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Dulac Criteria for Autonomous Systems Having an Invariant Affine Manifold

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For a class of higher dimensional autonomous systems that have an invariant affine manifold, conditions are derived to preclude the existence of periodic solutions on the invariant manifold. It is established that these conditions are robust under certain types of local perturbations of the vector field. As a consequence, each bounded semitrajectory on the invariant manifold is shown to converge to a single equilibrium using a C^1 closing lemma for this class. Applications to autonomous systems that are homogeneous of degree 1 are also considered.

1. INTRODUCTION

Let $X \subset \mathbf{R}^n$ be an open set and $x \mapsto f(x) \in \mathbf{R}^n$ a C^1 function defined for $x \in X$. We consider the autonomous system of ordinary differential equations

$$x' = f(x). \tag{1.1}$$

Let $x(t, x_0)$ denote the solution to (1.1) such that $x(0, x_0) = x_0$. We are interested in systems (1.1) that satisfy the following assumptions:

(H_1) There is a constant matrix B such that rank $B = r$ and the $(n - r)$-dimensional affine manifold

$$\Gamma = \{x \in X: B(x - \bar{x}) = 0\} \qquad \text{for some } \bar{x} \in X \tag{1.2}$$

satisfies $Bf(x) = 0$, $x \in \Gamma$.

(H_2) The Jacobian matrix $\partial f / \partial x$ can be written as

$$\frac{\partial f}{\partial x} = \nu(x) I_{n \times n} + A(x), \qquad x \in \Gamma \tag{1.3}$$

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and

$$BA(x) = 0, \qquad x \in \Gamma, \tag{1.4}$$

where $x \mapsto \nu(x)$ is a real-valued function.

A system (1.1) satisfying (H_1) and (H_2) will be called an autonomous system *having an invariant affine manifold*, since (H_1) is equivalent to that Γ is invariant with respect to (1.1). Let $a = B\bar{x}$. We denote the class of such systems by $\mathscr{S}_{B,a}$. When B and a are implied or when their appearance is not essential, we denote this class simply by $\mathscr{S}$. This is a wider class than that considered in [12], where (1.1) was assumed to satisfy:

(H'_1) The Jacobian matrix $\partial f/\partial x$ can be written as

$$\frac{\partial f}{\partial x} = -\nu I_{n\times n} + A(x), \qquad x \in X \tag{1.5}$$

where ν is some constant.

(H'_2) There is a constant matrix B such that rank $B = r$ and

$$BA(x) = 0, \qquad x \in X, \tag{1.6}$$

and the corresponding system (1.1) called an autonomous system *having an invariant linear subspace*. In this case, the existence of an invariant affine manifold was proved as a consequence of (1.5) and (1.6). Systems in $\mathscr{S}$ enjoy concrete examples arising from many different areas which are not covered by systems considered in [12]; for instance, the hypercycles which describe self-organization in Mathematical Biology (see [8, 11]), and some epidemiological models which allow variable total populations such as models with disease-related death and with vertical transmission (see [1, 3]). A general class of homogeneous systems of degree 1 (Section 3) can also be treated in this context.

In many cases, our prime interest in systems that belong to $\mathscr{S}$ is the behaviour of solutions that stay in the invariant manifold Γ. The main purpose of the present paper is to derive Dulac type conditions for the nonexistence of periodic solutions in Γ. Generalizations of the criteria of Bendixson and Dulac to higher dimensions have been obtained in the context of general autonomous systems by many authors [2, 14, 19, 21]. In our situation, one usually tries to restrict (1.1) to the invariant manifold Γ to get a lower dimensional system and then apply these general criteria. However, the effectiveness of this approach depends critically on the choice of coordinates for Γ. A different technique was developed in [12] which does not require this reduction in dimension and thus is independent of any choice of coordinates for Γ. As a consequence, Bendixson type conditions were derived more coherently and more economically. In the present paper, this technique is further explored in deriving Dulac type conditions for systems more general than those considered in [12]. We also present some applications where the theory in [12] is not applicable.

Autonomous convergence theorems were first obtained in [15, 21], and later in [16], for general autonomous systems in $\mathbf{R}^n$. Certain higher dimensional Bendixson–Dulac criteria were shown to be robust under C^1 perturbations of f and this robustness was used, together with the C^1 closing lemma of Pugh [20], to show that each bounded semitrajectory converges to an equilibrium under any of these higher dimensional Bendixson–Dulac criteria. In developing this type of result in Section 3 for systems which belong to $\mathscr{S}$, we prove in Lemma 3.3 that a local C^1 closing lemma can be applied to systems in $\mathscr{S}$, namely, we are able to show that the perturbation may be chosen from $\mathscr{S}$.

The paper is organized as follows: in Section 2, we derive Dulac criteria for systems in $\mathscr{S}$; in Section 3, we prove an autonomous convergence theorem on Γ, after we show the C^1 robustness of our Dulac criteria in a very specific sense and prove a local C^1 closing lemma for systems in $\mathscr{S}$ by applying Pugh's result; in Section 4, we present an application to autonomous homogeneous systems of degree 1.

2. DULAC CRITERIA

For a solution $x(t, x_0)$ to (1.1) with $x_0 \in \Gamma$, the linear variational equation takes the form

$$y'(t) = \nu\big(x(t, x_0)\big)y(t) + A\big(x(t, x_0)\big)y(t). \tag{2.1}$$

The change of variables $u(t) = y(t)\exp\{-\int_0^t \nu(x(s, x_0))\, ds\}$ leads to

$$u'(t) = A\big(x(t, x_0)\big)u(t) \tag{2.2}$$

satisfying the condition

$$BA\big(x(t, x_0)\big) = 0 \qquad \text{for } t \in \mathbf{R}. \tag{2.3}$$

Therefore the linear subspace $\ker B = \{u \in \mathbf{R}^n\colon Bu = 0\}$ of $\mathbf{R}^n$ is invariant with respect to the linear system (2.2) in the sense that $u(0) \in \ker B$ implies $u(t) \in \ker B$ for $t \in \mathbf{R}$, since $(Bu(t))' \equiv 0$. Linear differential systems with such invariance properties were investigated in [12]. We first recall a result proved there.

Let $\{w_{10}, \ldots, w_{r0}\}$ be an orthonormal basis for the orthogonal complement of $\ker B$ in $\mathbf{R}^n$, and $w_i(t)$ be the solution to (2.2) such that $w_i(0) = w_{i0}$. The following proposition is relation (4) in the proof of Theorem 2.5 in [12].

PROPOSITION 2.1. *Let $u_1(t), u_2(t)$ be two solutions to* (2.2). *Then*

$$\| u_1(t) \wedge u_2(t) \| \leq \| w_1(t) \wedge \cdots \wedge w_r(t) \wedge u_1(t) \wedge u_2(t) \|.$$

Here $\| \cdot \|$ is the euclidean norm and $\wedge$ denotes the exterior product in $\mathbf{R}^n$.

The exterior product $z(t) = w_1(t) \wedge \cdots \wedge w_r(t) \wedge u_1(t) \wedge u_2(t)$ satisfies the linear system

$$z'(t) = A^{[r+2]}\big(x(t, x_0)\big) z(t) \tag{2.4}$$

which is the $(r+2)$nd *compound equation* of (2.2). Here $A^{[r+2]}$ is the $(r+2)$nd *additive compound matrix* of A [17, 19]. It is a $\binom{n}{r+2} \times \binom{n}{r+2}$ matrix, and hence (2.4) is a $\binom{n}{r+2} \times \binom{n}{r+2}$ linear system. For a detailed study on compound matrices and compound equations, we refer the reader to [17, 19].

Let W be the euclidean unit ball in $\mathbf{R}^2$ and let $\overline{W}$ and ∂W be its closure and boundary, respectively. A function $\varphi \in \mathrm{Lip}(\overline{W} \to \Gamma)$ will be described as a *rectifiable* 2-*surface in* Γ; a function $\psi \in \mathrm{Lip}(\partial W \to \Gamma)$ is a *closed rectifiable curve in* Γ and will be called *simple* if it is one-to-one.

A method for deriving Dulac type conditions for general autonomous systems was developed in [14] by studying the evolution under (1.1) of certain functionals which are defined on rectifiable 2-surfaces. If Γ is simply connected, for a given simple closed curve in $\mathrm{Lip}(\partial W \to \Gamma)$, the set

$$\Sigma(\psi, \Gamma) = \left\{ \varphi \in \mathrm{Lip}(\overline{W} \to \Gamma) \colon \varphi(\partial W) = \psi(\partial W) \right\}$$

is not empty (see [14]). Consider a functional $\mathscr{A}$ defined on $\Sigma(\psi, \Gamma)$ by

$$\mathscr{A}\varphi = \int_{\overline{W}} \left| \frac{\partial \varphi}{\partial r_1} \wedge \frac{\partial \varphi}{\partial r_2} \right|,$$

where $|\cdot|$ is any vector norm in $\mathbf{R}^N$, $N = \binom{n}{r+2}$. For instance, when the norm is $|y| = |y^* y|^{1/2}$, where here and throughout the paper asterisk denotes transposition, then $\mathscr{A}\varphi$ is the usual surface area of $\varphi(\overline{W})$. Since Γ is affine, the following result can be proved as Proposition 2.2 in [14].

PROPOSITION 2.2. *Suppose ψ is a simple closed rectifiable curve on Γ. Then there exists a $\delta > 0$ such that*

$$\mathscr{A}\varphi \geq \delta$$

for all $\varphi \in \Sigma(\psi, \Gamma)$.

A subset $K \subset \Gamma$ is *absorbing* in Γ for (1.1) if each compact subset $D \subset \Gamma$ satisfies $x(t, D) \subset K$ for sufficiently large t. We make the following assumption:

(H_3) Γ is simply connected and there is a compact set K in the interior of Γ which is absorbing for (1.1).

Let $x \mapsto P(x)$ be a nonsingular $\binom{n}{r+2} \times \binom{n}{r+2}$ matrix-valued function which is C^1 in Γ. Then $|P^{-1}(x)|$ is uniformly bounded for $x \in K$. Let μ be the *Lozinskiĭ measure* (or the *logarithmic norm*) with respect to $|\cdot|$ defined by (cf. [6, p. 41], or [19])

$$\mu(F) = \lim_{h \to 0^+} \frac{|I + hF| - 1}{h}$$

for any $N \times N$ matrix F, $N = \binom{n}{r+2}$. Set

$$E = P_f P^{-1} + P \frac{\partial f^{[r+2]}}{\partial x} P^{-1} - r\nu I_{N \times N}. \tag{2.5}$$

Here P_f is the matrix obtained by replacing each entry p_{ij} of P by $(\partial p_{ij}^* / \partial x) f$, its directional derivative in the direction of f, and $\partial f^{[r+2]} / \partial x$ is the $(r+2)$nd additive compound matrix of the Jacobian matrix $\partial f / \partial x$. Define the quantity

$$\bar{q}_{r+2} = \limsup_{t \to \infty} \sup_{x_0 \in K} \frac{1}{t} \int_0^t \mu(E(x(s, x_0)))\, ds. \tag{2.6}$$

We note that, since $K \subset \Gamma$, the time average in (2.6) is only calculated along solutions on the affine manifold Γ. Since Γ is invariant, $\bar{q}_{r+2}$ is well defined.

A closed rectifiable curve $\psi \in \mathrm{Lip}(\partial W \to \Gamma)$ is *invariant* with respect to (1.1) if the subset $\psi(\partial W)$ of Γ is invariant. The following result gives a general Dulac criterion in a weak form for autonomous systems in $\mathscr{S}$.

THEOREM 2.3. *Assume that* (H_1), (H_2), *and* (H_3) *are satisfied. If*

$$\bar{q}_{r+2} < 0, \tag{2.7}$$

then no simple closed rectifiable curve in Γ *can be invariant with respect to* (1.1).

Proof. Let $\epsilon_0 = -\frac{1}{2}\bar{q}_{r+2} > 0$. There exists $T > 0$ such that

$$\int_0^t \mu(E(x(s, x_0)))\, ds < -\epsilon_0 t \tag{2.8}$$

for $t > T$, $x_0 \in K$.

Suppose that, on the contrary, there is a simple closed rectifiable curve $\psi \in \mathrm{Lip}(\partial W \to \Gamma)$ that is invariant with respect to (1.1). Let $\varphi \in \Sigma(\psi, \Gamma)$ and $\varphi_t(p) = x(t, \varphi(p))$, where $p = (p_1, p_2) \in W$. Then, from the invariance of ψ, we know $\varphi_t \in \Sigma(\psi, \Gamma)$ for all t, and hence

$$\mathscr{A}\varphi_t \geq \delta \tag{2.9}$$

for some $\delta > 0$ independent of t, by Proposition 2.2.

Now, $y_i(t) = \partial\varphi_t/\partial p_i = (\partial x(t, \varphi_t)/\partial x_0)(\partial\varphi/\partial p_i)$, $i = 1, 2$, are solutions to the linear variational equation (2.1) with respect to $x = \varphi_t(p)$, and hence $u_i(t) = (\partial\varphi_t/\partial p_i)\exp\{-\int_0^t \nu(\varphi_s(p))\, ds\}$ are solutions to the linear system (2.2) with $x(t) = \varphi_t(p)$. Let $w_1(t), \dots, w_r(t)$ be the solution to (2.2) as in Proposition 2.1. Then $u_1(t) \wedge u_2(t) \wedge w_1(t) \wedge \cdots \wedge w_r(t)$ satisfies the $(r+2)$nd compound equation (2.5). The following relation holds

$$y_1(t) \wedge y_2(t) = u_1(t) \wedge u_2(t) \exp\left\{2\int_0^t \nu(\varphi_s(p))\, ds\right\} \tag{2.10}$$

and by Proposition 2.1

$$\begin{aligned}\|u_1(t) \wedge u_2(t)\| &\exp\left\{2\int_0^t \nu(\varphi_s(p))\, ds\right\} \\ &\leq \|u_1(t) \wedge u_2(t) \wedge w_1(t) \wedge \cdots \wedge w_r(t)\| \exp\left\{2\int_0^t \nu(\varphi_s(p))\, ds\right\}.\end{aligned} \tag{2.11}$$

Direct differentiation yields that

$$v(t) = u_1(t) \wedge u_2(t) \wedge w_1(t) \wedge \cdots \wedge w_r(t) \exp\left\{2\int_0^t \nu(\varphi_s(p))\, ds\right\}$$

satisfies the differential system

$$v'(t) = \left(\frac{\partial f^{[r+2]}}{\partial x}(\varphi_t(p)) - r\nu(\varphi_t(p)) I_{N\times N}\right) v(t).$$

Here we use $\partial f^{[r+2]}/\partial x = A^{[r+2]} + I_{n\times n}^{[r+2]}$ from (1.3) and $I_{n\times n}^{[r+2]} = (r+2)I_{N\times N}$, $N = \binom{n}{r+2}$. Therefore $\Omega(t) = P(\varphi_t(p))v(t)$ satisfies the equation $\Omega'(t) = E(\varphi_t(p))\Omega(t)$, where E is given in (2.5). By a property of the Lozinskiĭ measure (see [6, p. 41]) and by (2.8)

$$|\Omega(t)| \leq |\Omega(0)| \exp\int_0^t \mu(E(\varphi_s(p)))\, ds \leq |\Omega(0)| \exp(-\epsilon_0 t), \tag{2.12}$$

for $p \in \overline{W}$ and $t > T$. Therefore $\Omega(t) \to 0$ as $t \to \infty$, uniformly for $p \in \overline{W}$. Since $\varphi_t(\overline{W}) \subset K$ for sufficiently large t, and $P^{-1}(x)$ is uniformly bounded for $x \in K$, this implies that $v(t) \to 0$ as $t \to \infty$, uniformly for $p \in \overline{W}$. Thus by (2.11) and (2.12), $|\partial\varphi_t/\partial p_1 \wedge \partial\varphi_t/\partial p_2| \to 0$ exponentially as $t \to \infty$, and the exponential rate is uniform for all $p \in \overline{W}$. As a consequence, $\mathscr{A}\varphi_t \to 0$ as $t \to \infty$. This contradicts (2.9), completing the proof of Theorem 2.3. ∎

Remarks. (i) Condition (2.7) rules out the existence in Γ of orbits of the following types: (a) periodic orbits, (b) homoclinic orbits, (c) heteroclinic cycles, since each of such orbits gives rise to an invariant simple closed rectifiable curve.

(ii) A sufficient condition for (2.7) is

$$\mu\left(P_f P^{-1} + P\frac{\partial f}{\partial x}^{[r+2]} P^{-1} - r\nu I_{N\times N}\right) < 0 \qquad \text{on } K \tag{2.13}$$

which is a general Dulac type condition. Note that conditions (2.7) and (2.13) provide the flexibility of a choice of N^2 arbitrary functions in addition to the choice of vector norms in deriving suitable conditions.

Setting $A = I$ in (2.13) leads to the condition

$$\mu\left(\frac{\partial f}{\partial x}^{[r+2]} - r\nu I_{N\times N}\right) < 0 \tag{2.14}$$

which is the Bendixson type condition obtained in [12]. Concrete expressions of (2.14) in terms of Lozinsskiĭ measures with respect to some common vector norms can be found in [12].

(iii) Conditions (2.13) and (2.14) take a simple form when $r = n - 2$. In this case, $r + 2 = n$, $N = \binom{n}{r+2} = 1$, and thus $P(x)$ is scalar-valued and $\partial f^{[r+2]}/\partial x = \operatorname{div}(f)$. For example, (2.14) will read

$$\operatorname{div}(f) < (n-2)\nu \qquad \text{on } \Gamma. \tag{2.15}$$

Also note that Γ is a 2-dimensional affine manifold in this case; any simple closed curve in Γ encloses a region in Γ. From the proof of Theorem 2.3 we can see that (2.15) need only hold for all $x \in \Gamma$ except a set of measure zero.

(iv) When $r = 0$, (H_1) and (H_2) pose no restriction on (1.1). In accordance, the condition (2.13) agrees with a Dulac type condition for general autonomous systems obtained in [14].

(v) Assume that (1.1) satisfies (H_1') and (H_2') with $\nu = 0$. It is proved in [12] that Bx gives r independent linear first integrals for (1.1). In the special case that $n = 3$, $r = 1$, it follows from Theorem 2.3 and Remark (iii) that no simple closed rectifiable curve on a level surface S defined by $Bx = c$ can be invariant with respect to (1.1) if $\operatorname{div}(f) < 0$ for almost all $x \in S$. This gives a result of Demidowitsch [7] for the case of a linear first integral.

(iv) For many systems which describe population dynamics, the feasible region X is usually the nonnegative cone $\mathbf{R}^n_+$ and the affine manifold Γ is the portion of a hypersurface bounded in $\mathbf{R}^n_+$, e.g., the simplex $\{x \in \mathbf{R}^n_+ : \sum_{i=1}^n x_i = 1\}$; the assumption (H_3) in such a case is equivalent to the uniform persistence of (1.1) in Γ (see [5]).

3. CONVERGENCE OF TRAJECTORIES AND GLOBAL STABILITY

In this section, we investigate further implications of condition (2.7) for the behaviour of solutions to (1.1) which stay on the invariant manifold Γ.

We begin by studying the robustness of condition (2.7) under certain smooth perturbations of f. Let $|\cdot|$ denote a vector norm on $\mathbf{R}^n$ and the operator norm which it induces for linear mappings from $\mathbf{R}^n$ to $\mathbf{R}^n$. The distance between two functions $f, g \in C^1(X \to \mathbf{R}^n)$ such that $f - g$ has compact support is

$$|f - g|_{C^1} = \sup\left\{ |f(x) - g(x)| + \left| \frac{\partial f}{\partial x}(x) - \frac{\partial g}{\partial x}(x) \right| : x \in X \right\}.$$

A function $g \in C^1(X \to \mathbf{R}^n)$ is called a C^1 *local ϵ-perturbation of f at* $x_0 \in X$ if there exists an open neighbourhood U of x_0 in X such that the support of $(f - g)$ $\operatorname{supp}(f - g) \subset U$ and $|f - g|_{C^1} < \epsilon$.

Given B and $\bar{x}$ in the assumption (H_1), set $a = B\bar{x}$. For $f \in \mathscr{S}_{B,a}$, let $g \in \mathscr{S}_{B,a}$ be a C^1 local perturbation of f. We consider the corresponding differential equation

$$x' = g(x). \tag{3.1}$$

Since $g \in \mathscr{S}_{B,a}$, the affine manifold Γ defined in (1.2) is necessarily invariant with respect to (3.1). We say that a Dulac criterion (2.7) is *robust under C^1 local perturbations of f at x_0* if, for each sufficiently small $\epsilon > 0$ and neighbourhood U of x_0, it is also satisfied by C^1 local ϵ-perturbations g such that $\operatorname{supp}(f - g) \subset U$.

Let $f_\Gamma = f|_\Gamma$. Then f_Γ defines a vector field on the affine manifold Γ from the assumption (H_1). Denote the differential equation $x' = f_\Gamma(x)$, $x \in \Gamma$ by $(1.1)_\Gamma$. A point $x_0 \in \Gamma$ is said to be *nonwandering* for $(1.1)_\Gamma$ if, for each sufficiently small relative neighbourhood V of x_0 in Γ, $x(t, V) \cap V \neq \emptyset$ for some t.

PROPOSITION 3.1. *Assume that* $f \in \mathscr{S}_{B,a}$. *Let* $\bar{q}_{r+2}$ *be defined in* (2.6) *for some vector norm* $|\cdot|$ *and matrix-valued function* P. *Then the Dulac criterion* $\bar{q}_{r+2} < 0$ *is robust under* C^1 *local perturbations in* $\mathscr{S}_{B,a}$ *at any nonequilibrium point* $x_0 \in \Gamma$ *that is nonwandering for* $(1.1)_\Gamma$.

Proof. This is essentially Proposition 3.3 of [16]. Since $\bar{q}_{r+2} < 0$ implies that $(1.1)_\Gamma$ has no periodic solutions, the minimum return time at a nonequilibrium nonwandering point x_0, as defined in (3.8) of [16], is $+\infty$. Also note that only trajectories on the invariant manifold Γ are involved in the calculation of $\bar{q}_{r+2}$ and that Γ is affine, the proposition can be proved as Proposition 3.3 in [16] by restricting both (1.1) and (3.1) to Γ. ∎

Next, we prove an adapted local C^1 closing lemma for systems in $\mathscr{S}_{B,a}$, in which the perturbations g belong to $\mathscr{S}_{B,a}$. The idea is to apply the general local closing lemma of Pugh [20] to the restricted system $(1.1)_\Gamma$ on Γ, and then extend the perturbation to $\mathbf{R}^n_+$ using a bump function. Since the extended perturbation has to be in $\mathscr{S}_{B,a}$, we carry out the construction of the bump function in detail.

LEMMA 3.2. *Let* $f \in \mathscr{S}_{B,a}$. *Suppose that* $x_0 \in \Gamma$ *is nonwandering for* $(1.1)_\Gamma$ *and that* $f(x_0) \neq 0$. *Then, for each* $\epsilon > 0$ *and each neighbourhood* U *of* x_0 *in* X, *there exists* C^1 ϵ-*perturbations* g *of* f *such that*

(1) $g \in \mathscr{S}_{B,a}$ *and* $\operatorname{supp}(f-g) \subset U$,

(2) *the system* (3.1) *has a nonconstant periodic solution whose trajectory lies in* Γ *and passes through* x_0.

Proof. Since Γ is affine, without loss of generality, we may assume that Γ lies on the coordinate plane $(x_1, \dots, x_{n-r})$ and that x_0 is at the origin. More precisely, write $X = X_1 \oplus X_2$, such that $X_1 \cong \mathbf{R}^{n-r}$, $X_2 \cong \mathbf{R}^r$. Then we assume $\Gamma \subset X_1$. If $x = (x_1, \dots, x_n)$, we write $x^{(1)} = (x_1, \dots, x_{n-r})$ and $x^{(2)} = (x_{n-r+1}, \dots, x_n)$ so that $x = (x^{(1)}, x^{(2)})$. In this setting, the matrix B may be chosen as the form $B = (0, I_{r\times r})$.

Now f_Γ is a mapping from Γ to X_1, and thus system $(1.1)_\Gamma$ is a $n-r$ dimensional autonomous system

$$\frac{d}{dt}x^{(1)} = f_\Gamma(x^{(1)}), \qquad x^{(1)} \in \Gamma \subset X_1. \tag{3.2}$$

Let $U_1(0,\delta) \subset \mathbf{R}^{n-r}$ be a euclidean ball centered at the origin and of radius δ. We choose δ small enough so that $U_1(0,\delta) \subset U$ and

$$V = U_1(0,\delta) \times \left\{x^{(2)} \in X_2 : |\rho x^{(2)}| < 1\right\} \subset U \qquad \text{for some } \rho > 1. \tag{3.3}$$

Apply to (3.2) the local C^1 Closing Lemma of Pugh as formulated in [10] (also see [20]). There exists a $h_\Gamma \in C^1(\Gamma \to X_1)$ such that

(a) $|h_\Gamma|_{C^1} < \epsilon/8\rho$ and $\operatorname{supp}(h_\Gamma) \subset U_1(0, \delta)$,

(b) $(d/dt)x^{(1)} = f_\Gamma(x^{(1)}) + h_\Gamma(x^{(1)})$ has a nonconstant periodic solution whose trajectory passes through 0.

Since this periodic trajectory necessarily stays in Γ, the lemma is proved if we can extend h_Γ to a function h defined on X so that $g = f + h$ satisfies Lemma 3.2(1). For $\rho > 1$ in (3.3), define a function $x \mapsto \alpha(x)$ from $\mathbf{R}^n$ to $\mathbf{R}$ by

$$\alpha(x) = \begin{cases} \left(1 - |\rho x^{(2)}|^2\right)^2, & |\rho x^{(2)}| \le 1 \\ 0, & \text{otherwise}, \end{cases}$$

where $x = (x^{(1)}, x^{(2)})$, and a function $x \mapsto h(x)$ from $\mathbf{R}^n$ to $\mathbf{R}^n$ by

$$h(x) = \begin{pmatrix} h_\Gamma(x^{(1)}) \\ 0 \end{pmatrix}.$$

Then $|\alpha| < 1$ and $|h|_{C^1} \le |h_\Gamma|_{C^1} < \epsilon/8\rho$. Moreover, the gradient

$$\nabla\alpha(x) = \begin{cases} -4\rho^2\left(1 - |\rho x^{(2)}|^2\right)(0, x^{(2)})^*, & |\rho x^{(2)}| \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

and $Bh(x) = 0$ for all $x \in X$. Set

$$g = f + \alpha h.$$

Then $g \in C^1(X \to \mathbf{R}^n)$, $g|_\Gamma = f_\Gamma + h_\Gamma$, and $\operatorname{supp}(f - g) \subset V \subset U$, where V is given in (3.3). In particular, $Bg(\Gamma) = 0$. Moreover

$$g - f = \alpha h$$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x} = \alpha \frac{\partial h}{\partial x} + h \nabla\alpha^*.$$

Thus

$$|g - f|_{C^1} \le |\alpha h| + \left|\alpha \frac{\partial h}{\partial x}\right| + |h \nabla\alpha^*|$$

$$\le \frac{\epsilon}{4\rho}(1 + 2\rho) < \epsilon,$$

and $B(\partial g/\partial x - \partial f/\partial x) = 0$ for all $x \in X$. Therefore $g \in \mathscr{S}_{B,a}$ and satisfies (1) and (2). ∎

PROPOSITION 3.3. *Under the assumptions* (H_1), (H_2), *and* (H_3), *if* $\bar{q}_{r+2} < 0$, *then the dimension of the stable manifold of any equilibrium of* $(1.1)_\Gamma$ *is at least* $n - r - 1$; *if an equilibrium of* $(1.1)_\Gamma$ *is not isolated, then its stable manifold has dimension* $n - r - 1$ *and it has a center manifold of dimension* 1 *which contains all nearby equilibria of* $(1.1)_\Gamma$.

Proof. Without loss of generality, we assume the same set up as in the proof of Lemma 3.2 such that $B = (0, I_{r\times r})$, $\Gamma \subset X_1 \cong \mathbf{R}^{n-r}$, and the Jacobian matrix $\partial f/\partial x$ may be written in the block form

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \nu(x) I_{r\times r} & 0 \\ A_1 & A_2 \end{bmatrix}.$$

It is easy to see that $(\partial f/\partial x)(x_0)$ has an eigenvalue $\nu(x_0)$ with r independent eigenvectors in X_1. At an equilibrium x_1 of $(1.1)_\Gamma$, $\bar{q}_{r+2} < 0$ implies

$$\mu\left[P\left(\frac{\partial f}{\partial x}^{[r+2]} - r\nu I_{N\times N}\right)P^{-1}\right] < 0 \tag{3.4}$$

since $f(x_1) = 0$ implies $P_{f(x_1)}(x_1) = 0$.

Let $\lambda_1, \dots, \lambda_{n-r}$ be the eigenvalues of $(\partial f/\partial x)(x_1)$ with respect to the invariant subspace X_1, such that $\operatorname{Re}\lambda_1 \geq \cdots \geq \operatorname{Re}\lambda_{n-r}$. Then

$$\lambda_1 + \lambda_2 + r\nu$$

is an eigenvalue of $(\partial f^{[r+2]}/\partial x)(x_1)$ (see [19]). Therefore the matrix $\partial f^{[r+2]}/\partial x - r\nu I_{N\times N}$ has an eigenvalue $\lambda_1 + \lambda_2$. Thus (3.4) implies (see [6, p. 41])

$$\operatorname{Re}\lambda_1 + \operatorname{Re}\lambda_2 < 0.$$

It then follows that $0 > \operatorname{Re}\lambda_2 \geq \cdots \geq \operatorname{Re}\lambda_{n-r}$; only $\lambda_1(x_1)$ can possibly have nonnegative real part. The stable manifold of x_1 in Γ has dimension at least $n - r - 1$. If x_1 is not isolated, $(\partial f/\partial x)(x_1)|_{X_1}$ has a singular matrix representation, $\lambda_1(x_1) = 0$ so that its stable manifold in Γ has dimension $n - r - 1$ and there is a 1-dimensional center manifold in Γ which contains all nearby equilibria in Γ, by the Center Manifold Theorem (cf. [9, p. 48]). ∎

THEOREM 3.4. *Under the assumptions* (H_1), (H_2), *and* (H_3), *if* $\bar{q}_{r+2} < 0$, *then*

(1) *Every nonwandering point of* $(1.1)_\Gamma$ *is an equilibrium.*

(2) *Every nonempty alpha or omega limit set of* $(1.1)_\Gamma$ *is a single equilibrium.*

Proof. Under the condition $\bar{q}_{r+2} < 0$, $(1.1)_\Gamma$ has no nonconstant periodic point by Theorem 2.3. Let $x_0 \in \Gamma$ be a nonwandering point for $(1.1)_\Gamma$ and $f(x_0) \neq 0$. Then by Lemma 3.2, there exist arbitrarily small local C^1 perturbations $g \in \mathscr{S}_{B,a}$ of f at x_0 such that (3.1) has a nonconstant periodic solution. However, any such g satisfies $\bar{q}_{r+2} < 0$, by Proposition 3.1, and thus cannot have any nonconstant periodic solution. This contradiction establishes (1).

To prove the assertion that each nonempty alpha or omega limit set is a single equilibrium, first observe that since each limit point is nonwandering, it is an equilibrium. Let $x_1 \in \omega(x_0)$, the omega limit set of x_0 for $(1.1)_\Gamma$. If x_1 is an isolated equilibrium for $(1.1)_\Gamma$, then $\{x_1\} = \omega(x_0)$, since $\omega(x_0)$ is connected. If x_1 is not isolated, then Proposition 3.3 implies that there is a 1-dimensional center manifold associated with x_1 in Γ containing all nearby equilibrium for $(1.1)_\Gamma$. Since $\omega(x_0)$ contains a continuum of equilibria, we may choose x_1 and a sufficiently small neighbourhood U of x_1 so that in U any of such local center manifolds consists entirely of equilibria and every trajectory of $(1.1)_\Gamma$ which intersects U is asymptotic to a trajectory in the center manifold. Thus $\lim_{t\to\infty} x(t, x_0) = x_1$ so that $\omega(x_0) = \{x_1\}$ in this case also. The proof that a nonempty alpha limit set is a single equilibrium is the same. ∎

COROLLARY 3.5. *Assume that* (H_1), (H_2), *and* (H_3) *are satisfied. Suppose* (1.1) *has a unique equilibrium* x_0 *in the interior of* Γ. *If* $\bar{q}_{r+2} < 0$ *for some vector norm* $|\cdot|$ *and matrix-valued function* $P(x)$, *then* x_0 *is globally asymptotically stable in the interior of* Γ.

Proof. Clearly $\{x_0\}$ is globally attracting in the interior of Γ since Theorem 3.4 implies that it is the omega limit set of every trajectory starting in the interior of Γ. Moreover, $\{x_0\}$ is stable since otherwise it would be both the alpha limit set and the omega limit set of some homoclinic trajectory $\gamma \subset \Gamma$, which gives rise to a simple closed invariant curve that is also rectifiable (see [15, the proof of Corollary 2.6]). This has been ruled out by $\bar{q}_{r+2} < 0$. ∎

4. HOMOGENEOUS SYSTEMS OF DEGREE 1

In this section, we consider autonomous system (1.1) under the assumption that $f(x)$ is homogeneous of degree 1, namely,

$$f(\lambda x) = \lambda f(x) \qquad \text{for } \lambda > 0,\ x \in \mathbf{R}^n. \tag{4.1}$$

Such systems have been used in modeling of population dynamics and the spread of infectious diseases (e.g., see [3, 18]), where components x_i of x

are restricted to be nonnegative. In the same spirit, we assume that each component f_i of f satisfies $f_i|_{x_i=0} \geq 0$ so that the nonnegative cone $\mathbf{R}^n_+$ is positively invariant with respect to (1.1).

Let $x(t, x_0)$ be a solution to (1.1) with $x_0 \in \mathbf{R}^n_+ \setminus 0$. It is customary in the study of homogeneous differential systems to consider the projected variable

$$\bar{y}(t) = \frac{x(t)}{N(t)}, \qquad \text{where } N(t) = \sum_{i=1}^{n} x_i(t). \tag{4.2}$$

Then $\sum_{i=1}^n \bar{y}_i(t) = 1$ for all t and $N'(t) = \sum_{i=1}^n f_i(x(t))$. We thus have the relation

$$\frac{N'(t)}{N(t)} = \sum_{i=1}^{n} f_i(\bar{y}(t)). \tag{4.3}$$

As a consequence, $\bar{y}(t)$ satisfies the following system of differential equations

$$y'(t) = -\left(\sum_{i=1}^{n} f_i(y) \right) y + f(y). \tag{4.4}$$

System (4.4) is well-defined in $\mathbf{R}^n_+$, though the variable $\bar{y}$ defined in (4.2) stays in the simplex

$$\Delta = \left\{ y \in \mathbf{R}^n_+ : \sum_{i=1}^{n} y_i = 1 \right\}. \tag{4.5}$$

In fact, by adding the equations of (4.4), one may check that Δ is invariant with respect to (4.4). When restricted to the invariant simplex Δ, system (4.4) describes the projected flow of the homogeneous system (1.1) onto Δ. In particular, for each solution $\bar{y}(t)$ of (4.4) with $\bar{y}(0) \in \Delta$, there exists a solution $x(t)$ to (1.1) such that $x(t)$ and $\bar{y}(t)$ satisfy the relation (4.2). In fact, by the uniqueness of solution, such a $x(t)$ may be obtained by setting the initial conditions $x_i(0) = \bar{y}_i(0)$. From these considerations, we want to study the behaviour of solutions of (4.4) on Δ.

Observe that Δ is an affine manifold in $\mathbf{R}^n_+$ defined by

$$(1, \ldots, 1) y = 1.$$

Thus the assumption (H_1) in Section 1 is satisfied with X being the interior of $\mathbf{R}^n_+$, Γ being Δ, $B = (1, \ldots, 1)$, $r = 1$, and $\bar{x} = (1/n, \ldots, 1/n)^*$.

The Jacobian matrix $J(y)$ of (4.4) can be written as

$$J(y) = \nu_1(y) I_{n\times n} + \Phi(y),$$

where

$$\nu_1(y) = -\sum_{i=1}^{n} f_i(y) \tag{4.6}$$

and

$$\Phi(y) = \frac{\partial f}{\partial y} - (y_1, \dots, y_n)^* \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial y_1}, \dots, \sum_{i=1}^{n} \frac{\partial f_i}{\partial y_n} \right). \tag{4.7}$$

Since

$$(1, \dots, 1)\Phi(y) = \left(1 - \sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial y_i}, \dots, \sum_{i=1}^{n} \frac{\partial f_i}{\partial y_n} \right)$$

one may see that (H_2) is also satisfied.

Theorem 2.3 can now be applied to system (4.4) to yield conditions which rule out the existence of solutions $x(t)$ to (1.1) for which $\bar{y}(t)$ is periodic in Δ. Assume that (4.4) is uniformly persistent in Δ and let K_1 be the compact absorbing set in Δ. We can define $\bar{q}_3$ as in (2.6) for $r = 1$, $\nu = \nu_1$, and for some $N \times N$ matrix-valued function $P(x)$ and some vector norm $|\cdot|$ in $\mathbf{R}^N$, $N = n/3$. The next result follows from Theorem 2.3 and the fact that $\bar{y}(t)$ is periodic if $x(t)$ associated with $\bar{y}(t)$ by (4.2) is periodic.

THEOREM 4.1. *If $\bar{q}_3 < 0$, then no simple closed rectifiable curve in Δ can be invariant with respect to* (4.3). *In particular*, (1.1) *has no periodic solutions in* $\mathbf{R}^n_+$.

Remark. The condition $\bar{q}_3 < 0$ also rules out the types of orbits of (1.1) listed in Remark (i) following the proof of Theorem 2.3.

For an application of Theorem 4.1 to a SEIRS epidemiological model with a homogeneous incidence function and a varying total population, we refer the reader to [13]. In the following, we consider a special case. Assume $n = 3$ and f takes the form

$$f(x) = Ax + (\text{diag } x)\phi(x), \tag{4.8}$$

where $A = (a_{ij})_{3\times 3}$ is a nonnegative matrix, namely,

$$a_{ij} \geq 0 \qquad i \neq j \tag{4.9}$$

and $\phi(x)$ is homogeneous of degree 0, that is, $\phi(\lambda x) = \phi(x)$, for $\lambda > 0$ and $x \in \mathbf{R}^3$. System (1.1) with f given in (4.8) has been used in population models (see [3, 4]). In this case, we have the following result.

THEOREM 4.2. *Assume that $n = 3$ and that f satisfies (4.8) and (4.9). Then the conclusions of Theorem 4.1 hold if*

$$\sum_{i=1}^{3} x_i \frac{\partial \phi_i}{\partial x_i} < 0 \qquad \text{when } \sum_{i=1}^{3} x_i = 1. \tag{4.10}$$

Proof. Set $P(y) = (y_1 y_2 y_3)^{-1}$, $n = 3$, $r = 1$, and $\nu = \nu_1$ in (2.5). Then the matrix E can be calculated with respect to system (4.4) as

$$E = P_g P^{-1} + \operatorname{div}(g) - \nu_1,$$

where g denotes the vector field of (4.4). Relation (4.3) leads to

$$\begin{aligned} P_g P^{-1} &= P_f P^{-1} + \left(\sum_{i=1}^{3} \frac{\partial P}{\partial y_i} y_i \right) \nu_1 P^{-1} \\ &= P_f P^{-1} - 3\nu_1 \end{aligned}$$

and

$$\operatorname{div}(g) = \operatorname{div}(f) - \sum_{j=1}^{3} \sum_{i=1}^{3} \frac{\partial f_i}{\partial y_j} y_j + 3\nu_1.$$

Differentiating (4.1) with respect to λ at $\lambda = 1$ leads to

$$\sum_{j=1}^{3} \frac{\partial f_i}{\partial y_j} y_j = f_i, \qquad i = 1, 2, 3.$$

Therefore

$$E = P_f P^{-1} + \operatorname{div}(f) = P^{-1} \operatorname{div}(Pf).$$

For f given in (4.8)

$$\operatorname{div}(Pf) = -\sum_{i \neq j} a_{ij} \frac{y_j}{y_i} + \sum_{i=1}^{3} \frac{\partial \phi_i}{\partial y_i} y_i,$$

and thus for $y \in K \subset S$

$$\mu(E) = P^{-1} \left(-\sum_{i \neq j} a_{ij} \frac{y_j}{y_i} + \sum_{i=1}^{3} \frac{\partial \phi_i}{\partial y_i} y_i \right) \leq -\delta_1 < 0$$

for some $\delta_1 > 0$. Therefore Theorem 4.2 follows from Theorem 2.3. ∎

Remarks. (1) When condition (4.10) is satisfied, a system (1.1) with $f(x) = (\text{diag } x)\phi(x)$ is said to be *self-regulating*. For 3-dimensional self-regulating systems, a result of Butler, Schmid, and Waltman [4] implies that 3-dimensional volumes decay exponentially along solutions of (1.1), whereas Theorem 4.2 implies that if (1.1) is also homogeneous of degree 1, then 2-dimensional surface areas as well as 3-dimensional volumes, when projected onto Δ, decay exponentially. Moreover, the projected dynamics of (1.1) onto Δ is trivial in the sense that every nonwandering point is an equilibrium and bounded solutions either converge to equilibria or approach the boundary.

(2) Homogeneous systems (1.1) with vector fields given in (4.8) was also considered in Busenberg and van den Driessche [2], where conditions which preclude the existence of periodic solutions and closed phase polygons Δ are derived without the uniform persistence assumption. One of the conditions given in [2] is

$$\frac{\partial \phi_i}{\partial x_i} + \frac{\partial \phi_j}{\partial x_j} - \left(\frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i} \right) \leq 0, \qquad i \neq j,\ i, j = 1, 2, 3, \text{ on } \Delta. \quad (4.11)$$

While (4.10) may be easier to verify than (4.11), it is not apparent that one implies the other.

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