Faculty of Engineering and Department of Physics

Engineering Physics 131

Midterm Examination

Monday February 24, 2020; 7:00 pm – 8:30 pm

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1. Closed book exam. No notes or textbooks allowed.
2. Formula sheets are included (may be removed).
3. The exam has **7** problems and is out of **50 points**. Attempt all parts of all problems.
4. *Questions 1 to 3 do not require detailed calculations and only the final answers to these questions will be marked.*
5. *For Questions 4 to 7, details and procedures to solve these problems will be marked.* Show all work in a neat and logical manner.
6. Write your solution directly on the pages with the questions. Indicate clearly if you use the backs of pages for material to be marked.
7. Only non-programmable calculator approved by the Faculty of Engineering permitted. All other electronic devices (e.g. cell phones) must be turned off and stowed away out of sight.

DO NOT separate the pages of the exam containing the problems.

LAST NAME: ______________________________

FIRST NAME: ______________________________

ID#: ______________

Please circle the name of your instructor:

EB01: Wheelock	**EB02: Jung**
EB03: Komrakova	**EB04: Rozmus**
EB05: Tang	**EB06: Wang**

Address all inquiries to an invigilator. Do not communicate with other candidates. If you become ill during the exam, contact an invigilator immediately. (You may not claim extenuating circumstances and request your paper to be cancelled after writing and handing in your examination.) You may not leave the exam until at least 30 minutes have elapsed.

End of Exam: When the signal is given to end the exam, students must promptly cease writing. If a student does not stop at the signal, the instructor has the discretion either not to grade the exam paper or to lower the grade on the examination. **If you do not submit your exam prior to 8:20 pm, please remain seated until we finish collecting ALL the exams from the entire class.**

Please do not write in the table below.

Question	**Value (Points)**	**Mark**
1	5	
2	5	
3	4	
4	6	
5	9	
6	10	
7	11	
Total	50	

1. [5 Points] This question is composed of 5 multiple choice questions, each having only one correct answer. Please circle your answer. (*1 mark for each question, no partial marks.*)

(1) If your net displacement is zero, then

A) the distance you travel must be zero.

B) your average speed must be zero.

C) your average velocity must be zero.

D) all of the above must be true.

(2) A car drives over a hill with its cruise control set to a constant 50 mph. Which one of the following statements is correct?

A) The car's speed is changing

B) The car's velocity is changing

C) Both the speed and velocity are changing

D) Neither the speed nor the velocity is changing

(3) From the top of a tall building you drop a ball from rest. Three seconds later, from the same location you drop another ball from rest. While the two balls are in the air, the distance between them:

A) increases as the square of time.

B) increases linearly with time.

C) stays constant.

D) none of the above.

Explanation: $y_1 = -\frac{1}{2}gt^2, \quad y_2 = -\frac{1}{2}g(t-3)^2 \quad \Rightarrow \quad y_2 - y_1 = -\frac{1}{2}g(-6t+9)$

(4) A car rounds a curve in the horizontal plane with its cruise control set to a constant 50 mph. Which one of the following statements is correct?

A) The car's tangential acceleration is zero but normal acceleration is non-zero.

B) The car's normal acceleration is zero but tangential acceleration is non-zero.

C) The car's tangential and normal accelerations are both zero.

D) The car's tangential and normal accelerations are both non-zero.

(5) You throw a tennis ball straight up (relative to yourself) while running at a constant speed. Neglecting air resistance, the ball will land

A) on you.

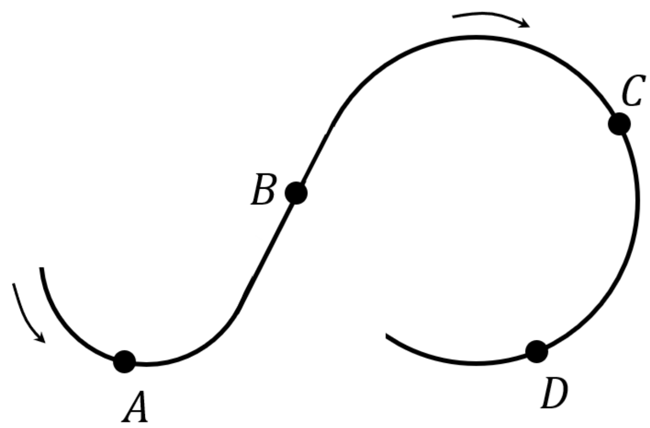
B) slightly ahead of you.

C) slightly behind you.

D) insufficient information to determine.

Explanation: $x = vt$ for both you and the tennis ball, where v = your speed.

2. [5 Points] A bug follows a trajectory shown in the figure below.



(1) In the figure above, draw vectors of tangential (**a**$_t$) and normal components (**a**$_n$) of its acceleration at the three points *A*, *B*, and *C* if:

- at point *A* the bug flies with constant speed;
- at point *B* the bug passes the inflection point of the trajectory with increasing speed; and
- at point *C* the bug is steadily decreasing its speed.

Clearly label **a**$_t$ and **a**$_n$ at each point. Indicate any zero components.

(*1 mark for the drawing at each of the three points. No partial marks.*)

(2) The magnitude of the bug's total acceleration at point *D* is described by the following equation:

$$a = \sqrt{\frac{mg}{C_d} kt^2 + bs^n}$$

where m is mass in kg, g is gravitational acceleration in m/s^2, C_d is the dimensionless drag coefficient, t is time in s; and s is the arc length in m measured from point *A* along the curve.

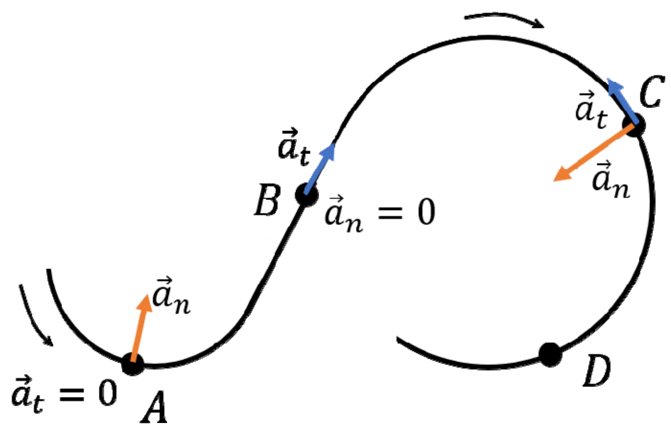
Express the dimensions of constant k in terms of the three fundamental dimensions [M], [L] and [T]:

$[k]$ = ____ *(1 mark, no partial marks.)*

Determine the value of n, if the dimension of constant b is $[L]^{5/2}[T]^{-2}$:

n = ____ (*1 mark, no partial marks.*)

Solution:



Dimensions of k:

The consistency of dimensions means that every term of the expression must have the same dimensions. Therefore, the term $\sqrt{\frac{mg}{C_d}}kt^2$ must have the dimensions of acceleration [L]/[T]².

$$\left[\sqrt{\frac{mg}{C_d}}kt^2\right] = \frac{[\mathrm{L}]}{[\mathrm{T}]^2}$$

$$[k] = \frac{[\mathrm{L}]}{[\mathrm{T}]^2}\left[\sqrt{\frac{C_d}{mg}}\right]\frac{1}{[T]^2}$$

$$[k] = \frac{[\mathrm{L}]}{[\mathrm{T}]^2}\sqrt{\frac{1}{[M]\frac{[\mathrm{L}]}{[\mathrm{T}]^2}}}\frac{1}{[T]^2}$$

$$[k] = \frac{[\mathrm{L}]^{\frac{1}{2}}}{[\mathrm{T}]^3[M]^{\frac{1}{2}}}$$

Value of n:

$$[bs^n] = \frac{[\mathrm{L}]}{[\mathrm{T}]^2}$$

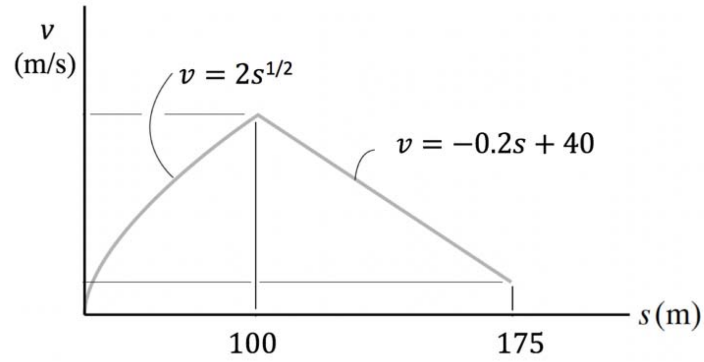
$$\frac{[L]^{\frac{5}{2}}}{[T]^2}[L]^n = \frac{[\mathrm{L}]}{[\mathrm{T}]^2}$$

$$[L]^{\frac{5}{2}+n} = [L]^1$$

$$\frac{5}{2} + n = 1$$

$$n = -\frac{3}{2}$$

3. [4 Points] A car drives along a straight road with velocity vs. position as shown in the diagram. Answer the following questions. (*1 mark for each question, no partial marks.*)



(1) On the interval $0 < s < 175$ m, for what values of s, if any, does <u>the magnitude of acceleration increase</u> with time? Circle your answer(s) below; if there is no such location, circle NONE.

$0 < s < 100$ m $\qquad$ $100 < s < 175$ m $\qquad$ NONE

(2) On the interval $0 < s < 175$ m, for what values of s, if any, is the acceleration <u>constant</u>? Circle your answer(s) below; if there is no such location, circle NONE.

$0 < s < 100$ m $\qquad$ $100 < s < 175$ m $\qquad$ NONE

(3) On the interval $0 < s < 175$ m, for what values of s, if any, does <u>the magnitude of acceleration decrease</u> with time? Circle your answer(s) below; if there is no such location, circle NONE.

$0 < s < 100$ m $\qquad$ $100 < s < 175$ m $\qquad$ NONE

(4) What is the maximum acceleration (on the interval $0 < s < 175$ m)?
Note: a negative number is a lower value than a positive number.

a_{max} = ____________2.00______________ m/s^2

Explanation:

For $0 < s < 100$: $a = v\frac{dv}{ds} = \left(2s^{1/2}\right)\left[2\left(\frac{1}{2}s^{-1/2}\right)\right] = 2.00$ m/s^2 ← a is constant

For $100 < s < 175$: $\quad |a| = \left|v\frac{dv}{ds}\right|$ ...where v is decreasing and slope $\left(\frac{dv}{ds}\right)$ is constant. Therefore the magnitude of acceleration is decreasing on this interval.

Note also that motion is in one direction (s is positive and increasing). Therefore if a is decreasing (or constant) as s increases, it follows that a is also decreasing (or constant) as t (time) increases.

a is negative for $100 < s < 175$ m, so a_{max} is the constant and positive acceleration for $0 < s < 100$ m.

4. [6 Points] A particle travels along the x-axis with its acceleration given by $a = (3t^2 - 4)$ m/s^2, where t is in seconds. When $t = 0$, the particle is at $x = -4$ m, and two seconds later its position is $x = -12$ m. Determine its position when $t = 5$ s.

Solution:

Acceleration is not constant, so we cannot use the equations of kinematics with constant acceleration. Also note that the initial velocity at $t = 0$ is not known.

$$dv = a\,dt$$

$$\int_{v_0}^{v} dv = \int_0^t (3t^2 - 4)dt$$

$$v - v_0 = \frac{3t^3}{3} - 4t = t^3 - 4t$$

$$ds = v\,dt$$

$$\int_{s_0}^{s} ds = \int_0^t (t^3 - 4t + v_0)\,dt$$

$$s - s_0 = \frac{t^4}{4} - \frac{4t^2}{2} + v_0 t = \frac{t^4}{4} - 2t^2 + v_0 t$$

Evaluate this equation at $t = 2$ s and solve for v_0. Note: $s = -12$ and $s_0 = -4$.

$$v_0 = \frac{s - s_0}{t} - \frac{t^4}{4t} + \frac{2t^2}{t} = \frac{s - s_0}{t} - \frac{t^3}{4} + 2t$$

$$= \frac{-12 - (-4)}{2} - \frac{2^3}{4} + 2(2) = -2\ m/s$$

Now calculate s when $t = 5$ s.

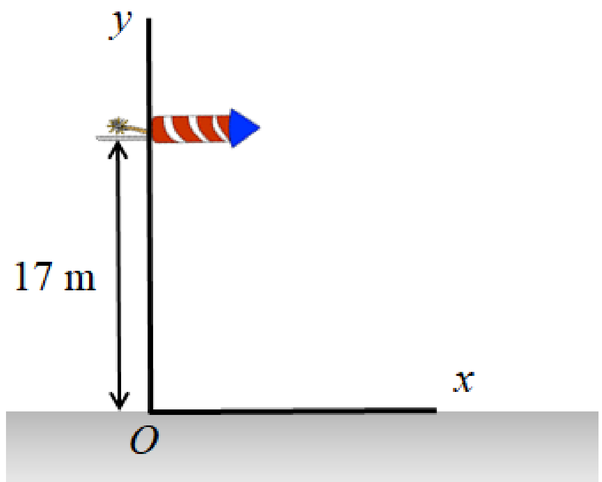
$$s = \frac{t^4}{4} - 2t^2 + v_0 t + s_0$$

$$s = \frac{5^4}{4} - 2(5)^2 - 2(5) - 4 = 92.25\ m$$

Answer: 92.3 m

5. [9 Points] A firework rocket is launched from rest at a height of $y = 17$ m above the ground, as shown in the figure below. In addition to gravity, the rocket is experiencing a horizontal thrust and hence acceleration, a_x. The magnitude of a_x is twice the magnitude of the acceleration due to gravity.

(1) [3 points] What trajectory, $y = y(x)$, will the rocket follow?
(2) [4 points] What will be its speed just before hitting the ground?
(3) [2 points] Find the horizontal distance between the launch and landing points.



Solution:

(1) $x(t) = \frac{a_x t^2}{2} = gt^2,\ y(t) = y_0 - \frac{gt^2}{2} = 17 - \frac{x}{2} \quad \Rightarrow \quad \boldsymbol{y = 17 - x/2}$

(2) - find v_y when the rocket hits the ground:

$$a_y\,dy = -gdy = v_y dv_y \ \Rightarrow gy_0 = \frac{v_y^2}{2} \Rightarrow \boldsymbol{v_y = (2gy_0)^{\frac{1}{2}} = 18.3\ m/s}$$

- find time until the rocket hits the ground:

$$y_0 = \frac{gt_*^2}{2} \Rightarrow t_* = \boldsymbol{(2y_0/g)^{1/2} = 1.86\ s}$$

- find v_x when the rocket hits the ground: $v_x = a_x t_* = 2gt_* = \mathbf{36.5\ m/s}$

- calculate speed: $v = \left(v_x^2 + v_y^2\right)^{1/2} = \mathbf{40.8\ m/s}$

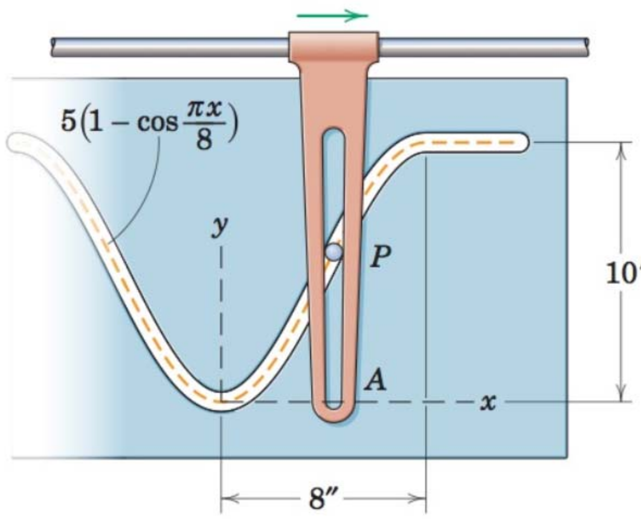
(3) horizontal distance X:

$y=0=17-X/2 \ \Rightarrow\ \mathbf{X=34\ m,}$ also $\left[X = a_x t_*^2/2 = gt_*^2 = \mathbf{34m}\right]$

6. **[10 Points]** The figure below shows a portion of a plate cam used in a control mechanism. The motion of pin P in the fixed slot is controlled by the vertical guide A, which travels horizontally to the right at a constant speed of 6 in/s over the central sinusoidal portion of the slot.

The sinusoidal curve is described by the equation $y = 5(1-\cos\frac{\pi x}{8})$, where x and y are both in inches (1″ = 1 inch), and the origin is at the lowest point of the curved slot. When the pin is at the position $x = 2$ in, determine

(1) [6 points] the magnitude of its normal acceleration;
(2) [4 points] the magnitude of its tangential acceleration.



Solution:

The normal component of acceleration is

$$a_n = \frac{v^2}{\rho} = \frac{v_x^2 + v_y^2}{\rho}$$

The x-component of the velocity of the pin P is equal to the velocity of the guide A that moves horizontally with a constant speed of 6 in/s. Therefore, $v_x = 6$ in/s.

The vertical component of the velocity is

$$v_y = \frac{dy}{dt} = \frac{d}{dt}\left[5\left(1-\cos\frac{\pi x}{8}\right)\right] = -5\frac{d}{dt}\left[\cos\frac{\pi x}{8}\right] = \frac{5\pi\dot{x}}{8}\sin\frac{\pi x}{8} = \frac{5\pi v_x}{8}\sin\frac{\pi x}{8}$$

At x=2 in, $v_y = \frac{5\pi\cdot 6}{8}\sin\frac{\pi\cdot 2}{8} = 8.33$ in/s.

The radius of curvature is

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|}$$

$$\frac{dy}{dx} = \frac{5\pi}{8}\sin\frac{\pi x}{8} \quad at\ x = 2 = 1.388$$

$$\frac{d^2y}{dx^2} = \frac{5\pi^2}{64}\cos\frac{\pi x}{8} = 0.545$$

$\rho = 9.186$ in

$$a_n = \frac{v_x^2+v_y^2}{\rho} = \frac{6^2+8.33^2}{9.186} = 11.47 \text{ in/s}^2$$

Tangential acceleration.

Since $v_x = const$, then $a_x = 0$.

$$a_y = \frac{dv_y}{dt} = \frac{d}{dt}\left(\frac{5\pi v_x}{8}\sin\frac{\pi x}{8}\right) = \frac{5\pi}{8}\frac{d}{dt}\left(v_x \sin\frac{\pi x}{8}\right) = \frac{5\pi^2}{64}v_x^2\cos\frac{\pi x}{8} = 19.628 \text{ in/s}^2.$$

Total acceleration: $a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_t^2 + a_n^2}$

$$a_t = \sqrt{a_y^2 - a_n^2} = 15.93 = 16 \text{ in/s}^2.$$

Second method:

$$a_t = \frac{dv}{dt} = \frac{d}{dt}\sqrt{v_x^2 + v_y^2} = \frac{1}{2}\frac{1}{\sqrt{v_x^2 + v_y^2}}\left(2v_x\dot{v}_x + 2v_y\dot{v}_y\right) = \frac{v_y\dot{v}_y}{\sqrt{v_x^2 + v_y^2}}$$
$$= \frac{v_y a_y}{\sqrt{v_x^2 + v_y^2}}$$

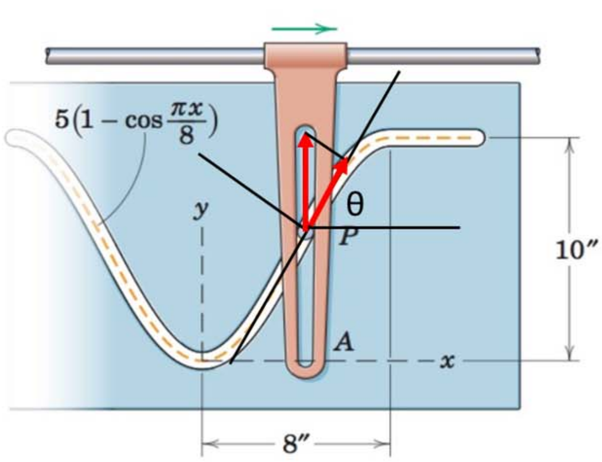
$$a_t = \frac{\frac{5\pi v_x}{8}\sin\frac{\pi x}{8}\frac{5\pi^2}{64}v_x^2\cos\frac{\pi x}{8}}{\sqrt{v_x^2+v_y^2}} = \frac{25\pi^3 v_x^3}{512}\frac{\sin\frac{\pi x}{8}\cdot\cos\frac{\pi x}{8}}{\sqrt{v_x^2+v_y^2}} = 15.93 \text{ in/s}^2.$$

Third method:

The slope

$$\frac{dy}{dx} = \frac{5\pi}{8}\sin\frac{\pi x}{8} = \tan\theta$$

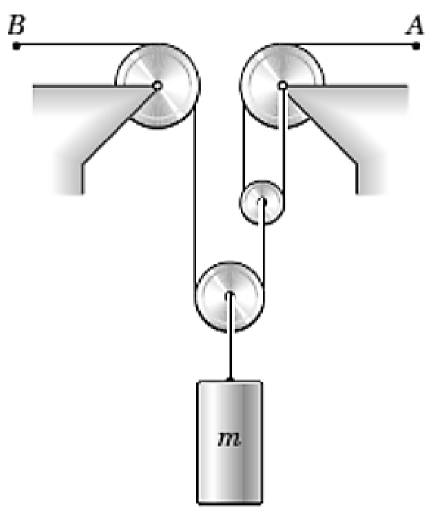
$$\theta = 54.23^\circ$$



$$a_t = a_y\cos\left(\frac{\pi}{2} - \theta\right) = 15.93 \text{ in/s}^2.$$

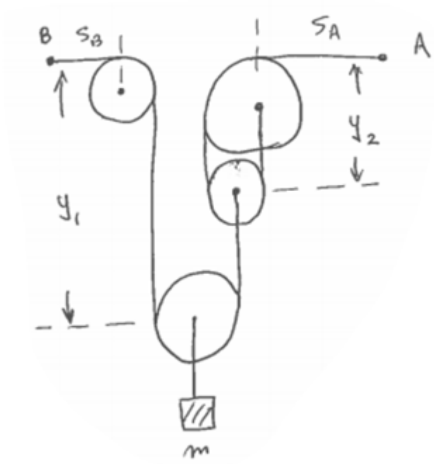
7. [11 Points] In the pulley system shown below, mass *m* moves vertically in response to the horizontal motion of points *A* and *B*, respectively at the end of the two inextensible cables. *B* moves to the **left** with a constant speed of 2 m/s, while *A* moves to the **left** (starting from rest at $t = 0$ s) with an acceleration of $(4t/3)$ m/s^2.

(1) [4 points] Determine the acceleration of mass *m* at time $t = 3$ s. Clearly specify both magnitude and direction of the acceleration.
(2) [4 points] Determine the velocity of mass *m* at time $t = 3$ s. Clearly specify both magnitude and direction of the velocity.
(3) [3 points] Determine the relative acceleration of mass *m* as seen by an observer who moves with *A*, at time $t = 3$ s. Clearly specify the magnitude of the relative acceleration and its direction in terms of the angle the relative acceleration makes with the horizontal.



Solution:

First define coordinates s_A, s_B, y_1, and y_2, representing the motion of A, B and two pulleys, all measured from fixed origins.



(1) Since the two cables are inextensible,

$$s_B + y_1 + (y_1 - y_2) = \text{constant}$$

$$s_A + 2y_2 = \text{constant.}$$

Therefore,

$$\dot{s}_B + 2\dot{y}_1 - \dot{y}_2 = 0$$

$$\dot{s}_A + 2\dot{y}_2 = 0.$$

The solution is

$$\dot{y}_2 = -\tfrac{1}{2}\,\dot{s}_A$$

$$\dot{y}_1 = -1/2\dot{y}_2 - 1/2\ \dot{s}_B = -1/4\ \dot{s}_A - \tfrac{1}{2}\ \dot{s}_B$$

Since $\ddot{s}_A = -4/3\ t\ m/s^2$, *at* $t = 3s$ *the acceleration of A is then*

$$a_A = \ddot{s}_A = -\ 4\ m/s^2 \leftarrow$$

Also, we know that $\ddot{s}_B = 0$. Therefore, the acceleration of the mass is

$$a_M = \ddot{y}_1 = -1/4\ \ddot{s}_A - \tfrac{1}{2}\ \ddot{s}_B = -1/4\ \ddot{s}_A = \underline{1\ m/s^2 \downarrow}$$

(2) The velocity of A at t = 3 s is

$$V_A = \dot{s}_A = \int_0^3 \left(-\frac{4t}{3}\right)dt = -\ 6\ m/s. \leftarrow$$

The velocity of the mass is then

$$V_M = \dot{y}_1 = -1/4(-6)-1/2(2) = \underline{0.5\ m/s \downarrow}$$

(3) The relative acceleration of mass *m* relative to A is

$$\vec{a}_{M/A} = \vec{a}_M - \vec{a}_A.$$

Making use of the amplitudes and directions of the velocities and accelerations determined in (1) and (2),

magnitude $a_{M/A} = (4^2+1^2)^{1/2} = \underline{4.12\ m/s^2}$

14.04^0

direction $\theta = tan^{-1}(1/4) = \underline{14.04^0.}$

Fundamental Equations of Dynamics

KINEMATICS

Particle Rectilinear Motion

Variable a	*Constant $a = a_c$*
$a = \dfrac{dv}{dt}$	$v = v_0 + a_c t$
$v = \dfrac{ds}{dt}$	$s = s_0 + v_0 t + \frac{1}{2} a_c t^2$
$a\,ds = v\,dv$	$v^2 = v_0^2 + 2a_c(s - s_0)$

Particle Curvilinear Motion

x, y, z Coordinates		*r, θ, z Coordinates*	
$v_x = \dot{x}$	$a_x = \ddot{x}$	$v_r = \dot{r}$	$a_r = \ddot{r} - r\dot{\theta}^2$
$v_y = \dot{y}$	$a_y = \ddot{y}$	$v_\theta = r\dot{\theta}$	$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$
$v_z = \dot{z}$	$a_z = \ddot{z}$	$v_z = \dot{z}$	$a_z = \ddot{z}$

n, t, b Coordinates

$v = \dot{s}$	$a_t = \dot{v} = v\dfrac{dv}{ds}$
	$a_n = \dfrac{v^2}{\rho} \quad \rho = \dfrac{[1 + (dy/dx)^2]^{3/2}}{\lvert d^2y/dx^2 \rvert}$

Relative Motion

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \quad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$

Rigid Body Motion About a Fixed Axis

Variable $\mathbf{a}$	*Constant* $\mathbf{a} = \mathbf{a}_c$
$\alpha = \dfrac{d\omega}{dt}$	$\omega = \omega_0 + \alpha_c t$
$\omega = \dfrac{d\theta}{dt}$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_c t^2$
$\omega\,d\omega = \alpha\,d\theta$	$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$

For Point P

$s = \theta r \quad v = \omega r \quad a_t = \alpha r \quad a_n = \omega^2 r$

Relative General Plane Motion—Translating Axes

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A(\text{pin})} \quad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A(\text{pin})}$

Relative General Plane Motion—Trans. and Rot. Axis

$\mathbf{v}_B = \mathbf{v}_A + \Omega \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$

$\mathbf{a}_B = \mathbf{a}_A + \dot{\Omega} \times \mathbf{r}_{B/A} + \Omega \times (\Omega \times \mathbf{r}_{B/A}) + 2\Omega \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$

KINETICS

Mass Moment of Inertia $I = \int r^2\,dm$

Parallel-Axis Theorem $I = I_G + md^2$

Radius of Gyration $k = \sqrt{\dfrac{I}{m}}$

Equations of Motion

Particle	$\Sigma\mathbf{F} = m\mathbf{a}$
Rigid Body (Plane Motion)	$\Sigma F_x = m(a_G)_x$ $\Sigma F_y = m(a_G)_y$ $\Sigma M_G = I_G\alpha$ or $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$

Principle of Work and Energy

$T_1 + U_{1-2} = T_2$

Kinetic Energy

Particle	$T = \frac{1}{2}mv^2$
Rigid Body (Plane Motion)	$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$

Work

Variable force	$U_F = \int F\cos\theta\,ds$
Constant force	$U_F = (F_c\cos\theta)\,\Delta s$
Weight	$U_W = -W\,\Delta y$
Spring	$U_s = -(\frac{1}{2}ks_2^2 - \frac{1}{2}ks_1^2)$
Couple moment	$U_M = M\,\Delta\theta$

Power and Efficiency

$P = \dfrac{dU}{dt} = \mathbf{F}\cdot\mathbf{v} \quad \epsilon = \dfrac{P_{\text{out}}}{P_{\text{in}}} = \dfrac{U_{\text{out}}}{U_{\text{in}}}$

Conservation of Energy Theorem

$T_1 + V_1 = T_2 + V_2$

Potential Energy

$V = V_g + V_e$, where $V_g = \pm Wy$, $V_e = +\frac{1}{2}ks^2$

Principle of Linear Impulse and Momentum

Particle	$m\mathbf{v}_1 + \Sigma\int \mathbf{F}\,dt = m\mathbf{v}_2$
Rigid Body	$m(\mathbf{v}_G)_1 + \Sigma\int \mathbf{F}\,dt = m(\mathbf{v}_G)_2$

Conservation of Linear Momentum

$\Sigma(\text{syst. } m\mathbf{v})_1 = \Sigma(\text{syst. } m\mathbf{v})_2$

Coefficient of Restitution $e = \dfrac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$

Principle of Angular Impulse and Momentum

Particle	$(\mathbf{H}_O)_1 + \Sigma\int \mathbf{M}_O\,dt = (\mathbf{H}_O)_2$ where $H_O = (d)(mv)$
Rigid Body (Plane motion)	$(\mathbf{H}_G)_1 + \Sigma\int \mathbf{M}_G\,dt = (\mathbf{H}_G)_2$ where $H_G = I_G\omega$ $(\mathbf{H}_O)_1 + \Sigma\int \mathbf{M}_O\,dt = (\mathbf{H}_O)_2$ where $H_O = I_O\omega$

Conservation of Angular Momentum

$\Sigma(\text{syst. } \mathbf{H})_1 = \Sigma(\text{syst. } \mathbf{H})_2$

APPENDIX A

Mathematical Expressions

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Hyperbolic Functions

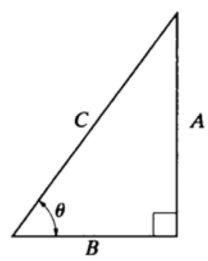
$\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\tanh x = \dfrac{\sinh x}{\cosh x}$

Trigonometric Identities

$\sin\theta = \dfrac{A}{C}$, $\csc\theta = \dfrac{C}{A}$

$\cos\theta = \dfrac{B}{C}$, $\sec\theta = \dfrac{C}{B}$

$\tan\theta = \dfrac{A}{B}$, $\cot\theta = \dfrac{B}{A}$



$\sin^2\theta + \cos^2\theta = 1$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos\theta = \pm\sqrt{\dfrac{1 + \cos 2\theta}{2}}$, $\sin\theta = \pm\sqrt{\dfrac{1 - \cos 2\theta}{2}}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$

Power-Series Expansions

$\sin x = x - \dfrac{x^3}{3!} + \cdots \qquad \sinh x = x + \dfrac{x^3}{3!} + \cdots$

$\cos x = 1 - \dfrac{x^2}{2!} + \cdots \qquad \cosh x = 1 + \dfrac{x^2}{2!} + \cdots$

Derivatives

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}(\cot u) = -\csc^2 u\frac{du}{dx}$$

$$\frac{d}{dx}(\sec u) = \tan u \sec u\frac{du}{dx}$$

$$\frac{d}{dx}(\csc u) = -\csc u \cot u\frac{du}{dx}$$

$$\frac{d}{dx}(\sin u) = \cos u\frac{du}{dx}$$

$$\frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}$$

$$\frac{d}{dx}(\tan u) = \sec^2 u\frac{du}{dx}$$

$$\frac{d}{dx}(\sinh u) = \cosh u\frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u\frac{du}{dx}$$

Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx) + C$$

$$\int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ba}}\ln\left[\frac{a+x\sqrt{-ab}}{a-x\sqrt{-ab}}\right] + C,\ ab < 0$$

$$\int \frac{x\,dx}{a+bx^2} = \frac{1}{2b}\ln(bx^2+a) + C,$$

$$\int \frac{x^2\,dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b\sqrt{ab}}\tan^{-1}\frac{x\sqrt{ab}}{a} + C,\ ab > 0$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left[\frac{a+x}{a-x}\right] + C,\ a^2 > x^2$$

$$\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3} + C$$

$$\int x\sqrt{a+bx}\,dx = \frac{-2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2} + C$$

$$\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3} + C$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right] + C,\ a > 0$$

$$\int x\sqrt{x^2 \pm a^2}\,dx = \frac{1}{3}\sqrt{(x^2 \pm a^2)^3} + C$$

$$\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x}{4}\sqrt{(a^2-x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right) + C,\ a > 0$$

$$\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2\ln\left(x + \sqrt{x^2 \pm a^2}\right)\right] + C$$

$$\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}\sqrt{(a^2-x^2)^3} + C$$

$$\int x^2\sqrt{x^2 \pm a^2}\,dx = \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\ln\left(x + \sqrt{x^2 \pm a^2}\right) + C$$

$$\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b} + C$$

$$\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2} + C$$

$$\int \frac{dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}}\ln\left[\sqrt{a+bx+cx^2} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right] + C,\ c > 0$$

$$= \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{-2cx-b}{\sqrt{b^2-4ac}}\right) + C,\ c > 0$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax) + C$$

$$\int x^2\cos(ax)\,dx = \frac{2x}{a^2}\cos(ax) + \frac{a^2x^2-2}{a^3}\sin(ax) + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) + C$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \cosh x\,dx = \sinh x + C$$