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# ON R.A. SMITH'S AUTONOMOUS CONVERGENCE THEOREM

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ABSTRACT. R.A. Smith [19] showed that his higher dimensional generalization of Bendixson's criterion for the nonexistence of periodic solutions to an autonomous differential equation implies that all bounded trajectories tend to an equilibrium. Here it is shown that a similar conclusion can be drawn from a generalized Dulac criterion. These conditions are also shown to have strong implications for the structure of invariant sets.

**1. Introduction.** Let the map $x \mapsto f(x)$ from an open set $\mathcal{D}$ in $\mathbf{R}^n$ to $\mathbf{R}^n$ be such that each solution $x(t)$ to the differential equation

$$dx/dt = f(x) \tag{1}$$

is uniquely determined by its initial value $x(0) = x_0$ and denote this solution $x(t, x_0)$.

A point $x_0 \in \mathcal{D}$ is *wandering* for (1) if there exists a neighborhood $\mathcal{U}$ of $x_0$ and $T > 0$ such that $\mathcal{U} \cap x(t, \mathcal{U})$ is empty for all $t \geq T$. Thus, for example, any equilibrium, $\alpha$ limit point or $\omega$ limit point is nonwandering.

The closing lemma of Pugh [**15, 16, 17**] shows that if $f$ is $C^r$ and $x_0$ is a nonwandering point which is not an equilibrium for (1), then there are differential equations arbitrarily $C^r$-close to (1) which have nonconstant periodic solutions. Suppose now that $f$ satisfies a condition which precludes the existence of periodic solutions to (1). If the condition is sufficiently robust that it is also satisfied by functions which are $C^r$-close to $f$, then the closing lemma implies that every nonequilibrium point of (1) is wandering. In particular, every $\alpha$ or $\omega$ limit set consists entirely of equilibria; it follows that if the zeros of $f$ are isolated, then

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every semi-trajectory either tends to an equilibrium or the boundary of $\mathcal{D}$.

Smith's generalization [**19**] to higher dimensions of Bendixson's criterion, $n = 2$, for the nonexistence of nonconstant periodic solutions of (1) has the required robustness in $C^1$. He uses this fact to conclude that, if $\mathcal{D} = \mathbf{R}^n$ and his criterion is satisfied, all bounded solutions of (1) tend to an equilibrium even without the assumption that equilibria are isolated. In [**10, 14**] we gave other higher dimensional versions of Bendixson's criterion and its extension, Dulac's criterion. We will show that these also imply that trajectories which do not tend to the boundary of $\mathcal{D}$ end at equilibria and, even more, any $\alpha$ or $\omega$ limit set which is not empty consists of a single equilibrium.

Theorem 2.5 is our generalization of Smith's autonomous convergence theorem; the proof is based on Pugh's closing lemma and the center manifold theorem. Theorem 2.10 shows that systems which satisfy the generalized Dulac criterion cannot have invariant sets which are very complex. In Section 3 we show how similar conclusions can be drawn even when the Dulac inequality is not strict.

**2. Wandering point theorems.** We begin by formulating the special case of the closing lemma which we use. Let $|\cdot|$ denote a norm on $\mathbf{R}^n$ and the operator norm which it induces for linear maps from $\mathbf{R}^n$ to $\mathbf{R}^n$. The distance between two functions $f, g \in C^1(\mathcal{D} \to \mathbf{R}^n)$ such that $f - g$ has compact support is

$$\sup\{|f(x) - g(x)| + |Df(x) - Dg(x)| : x \in \mathcal{D}\}.$$

For a function $g$ which is $C^1$-close to $f$, we consider the corresponding differential equation

$$dx/dt = g(x). \tag{2}$$

**Lemma 2.1.** *Let $f \in C^1(\mathcal{D} \to \mathbf{R}^n)$. Suppose that $x_0$ is a nonwandering point for* (1) *and that $f(x_0) \neq 0$. Then, for each neighborhood $\mathcal{U}$ of $x_0$, there exist functions $g \in C^1(\mathcal{D} \to \mathbf{R}^n)$ arbitrarily $C^1$-close to $f$ such that*

(a) *$f(x) = g(x)$, if $x \in \mathcal{D}\backslash\mathcal{U}$ and*

(b) *the system* (2) *has a nontrivial periodic solution whose trajectory intersects* $\mathcal{U}$.

Now we consider the implications for (1) of the existence of $V \in C^1(\mathcal{D} \to \mathbf{R})$ which satisfies

$$(3) \qquad (\partial V/\partial x)^* f(x) < 0, \quad \text{if } f(x) \neq 0,$$

where $\partial/\partial x$ is the gradient operator and the asterisk denotes transposition.

**Proposition 2.2.** *If there exists a real-valued function* $x \mapsto V(x)$ *which satisfies* (3), *then every nonwandering point of* (1) *is an equilibrium.*

*Proof.* This may be shown directly by observing that all trajectories depart from a small neighborhood of a nonequilibrium and, since (3) implies that $V$ is strictly decreasing along trajectories, no trajectory returns. In the spirit of the present development, it may also be deduced from the $C^0$ version of Lemma 2.1. Evidently, if $x_0$ is nonwandering for (1) and $f(x_0) \neq 0$, the set $\mathcal{U}$ in Lemma 2.1 may be chosen so that it contains no zeros of $f$ and $\overline{\mathcal{U}} \subset D$ is compact. Then the functions $g$ may be chosen sufficiently $C^0$-close to $f$ that $(\partial V/\partial x)^* g(x) < 0$, if $g(x) \neq 0$, and therefore $V(x(t))$ decreases strictly for all nonequilibrium solutions $x(t)$ of (2) so that no such solution can be periodic. Thus the $C^0$ closing lemma implies that every nonwandering point of (1) is an equilibrium. □

**Corollary 2.3.** *Suppose that* $\mathcal{D}$ *is simply connected and a* $C^1$ *function* $x \mapsto a(x) \in \mathbf{R}^n$ *exists such that*

(a) $a^* f(x) < 0$ *if* $f(x) \neq 0$,

(b) $\partial a_i/\partial x_j - \partial a_j/\partial x_i = 0$ *in* $\mathcal{D}$ *for all* $i, j = 1, \dots, n$.

*Then every nonwandering point of* (1) *is an equilibrium.*

This follows from the fact that (b) implies $\partial V/\partial x = a$ for some scalar function $V$ and then (a) shows that (3) is satisfied. For example, if $f \in C^1$ and $\partial f_i/\partial x_j - \partial f_j/\partial x_i = 0$, $i, j = 1, \dots, n$, then we may choose $a = -f$ in Corollary 2.3. Corollary 2.3 may also be proved using Stokes' theorem.

Let $\mathcal{B}$ denote the Euclidean unit ball in $\mathbf{R}^2$ and $\overline{\mathcal{B}}$, $\partial\mathcal{B}$ its closure and boundary respectively. A function $\varphi \in \mathrm{Lip}\,(\overline{\mathcal{B}} \to \mathcal{D})$ will be considered a *simply connected rectifiable* 2*-surface in* $\mathcal{D}$ or, briefly, a *surface* in $\mathcal{D}$; a function $\psi \in \mathrm{Lip}\,(\partial\mathcal{B} \to \mathcal{D})$ is a *closed rectifiable curve* in $\mathcal{D}$, will be called *simple* if it is one-to-one, and we will write $\psi = \partial\varphi$ if $\varphi(\partial\mathcal{B}) = \psi(\partial\mathcal{B})$ and $\varphi|_{\partial\mathcal{B}}$ is simple.

Let $x \mapsto A(x)$ be a nonsingular $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function which is $C^1$ on $\mathcal{D}$, and let $|\cdot|$ be a norm on $\mathbf{R}^{\binom{n}{2}}$. We consider a functional $\mathcal{S}$ on the surfaces in $\mathcal{D}$ defined by

$$\mathcal{S}\varphi = \int_{\mathcal{B}} \left| A(\varphi)\frac{\partial\varphi}{\partial u_1} \wedge \frac{\partial\varphi}{\partial u_2} \right| \tag{4}$$

if $u = (u_1, u_2)$ and $u \mapsto \varphi(u)$ is Lipschitzian on $\overline{\mathcal{B}}$. If $|y|^2 = y^*y$ and $A(x) = I$, then $S\varphi$ is the usual surface area of $\varphi(\overline{\mathcal{B}})$ counting multiplicities. The set $\mathcal{D}$ has the *minimum property* with respect to $\mathcal{S}$ if, for each simple closed rectifiable curve $\psi$ in $\mathcal{D}$, there is a sequence of surfaces $\varphi^k$ in $\mathcal{D}$ which is a minimizing sequence for $\mathcal{S}\varphi$ with respect to all surfaces $\varphi$ in $\mathcal{D}$ with $\psi = \partial\varphi$ and such that $\cup_k \varphi^k(\overline{\mathcal{B}})$ has compact closure in $\mathcal{D}$. If $n = 2$, then any simply connected open set $\mathcal{D}$ has the minimum property with respect to any $\mathcal{S}$, since $\mathcal{S}\varphi = \int_{\varphi(\mathcal{B})} |A|$. When $n \geq 2$, for example, if $\mathcal{S}$ is the usual surface area, then any convex open set $\mathcal{D}$ has the minimum property. The set $\mathcal{D} = \mathbf{R}^n$ has the minimum property with respect to $\mathcal{S}$ if $A = I$ and $|\cdot|$ is any absolute norm. In fact, if $\varphi^k$ is a sequence with $\psi = \partial\varphi^k$ which minimizes $\mathcal{S}$ in this case, then for any interval $\mathcal{I} \subset \mathbf{R}^n$ with $\psi(\partial\mathcal{B}) \subset \mathcal{I}$ we can obtain a sequence $\tilde{\varphi}^k$ with $\mathcal{S}\tilde{\varphi}^k = \mathcal{S}\varphi^k$ and $\tilde{\varphi}^k(\overline{\mathcal{B}}) \subset \mathcal{I}$, by reflection in the sides of the interval $\mathcal{I}$.

For a simply connected open set $\mathcal{D}$ which has the minimum property with respect to $\mathcal{S}$, we will assume that the generalized Dulac condition [**10**, Theorem 3.4] is satisfied:

$$\mu(A_f A^{-1} + A(\partial f/\partial x)^{[2]} A^{-1}) < 0 \quad \text{in } \mathcal{D}. \tag{5}$$

Here $\mu$ is the Lozinskiĭ measure corresponding to the norm $|\cdot|$ on $\mathbf{R}^{\binom{n}{2}}$ considered in (4) [**1**, p. 41, **10**], $A_f = (DA)(f)$ or, equivalently,

$A_f$ is the matrix obtained by replacing each entry $a_{ij}$ in $A$ by its directional derivative in the direction $f$, $(\partial a_{ij}/\partial x)^* f$, and $\partial f^{[2]}/\partial x$ is a $\binom{n}{2} \times \binom{n}{2}$ matrix, the second additive compound of the Jacobian matrix $\partial f/\partial x$ [**14, 18**]. The second additive compound matrix of an $n \times n$ matrix is given in the Appendix for each of the cases $n = 2, 3, 4, 5$. For readers unfamiliar with the Lozinskiĭ measure, the condition (5) is equivalent to assuming that $V(x, y) = |A(x)y|$ is a Liapunov function whose derivative with respect to the $n + \binom{n}{2}$ dimensional system

$$\frac{dx}{dt} = f(x), \qquad \frac{dy}{dt} = \frac{\partial f}{\partial x}^{[2]}(x)y \tag{6}$$

is negative definite.

A set $\mathcal{D}_0$ is *absorbing* with respect to (1) if solutions exist for all $t \geq 0$ and each bounded subset $\mathcal{D}_1$ of $\mathcal{D}$ satisfies $x(t, \mathcal{D}_1) \subset \mathcal{D}_0$ for all sufficiently large $t$. When $\mathcal{D}$ does not necessarily have the minimum property with respect to $\mathcal{S}$, it will be assumed that there is a set $\mathcal{D}_0 \subset \mathcal{D}$ which is absorbing with respect to (1) such that

$$\mu(A_f A^{-1} + A(\partial f/\partial x)^{[2]} A^{-1}) \leq -\delta < 0 \quad \text{in } \mathcal{D}_0. \tag{7}$$

**Proposition 2.4.** *If* (5) *is satisfied in* $\mathcal{D}$, *then the dimension of the stable manifold of any equilibrium is at least* $(n-1)$. *If an equilibrium is not isolated, then its stable manifold has dimension* $(n-1)$ *and it has a local center manifold of dimension* 1 *which contains all nearby equilibria.*

*Proof.* For the definitions of center manifold and stable or unstable manifold, see [**6**]. If $x_1$ is an equilibrium, then

$$\mu\left(A\frac{\partial f}{\partial x}^{[2]} A^{-1}\right) = \mu\left(A_f A^{-1} + A\frac{\partial f^{[2]}}{\partial x} A^{-1}\right) < 0 \tag{8}$$

at $x_1$, since $f(x_1) = 0$ implies $A_{f(x_1)}(x_1) = 0$. If $\nu_i(x_1)$ are the eigenvalues of $(\partial f/\partial x)(x_1)$ with $\operatorname{Re}\nu_1(x_1) \geq \operatorname{Re}\nu_2(x_1) \geq \cdots \geq \operatorname{Re}\nu_n(x_1)$, then $\nu_i(x_1) + \nu_j(x_1)$, $i \neq j$, are the eigenvalues of $(\partial f/\partial x)^{[2]}(x_1)$ [**11**, p. 505] and hence of $A(\partial f/\partial x)^{[2]} A^{-1}(x_1)$. Thus (8) implies that

$\operatorname{Re}[\nu_i(x_1) + \nu_j(x_1)] \le \mu[A(\partial f/\partial x)^{[2]}A^{-1}(x_1)] < 0$ [**1**, p. 41]; therefore, $0 > \operatorname{Re}\nu_2(x_1) \ge \cdots \ge \operatorname{Re}\nu_n(x_1)$ and only $\nu_1(x_1)$ can possibly have nonnegative real part; the stable manifold has dimension at least $n-1$. If the equilibrium $x_1$ is not isolated, $(\partial f/\partial x)(x_1)$ is a singular matrix, $0 = \nu_1(x_1)$ so the stable manifold has dimension $(n-1)$ and there is a one-dimensional center manifold (cf. [**8**, p. 48]). Since all positive semi-trajectories near $x_1$ are asymptotic to a trajectory in the center manifold, all equilibria near $x_1$ are in the center manifold. □

**Theorem 2.5.** *Suppose that $\mathcal{D}$ is simply connected, $f$ is of class $C^1$ on $\mathcal{D}$ and there exists a $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function $A$ which is also $C^1$ on $\mathcal{D}$ and such that either* (a) *or* (b) *is satisfied*:

(a) *$\mathcal{D}$ has the minimum property with respect to $\mathcal{S}$ and* (5) *is satisfied.*

(b) *$\mathcal{D}_0$ is an absorbing subset of $\mathcal{D}$ with respect to* (1) *and* (7) *is satisfied.*

*Then*:

(c) *Every nonwandering point of $\mathcal{D}$ is an equilibrium.*

(d) *Every nonempty $\alpha$ or $\omega$ limit set in $\mathcal{D}$ is a single equilibrium.*

(e) *Any equilibrium in $\mathcal{D}$ is the alpha limit set of at most two distinct nonequilibrium trajectories.*

*Proof.* If (a) or (b) is satisfied by $f$, then a similar condition is also satisfied by all perturbations $g$ of $f$ considered in Lemma 2.1 which are sufficiently $C^1$-close to $f$. From [**10**, Theorems 3.4 and 3.6], no such perturbation can have a nontrivial periodic solution to (2). Therefore, every nonwandering point is an equilibrium for (1).

To prove the assertion that each nonempty $\alpha$ or $\omega$ limit set is a single equilibrium, first observe that since each limit point is nonwandering, it is an equilibrium. Let $x_1 \in \omega(x_0)$, the $\omega$ limit set of $x_0$; if $x_1$ is an isolated equilibrium, then $\{x_1\} = \omega(x_0)$. If $x_1 \in \omega(x_0)$ is not an isolated equilibrium, then Proposition 2.4 implies that there is a one-dimensional center manifold containing all nearby equilibria associated with $x_1$. Since $\omega(x_0)$ is locally a continuum of equilibria, $x_1 \in \omega(x_0)$ may be chosen so that a local center manifold at $x_1$ in a small neighborhood $\mathcal{U}$ of $x_1$ consists entirely of equilibria. Moreover, every trajectory which intersects $\mathcal{U}$ is asymptotic to a trajectory in this center manifold. Thus, $\lim_{t\to\infty} x(t, x_0) = x_1$ so that $\omega(x_0) = \{x_1\}$ in this case also. The proof that a nonempty alpha limit set is an equilibrium $x_2$ is the same. Moreover, since the stable manifold of $x_2$ has dimension $(n-1)$, $x_2$ has either a one-dimensional center manifold or a one-dimensional unstable manifold. Since all trajectories near the center manifold are asymptotic to a trajectory in that manifold [**8**, p. 48] and the stable manifold is asymptotic to $x_2$, at most two nonequilibrium trajectories can share $x_2$ as their $\alpha$ limit. The uniqueness of the unstable manifold implies the same conclusion in the other case. □

The equilibrium $\omega(x_0)$ need not be isolated as is seen from the example

$$dx_1/dt = -x_1, \qquad dx_2/dt = 0.$$

Here $(\partial f/\partial x)^{[2]} = \operatorname{div} f = -1 < 0$ so that (5) is satisfied with $A = 1$. $\mathcal{D} = \mathbf{R}^2$ has the minimum property with respect to $\mathcal{S}$, which is the area in the plane in this case, and the conditions of Theorem 2.5 are all satisfied. Each solution satisfies $\lim_{t\to\infty}(x_1(t), x_2(t)) = \lim_{t\to\infty}(x_1(0)e^{-t}, x_2(0)) = (0, x_2(0))$ a nonisolated equilibrium.

Theorem 2.5 gives a global asymptotic stability criterion for systems with unique equilibria.

**Corollary 2.6.** *Suppose*

(a) *$\mathcal{D}$ is simply connected,*

(b) *$\mathcal{D}_0$ is a compact set which is absorbing with respect to* (1),

(c) *inequality* (5) *is satisfied for some matrix $A$,*

(d) *the system* (1) *has a unique equilibrium $x_0$.*

*Then $x_0$ is globally asymptotically stable.*

*Proof.* Clearly $\{x_0\}$ is globally attracting since Theorem 2.5 implies it is the $\omega$ limit set of every trajectory. Moreover, it is stable since otherwise it would be both the $\alpha$ limit set and the $\omega$ limit set of some homoclinic trajectory $\gamma = \{x(t) : t \in (-\infty, \infty)\}$. In this circumstance we assert that $\mathcal{C} = \gamma \cup \{x_0\}$ is the trace of a rectifiable simple closed curve. This curve is invariant with respect to (1), $x(t, \mathcal{C}) = \mathcal{C}$, and existence of such an invariant curve is precluded by the generalized

Dulac condition as shown in [**10**]. It remains to prove that $\mathcal{C}$ is rectifiable. Since $\gamma$ is in the $C^1$ center manifold or unstable manifold of $x_0$, and this is one-dimensional, it is only necessary to show that $\gamma_+ = \{x(t) : t \in [1, \infty)\}$ is the trace of a rectifiable curve. This is rectifiable if it also approaches its $\omega$ limit through a center manifold. If it does not approach $x_0$ through a center manifold, then, by the center manifold theorem, it approaches $x_0$ exponentially in time. Thus, $|f(x(t))| \leq Ce^{-\lambda t}$ for some constants $C$, $\lambda > 0$, since $f(x_0) = 0$. Considering $\tau(s) = (1-s)^{-1}$, $y(s) = x(\tau(s))$, $s \in [0,1)$, we find $y'(s) = f(x(\tau(s)))\tau'(s)$ so that $|y'(s)| \leq Ce^{-\lambda\tau(s)}\tau'(s) = Ce^{-\lambda(1-s)^{-1}}(1-s)^{-2}$ and $y'$ is bounded with $y[0,1) = \gamma_+$. $\square$

**Corollary 2.7.** *The conclusion of Theorem* 2.5 *holds if* $\mathcal{D} = \mathbf{R}^n$ *and the generalized Bendixson criterion* $\mu(\partial f^{[2]}/\partial x) < 0$ *is satisfied in* $\mathbf{R}^n$, *where* $\mu$ *is the Lozinskiĭ measure corresponding to an absolute norm.*

Conditions (i), (ii), (iii) of [**10**, Theorem 3.3] give concrete examples of the condition $\mu(\partial f/\partial x)^{[2]} < 0$. The conditions (iv), (v), (vi) of this theorem are examples of $\mu(-\partial f/\partial x)^{[2]} < 0$ which, as we see below, has similar consequences.

Denote by $(1)_-$ the system (1) with $f$ replaced by $-f$. The trajectories of $(1)_-$ are the same as those of (1) with the direction of the flow reversed. From this we deduce the following corollary.

**Corollary 2.8.** *If the system* $(1)_-$ *satisfies the conditions of Theorem* 2.5 *or Corollary* 2.6, *then the same conclusions may be drawn for* (1) *except that the statements about* $\alpha$ *and* $\omega$ *limit sets should be interchanged.*

Even in the case $n = 2$, this result gives a somewhat stronger conclusion than that usually drawn from Bendixson's criterion.

**Corollary 2.9.** *Suppose* $\mathcal{D} \subset \mathbf{R}^2$ *is simply connected and*

$$\operatorname{div} f(x) < 0 \quad \textit{in } \mathcal{D}.$$

*Then every nonwandering point with respect to* (1) *is an equilibrium*;

*every nonempty $\alpha$ or $\omega$ limit set is a single isolated equilibrium; any equilibrium is the $\alpha$ limit set of at most two nonequilibrium trajectories.*

A subset $\mathcal{K}$ of $\mathcal{D}$ is *positively* (*negatively*) *invariant* with respect to (1) if $x(t,\mathcal{K}) \subset \mathcal{K}$ for all $t \geq 0$, $t \leq 0$, and is *invariant* if $x(t,\mathcal{K}) = \mathcal{K}$ for all $t$. The $\alpha$ and $\omega$ limit sets of a trajectory are important examples of invariant sets, and we have seen in the preceding discussion that these are very simple for systems which satisfy Dulac's condition or its higher dimensional generalizations. In fact, we will show that any compact set which is invariant in such a system is at most one-dimensional.

**Theorem 2.10.** *Suppose that $f$ satisfies the conditions of Theorem* 2.5 *and $\mathcal{K} \subset \mathcal{D}$ is a compact set which is invariant with respect to* (1). *Then its Hausdorff dimension satisfies*

$$\dim_H \mathcal{K} \leq 1.$$

*In particular, if $\mathcal{K}$ is also connected, then* $\dim_H \mathcal{K} = 0$ *or* 1 *depending on whether $\mathcal{K}$ contains one point or more than one point.*

*Proof.* Since $\mathcal{K}$ is compact, Theorem 2.5 implies that every trajectory in $\mathcal{K}$ is either an equilibrium or is asymptotic at each end to some equilibrium. Let $\mathcal{K}_0$ be the set of equilibria in $\mathcal{K}$ and $\mathcal{K}_0'$ its set of cluster points. If $x \in \mathcal{K}_0'$, then Proposition 2.4 implies that there is a neighborhood $\mathcal{U}(x)$ of $x$ such that all equilibria in $\mathcal{U}(x)$ lie in a one-dimensional center manifold $\sigma(x)$ at $x$. A finite set of these neighborhoods $\mathcal{U}(x_i)$, $i = 1, \dots, N$, covers the compact set $\mathcal{K}_0'$. The set $\mathcal{K}$ is composed of complete trajectories of the following three types:

(i) trajectories in one of the one-dimensional manifolds $\sigma(x_i)$, $i = 1, \dots, N$,

(ii) the finite set of equilibria $\mathcal{K}_0 \setminus \cup_i \sigma(x_i)$,

(iii) nonequilibrium trajectories whose $\alpha$ and $\omega$ limit sets are each single equilibria of types (i) and (ii).

Any trajectory or any smooth arc has Hausdorff measure zero in dimension $s > 1$ (see [**5**, p. 28]). From this it follows that the set of all trajectories of types (i) and (ii) has $s$-dimensional measure zero. Moreover, there are at most finitely many trajectories of type (iii).

Otherwise, there would exist a rectifiable simple closed curve composed of trajectories of types (ii) and (iii) together with a finite number of invariant arcs from $\sigma(x_i)$, $i = 1, \dots, N$. It was shown in [**10**] that the conditions of Theorem 2.5 preclude the existence of such invariant curves. We therefore conclude that the set of trajectories of type (iii) also has $s$-dimensional measure zero. Thus, $\mathcal{K}$ has zero $s$-dimensional measure, if $s > 1$, and $\dim_H \mathcal{K} \leq 1$ [**5**, p. 29]. If $\mathcal{K}$ is also connected and contains more than one point, then the sum of the diameters of the sets in any open cover exceeds the distance between any pair of points in $\mathcal{K}$ which allows us to conclude that $\dim_H \mathcal{K} \geq 1$ and therefore $\dim_H \mathcal{K} = 1$.

Smith [**19**, Theorem 7] shows that if $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_n(x)$ are the eigenvalues of $(1/2)[(\partial f/\partial x)^*(x) + (\partial f/\partial x)(x)]$, then each bounded semi-trajectory of (1) converges to an equilibrium if $\lambda_1(x) + \lambda_2(x) < 0$ in $\mathbf{R}^n$. This also follows from Corollary 2.5 of this paper since $\mu((\partial f/\partial x)^{[2]}) = \lambda_1 + \lambda_2$ if $|\cdot|$ is the Euclidean norm on $\mathbf{R}^{\binom{n}{2}}$. Our result shows that the same conclusion can be drawn if the boundedness assumption is replaced by one of existence of an $\alpha$ or $\omega$ limit point of the semi-trajectory. The domain $\mathbf{R}^n$ may be replaced by any convex open set $\mathcal{D}$ since such sets have the minimum property with respect to $\mathcal{S}$ which is the usual surface area in this case. The domain may also be any open set $\mathcal{D}$ which is simple connected and has an absorbing subset $\mathcal{D}_0$ in which $\lambda_1(x) + \lambda_2(x) \leq -\delta < 0$ holds. Analogous results may be inferred from conditions of the form $\lambda_{n-1}(x) + \lambda_n(x) > 0$, since $-\mu(-(\partial f/\partial x)^{[2]}) = \lambda_{n-1} + \lambda_n$. Smith's proof shows that his condition implies $\dim_H \mathcal{K} < 2$ for any compact invariant $\mathcal{K}$; in fact, we see from Theorem 2.9 that his condition implies $\dim_H \mathcal{K} \leq 1$. □

An earlier result of Hartman and Olech [**7**] is somewhat related to observations of this paper. They show that if $x = 0$ is the only equilibrium of (1) and it is locally asymptotically stable, then it is globally asymptotically stable provided $\lambda_1(x) + \lambda_2(x) \leq 0$ in $\mathbf{R}^n$ and $\int_0^\infty p = \infty$ where $p(u) = \min\{|f(x)| : |x| = u\}$.

**3. A Weak Dulac Condition.** If the condition (3) is replaced by

$$(\partial V/\partial x)^* f(x) \leq 0, \qquad x \in \mathcal{D} \tag{9}$$

then, instead of Proposition 2.2, we can conclude the following:

**Proposition 3.1.** *If* (9) *is satisfied, then every nonwandering point is in the set* $S = \{x : (\partial V/\partial x)^* f(x) = 0\}$.

This is a slightly stronger result than can be inferred from the invariance principle of LaSalle [**9**, p. 30] which states in this case that every $\omega$ limit point is in the maximal invariant subset of $S$. The statement follows from the fact that (9) is sufficiently robust that it is also satisfied by systems which are sufficiently close to (1) in the sense of Lemma 2.1 and that the inequality is strict at some points of the periodic trajectory of that lemma.

A similar conclusion may be drawn if a weakened version of (5) is used in Theorem 2.5:

$$\mu\bigg(A_f A^{-1} + A\frac{\partial f}{\partial x}^{[2]} A^{-1}\bigg) \leq 0 \quad \text{in } \mathcal{D}. \tag{10}$$

Let $\mathcal{D}_1$ be the subset of $\mathcal{D}$ on which (10) is strict and $\mathcal{D}_2 = \mathcal{D}\backslash\mathcal{D}_1$.

**Theorem 3.2.** *Suppose* $\mathcal{D}$ *is simply connected,* $f$ *is of class* $C^1$ *and there is a matrix* $A$ *of class* $C^1$ *such that either* (a) *or* (b) *holds*:

(a) $\mathcal{D}$ *has the minimum property with respect to* $\mathcal{S}$ *and* (10) *is satisfied on* $\mathcal{D}$.

(b) $\mathcal{D}_0$ *is an absorbing compact subset of* $\mathcal{D}$ *and* (10) *is satisfied on* $\mathcal{D}_0$.

*Then the conclusions* $(c), (d), (e)$ *of Theorem 2.5 hold if in those statements* $\mathcal{D}$ *is replaced by* $\mathcal{D}_1$.

This may be seen by again supposing that $x_0 \in \mathcal{D}_1$ is nonwandering and $f(x_0) \neq 0$. Then, choosing the neighborhood $\mathcal{U}$ of $x_0$ in Lemma 2.1 a sufficiently small subset of the region where (10) is strict, we find a system $C^1$-close to (1) which also satisfies (10), strictly near $x_0$ and has a nontrivial periodic trajectory intersecting $\mathcal{U}$. Then, using Criteria 3.1 and 3.2 of [**10**] in cases (a) and (b), respectively, we find a contradiction as before. Thus, we must have $f(x_0) = 0$ if $x_0 \in \mathcal{D}_1$ is nonwandering.

The Corollaries 2.6, 2.7 and 2.8 may also be modified in this way. Similarly, the considerations leading to Theorem 2.10, suitably altered, lead to a modification of that result.

**Theorem 3.3.** *Suppose that $\mathcal{D}$ is simply connected,*

(a) *$f$ satisfies condition* (b) *of Theorem* 3.1*,*

(b) *$\mathcal{D}_1$ is invariant and*

(c) *at most a finite number of trajectories in $\mathcal{K} \cap \mathcal{D}_1$ have a limit point in $\mathcal{D}_2$.*

*Then* $\dim_H(\mathcal{K} \cap \mathcal{D}_1) \le 1$.

This result is established by applying the proof of Theorem 2.10 to the complement in $\mathcal{K} \cap \mathcal{D}_1$ of the at most one-dimensional set in $\mathcal{K} \cap \mathcal{D}_1$ consisting of those trajectories with limit points in $\mathcal{D}_2$.

The condition (c) may of course be difficult to establish in general and may require the use of more than one functional $\mathcal{S}$ or function $V$ or extensions of the ideas considered here.

Corollary 2.6 and Theorems 2.10 and 3.2 are somewhat surprising in view of estimates on Hausdorff dimensions of attractors due to Smith [**19**], Temam [**20**], Boichenko and Leonov [**2**], Eden, Foias and Temam [**4**]. These results, which seem to give good estimates on attractors whose dimension is greater than 2 even in delicate cases such as the Lorenz system, would only give 2 as an upper bound in some cases where Theorems 2.10 and 3.2 would give a bound 1 and Corollary 2.6 would give a value 0. For example, in the terminology of [**4**], conditions such as (5) would imply that $\mu_1(x) + \mu_2(x) < 0$, where $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_n$ are the local Liapunov exponents which in turn implies that any invariant compact set has Hausdorff dimension less than 2.

Now consider the dissipative system

$$dx_1/dt = x_1 - x_1^3, \qquad dx_2/dt = -x_2 \tag{11}$$

of Eckmann and Ruelle cited by Eden [**3**]. Here the global attractor is $\mathcal{K} = [-1, 1] \times \{0\}$ so that $\dim_H \mathcal{K} = 1$.

If $V(x) = x_2^2$, we find $(\partial V/\partial x)^* f(x) = -2x_2^2$ so we could infer independently from this that $\mathcal{K}$ has dimension at most 1 since Proposition 3.1 implies it is located on the $x_1$-axis. Since there is more than one equilibrium, we conclude that $\dim_H K = 1$. Alternatively, consider

TABLE 1. The matrix $A^{[2]}$, $n = 2, 3, 4, 5$.

$$n=2:(1)=(1,2) \qquad a_{11}+a_{22} = \operatorname{tr} A$$

$$\begin{array}{r} n=3:1=(1,2)\\ 2=(1,3)\\ 3=(2,3) \end{array} \begin{bmatrix} a_{11}+a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11}+a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22}+a_{33} \end{bmatrix}$$

$$\begin{array}{r} n=4:1=(1,2)\\ 2=(1,3)\\ 3=(1,4)\\ 4=(2,3)\\ 5=(2,4)\\ 6=(3,4) \end{array} \begin{bmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{bmatrix}$$

$$\begin{array}{r} n=5:1=(1,2)\\ 2=(1,3)\\ 3=(1,4)\\ 4=(1,5)\\ 5=(2,3)\\ 6=(2,4)\\ 7=(2,5)\\ 8=(3,4)\\ 9=(3,5)\\ 10=(4,5) \end{array} \begin{bmatrix} a_{11}+a_{22} & a_{23} & a_{24} & a_{25} & -a_{13} & -a_{14} & -a_{15} & 0 & 0 & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{35} & a_{12} & 0 & 0 & -a_{14} & -a_{15} & 0 \\ a_{42} & a_{43} & a_{11}+a_{44} & a_{45} & 0 & a_{12} & 0 & a_{13} & 0 & -a_{15} \\ a_{52} & a_{53} & a_{54} & a_{11}+a_{55} & 0 & 0 & a_{12} & 0 & a_{13} & a_{14} \\ -a_{31} & a_{21} & 0 & 0 & a_{22}+a_{33} & a_{34} & a_{35} & -a_{24} & -a_{25} & 0 \\ -a_{41} & 0 & a_{21} & 0 & a_{43} & a_{22}+a_{44} & a_{45} & a_{23} & 0 & -a_{25} \\ -a_{51} & 0 & 0 & a_{21} & a_{53} & a_{54} & a_{22}+a_{55} & 0 & a_{23} & a_{24} \\ 0 & -a_{41} & a_{31} & 0 & -a_{42} & a_{32} & 0 & a_{33}+a_{44} & a_{45} & -a_{35} \\ 0 & -a_{51} & 0 & a_{31} & -a_{52} & 0 & a_{32} & a_{54} & a_{33}+a_{55} & a_{34} \\ 0 & 0 & -a_{51} & a_{41} & 0 & -a_{52} & a_{42} & -a_{53} & a_{43} & a_{44}+a_{55} \end{bmatrix}$$

$A(x) = x_2^2 + 1$ so that

$$A_f A^{-1} + A \frac{\partial f}{\partial x}^{[2]} A^{-1} = -2x_2^2(x_2^2+1)^{-1} - 3x_1^2$$

and the weak Dulac condition (10) is satisfied with $\mathcal{D}_2 = \{(0,0)\}$. At most a finite number of trajectories can have limit points in $\mathcal{D}_2$ since otherwise we could find an invariant rectifiable simple closed curve which cannot exist by Criterion 3.2 of [**10**]. Theorem 3.3 therefore implies $\dim_H \mathcal{K} \leq 1$.

## Appendix

If $A = [a_{ij}]$ is an $n \times n$ matrix, its second additive compound $A^{[2]}$ is the $\binom{n}{2} \times \binom{n}{2}$ matrix defined as follows. For any integer $i = 1, \dots, \binom{n}{2}$, let $(i) = (i_1, i_2)$ be the $i$-th member in the lexicographic ordering of the integer pairs $(i_1, i_2)$ such that $1 \leq i_1 < i_2 \leq n$. Then the element in the $i$-row and $j$-column of $A^{[2]}$ is

$$\begin{cases} a_{i_1 i_1} + a_{i_2 i_2}, & \text{if } (j) = (i) \\ (-1)^{r+s} a_{i_r j_s}, & \text{if exactly one entry } i_r \text{ of } (i) \text{ does not} \\ & \text{occur in } (j) \text{ and } j_s \text{ does not occur in } (i) \\ 0, & \text{if neither entry from } (i) \text{ occurs in } (j). \end{cases}$$

The table on page 377 gives $A^{[2]}$ in the cases $n = 2, 3, 4, 5$.

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