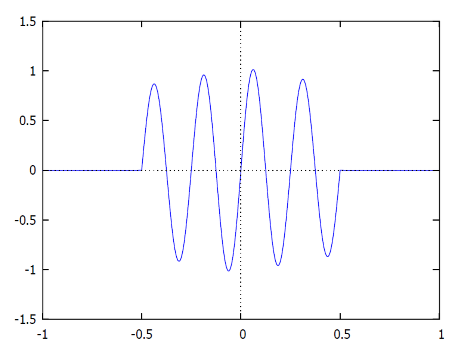


Fig. 3: Solution to the particle in an infinite potential well

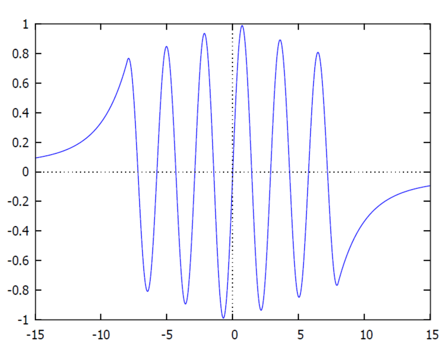


Fig. 4: Solution to the particle in a finite potential well

Fractional analysis

Definition: Caputo Fractional Derivative
The left and right CFD of order $\alpha \in \mathbb{R}^+$ and starting point a of f are

$${}_s^C D_b^\alpha f(s) = \frac{-1}{\Gamma(1-\alpha)}\int_s^b (z-s)^{-\alpha}f'(z)dz,$$

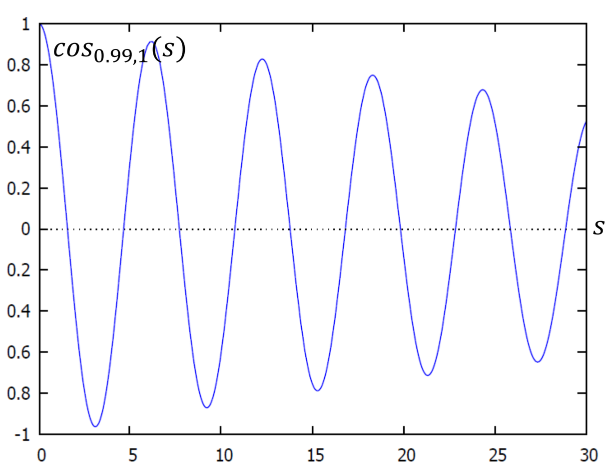
$${}_a^C D_s^\alpha f(s) = \frac{1}{\Gamma(1-\alpha)}\int_a^s (s-z)^{-\alpha}f'(z)dz.$$

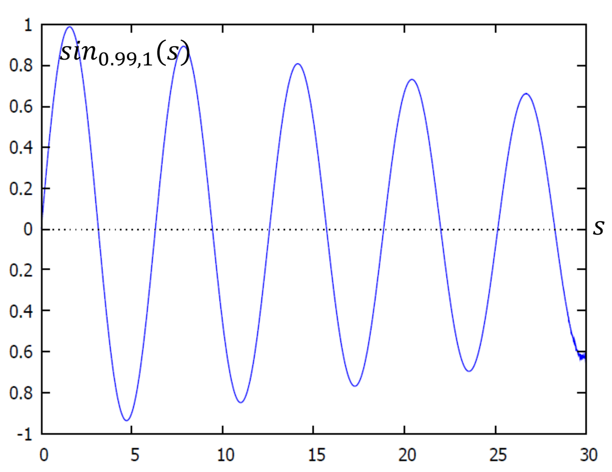
$$\frac{\partial^\alpha}{\partial t^\alpha}\Psi(x,t) = {}_0^C D_t^\alpha \Psi(x,t)$$

$$\frac{\partial^\alpha}{\partial x^\alpha}\Psi(x,t) = \begin{cases} {}_x^C D_0^\alpha \Psi(x,t), & x<0 \\ {}_0^C D_x^\alpha \Psi(x,t), & x \geq 0 \end{cases}$$

Definition: Fractional sine and cosine functions
The fractional sine and cosine functions of parameter $\alpha \in \mathbb{R}^+$ are

$$cos_{\alpha,\beta}(s) = \sum_{j=0}^{\infty}\frac{(-1)^j s^{2j}}{\Gamma(\alpha(2j)+\beta)} \qquad sin_{\alpha,\beta}(s) = \sum_{j=0}^{\infty}\frac{(-1)^j s^{2j+1}}{\Gamma(\alpha(2j+1)+\beta)}$$





Free particle with zero angular momentum (independent of t)

The FSE with $V(x) = 0$ reads $\frac{\hbar^{2\alpha}}{2\mu^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\Psi(x,t) + \varepsilon^\alpha\Psi(x,t) = 0,$

with solution $\psi(x) = \begin{cases} AE_\alpha(ik^\alpha(-x)^\alpha) + BE_\alpha(-ik^\alpha(-x)^\alpha), & x<0 \\ AE_\alpha(ik^\alpha x^\alpha) + BE_\alpha(-ik^\alpha x^\alpha), & x \geq 0, \end{cases}$

wave number $k^2 = \frac{2\mu\varepsilon}{\hbar^2}$ and priviliged point/singularity at 0.

Particle in an infiinite potential well (independent of t)

The FSE with $V(x) = \begin{cases} 0, & x \in [-a,a] \\ \infty, & else \end{cases}$ reads

$$\frac{\hbar^{2\alpha}}{2\mu^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\Psi(x,t) + [\varepsilon^\alpha - V(x)]\Psi(x,t) = 0,$$

with either even or odd solutions

$$\psi(x) = \begin{cases} Asin_\alpha(k^\alpha(-x)^\alpha), & x \in [-a,0] \\ -Asin_\alpha(k^\alpha x^\alpha), & x \in [0,a] \\ 0, & else. \end{cases}$$

A serves as normalization constant and k can be obtained with the continuity condition,

$$sin_\alpha(k^\alpha a^\alpha) = 0.$$

This also implies that the energy states are quantized, but unlike the classical approach, there are finite many energy states, depending on α (α can be fixed in order to obtain as many states as desired).
For a potential $W(x) = V(x-c)$ the exact same result is obtained by shifting the center of derivation.

Particle in a finite potential well

The FSE with $V(x) = \begin{cases} {V_0}^\alpha, & x \in [-a,a] \\ 0, & else \end{cases}$ reads

$$\frac{\hbar^{2\alpha}}{2\mu^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\alpha}{\partial x^\alpha}\Psi(x,t) + [\varepsilon^\alpha - V(x)]\Psi(x,t) = 0.$$

For $\varepsilon^\alpha < {V_0}^\alpha$ there are either even or odd solutions

$$\kappa^{2\alpha} = \frac{2\mu^\alpha({V_0}^\alpha + \varepsilon^\alpha)}{\hbar^{2\alpha}} \qquad \psi(x) = \begin{cases} E_\alpha(k^\alpha(-x)^\alpha), & x \in (-\infty,-a) \\ -Asin_\alpha(\kappa^\alpha(-x)^\alpha), & x \in [-a,0] \\ Asin_\alpha(\kappa^\alpha x^\alpha), & x \in [0,a] \\ E_\alpha(k^\alpha x^\alpha), & x \in (a,\infty) \end{cases}$$

To obtain the discrete energy states, the following transcendental equation can be used.

$$\frac{E_{\alpha,\alpha}(k^\alpha a^\alpha)}{E_\alpha(k^\alpha a^\alpha)} = -\frac{\kappa^\alpha}{k^\alpha}\frac{cos_{\alpha,\alpha}(\kappa^\alpha a^\alpha)}{sin_\alpha(\kappa^\alpha a^\alpha)}$$

Future work

Operators $\hat{p}$ and $\hat{E}$ are not self-adjoint in $L^2(\mathbb{R})$. A suitable space of functions, in which these operators are self-adjoint, has to be constructed.

Study of scattering phenomena (finite potential well with $\varepsilon^\alpha > {V_0}^\alpha$) and quantum tunneling (finite potential barrier with $\varepsilon^\alpha < {V_0}^\alpha$), physical implications of the singularity point, amongst others.

Study of applications of the fractional quantum mechanical model to physical mathematics and quantum mechanics.

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