SIAM J. MATH. ANAL.
Vol. 27, No. 4, pp. 1070–1083, July 1996

012

A GEOMETRIC APPROACH TO GLOBAL-STABILITY PROBLEMS*

MICHAEL Y. LI† AND JAMES S. MULDOWNEY‡

Abstract. A new criterion for the global stability of equilibria is derived for nonlinear autonomous ordinary differential equations in any finite dimension based on recent developments in higher-dimensional generalizations of the criteria of Bendixson and Dulac for planar systems and on a local version of the C^1 closing lemma of Pugh. The classical result of Lyapunov is obtained as a special case.

Key words. global stability, Bendixson criterion, compound equations

AMS subject classifications. 34D20, 34C35

1. Introduction.

Let the map $x \mapsto f(x)$ from an open subset $D \subset \mathbf{R}^n$ to $\mathbf{R}^n$ be such that each solution $x(t)$ to the differential equation

$$x' = f(x) \tag{1.1}$$

is uniquely determined by its initial value $x(0) = x_0$, and denote this solution by $x(t, x_0)$.

An equilibrium point $\bar{x} \in D$ of (1.1) is said to be *locally stable*—or simply *stable*—if, for each neighbourhood U of $\bar{x}$, there exists a neighbourhood V of $\bar{x}$ such that $x(t, V) \subset U$ for all $t > 0$; it is said to *attract points* in a neighbourhood W if $x(t, x_0) \to \bar{x}$ as $t \to \infty$ for each $x_0 \in W$. It is important to note that this convergence may not be uniform for $x_0 \in W$, and $\bar{x}$ may fail to be stable when it attracts points in a neighbourhood. This can be demonstrated by an example of $\bar{x}$ with a homoclinic orbit. However, if a stable $\bar{x}$ attracts points in a bounded set W, then the attraction is uniform with respect to $x_0 \in W$. In this case, $\bar{x}$ is said to attract W. We say $\bar{x}$ is *asymptotically stable* if it is stable and attracts a neighbourhood. The *basin of attraction* of $\bar{x}$ is the union of all points which it attracts. If $\bar{x}$ is asymptotically stable, its basin of attraction is an open subset of D and contains a neighbourhood of $\bar{x}$. An equilibrium $\bar{x}$ is said to be *globally asymptotically stable*—or simply *globally stable*—with respect to an open set D_1 if it is asymptotically stable and its basin of attraction contains D_1.

The local stability of an equilibrium $\bar{x}$ can be routinely verified by construction of a Lyapunov function in a small neighbourhood of $\bar{x}$ or by linearizing (1.1) at $\bar{x}$ if f is C^1. The primary interest of this paper is in the problem of global stability. Note that if $\bar{x}$ is globally stable with respect to D_1, then $\bar{x}$ is necessarily the only equilibrium in D_1 and there exists compact neighbourhood K of $\bar{x}$ such that each compact subset $F \subset D_1$ satisfies $x(t, F) \subset K$ for sufficiently large t. Such a K is called *absorbing* in D_1 for (1.1). An open set $D \subset \mathbf{R}^n$ is *simply connected* if each closed curve in D can be

* Received by the editors April 22, 1994; accepted for publication (in revised form) October 21, 1994. This research was supported by Natural Sciences and Engineering Research Council of Canada grant NSERC A7197.

† Centre de Recherches Mathématiques, Université de Montréal, Montréal, PQ H3C 3J7, Canada. Current address: Center for Dynamical Systems, Georgia Institute of Technology, Atlanta, GA 30332. The research of this author was partially supported by the Dissertation Fellowship at the University of Alberta and by the Natural Sciences and Engineering Research Council of Canada Postdoctoral Fellowship.

‡ Department of Mathematics, University of Alberta, Edmonton, AL T6G 2G1, Canada.

continuously deformed to a point within D. Without loss of generality, we formulate the problem as follows:

THE GLOBAL-STABILITY PROBLEM. *Assume that*

(H_1) *D is simply connected;*

(H_2) *there is a compact absorbing set $K \subset D$;*

(H_3) *$\bar{x}$ is the only equilibrium of* (1.1) *in D.*

Find conditions under which the global stability of $\bar{x}$ with respect to D is implied by its local stability.

The difficulty associated with this problem is largely due to the lack of practical tools. The method of Lyapunov functions is most commonly used (see [5], [6]); its application is often hindered by the fact that in many cases global Lyapunov functions are difficult to construct and there is practically no general approach to the construction of such functions. The application of the theory of monotone flows [8], [21] is an alternative which has been successfully implemented in recent years.

A new approach to the global-stability problem has emerged from a series of papers on higher-dimensional generalizations of the criteria of Bendixson and Dulac for planar systems and on so-called autonomous convergence theorems. Assume that (1.1) satisfies a condition in D which precludes the existence of periodic solutions and suppose that this condition is robust in the sense that it is also satisfied by ordinary differential equations that are C^1-close to (1.1); then every nonwandering point of (1.1) is an equilibrium, since otherwise, by the C^1 closing lemma of Pugh [19], [20], we can perturb (1.1) near such a nonequilibrium nonwandering point to get a periodic solution. As a special case, every omega limit point of (1.1) is an equilibrium. In the context of our global-stability problem, this implies that $\bar{x}$ attracts points in D. As a consequence its global stability is implied by the local stability.

Higher-dimensional generalizations of Bendixson's criterion have been obtained in papers of R. A. Smith [22] and J. S. Muldowney [17]. This was further developed and the generalized Dulac criteria derived in [11] based on the study of evolution of general surface functionals under (1.1). Smith used the fact that his condition has the required robustness to imply that all bounded trajectories converge to equilibria. Such results are called, after Smith [22], autonomous convergence theorems, and they are further explored in [12] and proved under the generalized Dulac conditions developed in [11]. In the special case that (1.1) has a unique equilibrium $\bar{x}$ in D, it is proved in [12] that each of these generalized Dulac conditions also implies the local stability of $\bar{x}$ and hence its global stability with respect to D. As shown in [12], the traditional method of Lyapunov functions can also be interpreted in this context.

Each of these conditions will be called a *Bendixson criterion* in this paper. Our main purpose is to introduce a new Bendixson criterion for (1.1) which is an extension of the generalized Dulac conditions in [11] and [12]. Roughly speaking, instead of requiring these generalized Dulac conditions to hold pointwise in D as in [11] and [12], we require that they hold after being averaged over time along all the trajectories. Because of this time average along trajectories, it is not always true that our Bendixson criterion is robust under small C^1 perturbations of f. We introduce the notion of robustness of a Bendixson criterion under *local* C^1 perturbations of f in the sense that it is also satisfied by g which is C^1-close to f and differs from f only near a point. We prove that our Bendixson criterion is robust in this weaker sense. Using a local version of Pugh's closing lemma [7], we develop the theory of [12] under weaker conditions.

For autonomous systems which possess the Poincaré–Bendixson property, a different method for proving global stability was recently used in [1] for a planar system associated with a chemostat model and in [13] for a three-dimensional competitive system arising from an epidemiological model. In this case, the key step is to rule out periodic trajectories. This was accomplished by proving that periodic trajectories are orbitally asymptotically stable whenever they exist using the stability criterion of Poincaré for planar systems (as in [1]) and its higher-dimensional generalizations (as in [13]) developed in [17].

The present paper is arranged as follows: in the next section, we establish the general framework; in §3, we introduce our Bendixson criterion in Theorem 3.1 and prove a new global-stability result in Theorem 3.5; in §4, as an example, we consider a global-stability problem arising in an epidemiological model.

2. A general principle for global stability. We begin by formulating the local version of the C^1 closing lemma of Pugh as in [7]. Let $|\cdot|$ denote a vector norm on $\mathbf{R}^n$ and the operator norm which it induces for linear mappings from $\mathbf{R}^n$ to $\mathbf{R}^n$. The distance between two functions f, $g \in C^1(D \to \mathbf{R}^n)$ such that $f-g$ has compact support is

$$|f-g|_{C^1} = \sup\left\{|f(x)-g(x)| + \left|\frac{\partial f}{\partial x}(x) - \frac{\partial g}{\partial x}(x)\right| : x \in D\right\}.$$

Here and throughout the paper, $\frac{\partial f}{\partial x}$ denotes the Jacobian matrix of a mapping f. A function $g \in C^1(D \to \mathbf{R}^n)$ is called a C^1 *local ϵ-perturbation* of f at $x_0 \in D$ if there exists an open neighbourhood U of x_0 in D such that the support $\operatorname{supp}(f-g) \subset U$ and $|f-g|_{C^1} < \epsilon$. For such g, we consider the corresponding differential equation

$$x' = g(x). \tag{2.1}$$

A point $x_0 \in D$ is *wandering* for (1.1) if there exists a neighbourhood U of x_0 and $T > 0$ such that $U \cap x(t, U)$ is empty for all $t > T$. Thus, for example, any equilibrium, alpha limit point, or omega limit point is nonwandering.

LEMMA 2.1. *Let $f \in C^1(D \to \mathbf{R}^n)$. Suppose that x_0 is a nonwandering point for (1.1) and that $f(x_0) \neq 0$. Then, for each neighbourhood U of x_0 and $\epsilon > 0$, there exists a C^1 local ϵ-perturbation g of f at x_0 such that*

(1) $\operatorname{supp}(f-g) \subset U$ *and*

(2) *the system* (2.1) *has a nonconstant periodic solution whose trajectory passes through x_0.*

A *Bendixson criterion* for (1.1) is a condition satisfied by f which precludes the existence of nonconstant periodic solutions to (1.1). A Bendixson criterion is said to be *robust under C^1 local perturbations of f at x_0* if, for each sufficiently small $\epsilon > 0$ and neighbourhood U of x_0, it is also satisfied by each C^1 local ϵ-perturbations g such that $\operatorname{supp}(f-g) \subset U$.

Given in the following are some examples of Bendixson criteria.

(1) When $n = 2$, $D = \mathbf{R}^2$, the classical result of Bendixson states that if

$$\operatorname{div}(f) < 0 \quad \text{in } \mathbf{R}^2, \tag{2.2}$$

then (1.1) has no nonconstant periodic solutions. Bendixson's criterion (2.2) was later generalized to the Dulac criterion

$$\operatorname{div}(\alpha f) < 0, \tag{2.3}$$

where $x \mapsto \alpha(x)$ is some scalar-valued function. Conditions (2.2) and (2.3) can be replaced by $\operatorname{div}(f) > 0$ and $\operatorname{div}(\alpha f) > 0$, respectively, so that it is not the sign but having constant sign throughout D that is important.

(2) Let W be the Euclidean unit ball in $\mathbf{R}^2$ and let $\overline{W}$ and ∂W be its closure and boundary, respectively. If $D \subset \mathbf{R}^n$, a function $\varphi \in \operatorname{Lip}(\overline{W} \to D)$ will be described as a *simply connected rectifiable 2-surface in* D; a function $\psi \in \operatorname{Lip}(\partial W \to D)$ is a *closed rectifiable curve in* D and will be called *simple* if it is one to one. Let $|\cdot|$ denote a vector norm in $\mathbf{R}^N$ as well as the matrix norm which it induces for $N \times N$ matrices. The *Lozinskiĭ measure* $\mu(E)$ of a $N \times N$ matrix E with respect to the norm $|\cdot|$ is defined as

$$\mu(E) = \lim_{h \to 0+} \frac{|I + hE| - 1}{h}$$

(see [3, p. 41]). Lozinskiĭ measures have been used for estimation of eigenvalues of matrices. They also arise in the stability analysis of linear differential systems when certain vector norm of solutions are used as Lyapunov functions. Readers are refered to [3] for their properties and applications. Consider a nonsingular $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function $x \mapsto A(x)$ which is C^1 in D and a vector norm $|\cdot|$ on $\mathbf{R}^{\binom{n}{2}}$. Let μ be the Lozinskiĭ measure with respect to $|\cdot|$. Under assumptions (H_1) and (H_2), it is proved in [11] that if

$$\mu\left(A_f A^{-1} + A \frac{\partial f}{\partial x}^{[2]} A^{-1}\right) \le -\delta < 0 \qquad \text{on} \quad K, \tag{2.4}$$

then no simple closed rectifiable curve in D can be invariant with respect to (1.1). Here $A_f = (DA)(f)$ or, equivalently, A_f is the matrix obtained by replacing each entry a_{ij} in A by its directional derivative in the direction of f, $\frac{\partial a_{ij}}{\partial x}^* f$, and $\frac{\partial f}{\partial x}^{[2]}$ is the second additive compound matrix of $\frac{\partial f}{\partial x}$ (see [15], [17]). For readers unfamiliar with the Lozinskiĭ measure, the condition (2.4) is equivalent to assuming that $V(x, y) = |A(x)y|$ is a Lyapunov function whose derivative with respect to the $n + \binom{n}{2}$-dimensional system

$$\frac{dx}{dt} = f(x), \qquad \frac{dy}{dt} = \frac{\partial f}{\partial x}^{[2]}(x)\, y$$

is negative definite. It rules out not only periodic trajectories but also homoclinic trajectories and heteroclinic loops since each case gives rise to a simple closed rectifiable invariant curve.

Setting $A = I$ in (2.4) leads to the following condition:

$$\mu\left(\frac{\partial f}{\partial x}^{[2]}\right) < 0, \tag{2.5}$$

which was first obtained in [17]. If the norm $|\cdot|$ is such that $|y| = |y^* y|^{\frac{1}{2}}$, then calculation of the Lozinskiĭ measure μ in (2.5) according to [2] or [17] yields

$$\lambda_1 + \lambda_2 < 0, \tag{2.6}$$

where $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ are eigenvalues of $\frac{1}{2}(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}^*)$, a condition which was first established in [22]. Note that when $n = 2$, condition (2.5) is the classical Bendixson criterion. The criterion (2.4) provides the flexibility of a choice of $\binom{n}{2} \times \binom{n}{2}$ arbitrary functions in addition to the choice of vector norms $|\cdot|$ in deriving suitable conditions.

(3) Let $x \mapsto V(x)$ be a scalar-valued function which is C^1 in D. Then the condition

$$\frac{\partial V}{\partial x} f(x) < 0 \qquad \text{if} \qquad f(x) \neq 0 \tag{2.7}$$

is a Bendixson criterion since $V(x)$ is strictly decreasing along each solution of (1.1). Such a function is usually called a *global Lyapunov function* for (1.1).

Suppose that f satisfies a Bendixson criterion which is robust under C^1 local perturbations of f at all nonwandering points of (1.1) which are not equilibria. Then, for each C^1 local ϵ-perturbation g of f at such a nonwandering point, when ϵ is sufficiently small, (2.1) can not have any nonconstant periodic solutions. Therefore, Lemma 2.1 implies that every nonequilibrium point of (1.1) must be wandering. We thus have the following result.

PROPOSITION 2.2. *Suppose a Bendixson criterion for* (1.1) *is robust under* C^1 *local perturbations of* f *at all nonequilibrium nonwandering points to* (1.1). *Then every nonequilibrium point of* (1.1) *is wandering.*

Suppose $D = \mathbf{R}^n$ and all solutions to (1.1) are forwardly bounded. Then for each $x_0 \in \mathbf{R}^n$, $\omega(x_0)$ is nonempty and compact. If we assume that (1.1) has a unique equilibrium $\bar{x}$ in $\mathbf{R}^n$, then the conditions of Proposition 2.2 imply that $\omega(x_0) = \bar{x}$ for all $x_0 \in \mathbf{R}^n$. If $\bar{x}$ is also stable, then it is globally stable with respect to D. This provides a solution to the global-stability problem.

THEOREM 2.3 (global-stability principle). *Assume that*

(1) $D = \mathbf{R}^n$ *and all solutions to* (1.1) *are forwardly bounded;*

(2) $\bar{x} \in \mathbf{R}^n$ *is the unique equilibrium of* (1.1) *in* $\mathbf{R}^n$; *and*

(3) (1.1) *satisfies a Bendixson criterion that is robust under* C^1 *local perturbations of* f *at each nonwandering point* x_1 *for* (1.1) *such that* $f(x_1) \neq 0$.

Then $\bar{x}$ *is globally stable in* $\mathbf{R}^n$ *provided it is stable.*

If $D \subset \mathbf{R}^n$ is an open subset, results like Theorem 2.3 also hold under the assumption (H_2) that D contains a compact absorbing set K. In this case, the trajectory of each solution to (1.1) eventually enters and remains in K; it does not approach the boundary of D. Condition (3) of Theorem 2.3 implies that its omega limit set is the singleton $\{\bar{x}\}$. Therefore, we have the following local version of Theorem 2.3.

THEOREM 2.4. *Suppose that assumptions* (H_2) *and* (H_3) *hold and that* (1.1) *satisfies a Bendixson criterion that is robust under* C^1 *local perturbations of* f *at all nonequilibrium nonwandering points for* (1.1). *Then* $\bar{x}$ *is globally asymptotically stable with respect to* D *provided it is stable.*

In many cases, a Bendixson criterion would imply that the unique equilibrium $\bar{x}$ is locally stable. This is the case for conditions (2.4) and (2.7). The following result, which contains the classical global-stability result of Lyapunov (see [5]), was proved in [12].

THEOREM 2.5. *Under assumptions* (H_1), (H_2), *and* (H_3), $\bar{x}$ *is globally asymptotically stable in* D *provided that either* (2.4) *or* (2.7) *holds.*

Remark. Another generalization of the global stability result of Lyapunov is LaSalle's invariance principle, [10, Chap. 2, Thm. 6.4], in which properties of limit sets are obtained when (2.7) is replaced by the weak inequality $\frac{\partial V}{\partial x} f(x) \leq 0$ in D. For discussions on the relation of LaSalle's result with autonomous convergence theorems, we refer the reader to [12].

3. The quantity $\overline{q}_2$.

In this section, we develop some new Bendixson criteria which are robust under local C^1 perturbations. Assume that (1.1) has a compact absorbing set $K \subset D$. Then every solution $x(t, x_0)$ of (1.1) exists for all $t > 0$. The following quantities are well defined:

$$\bar{q}_2 = \limsup_{t\to\infty} \sup_{x_0\in K} \frac{1}{t}\int_0^t \mu(B(x(s,x_0)))\,ds, \tag{3.1}$$

where

$$B = A_f A^{-1} + A\frac{\partial f^{[2]}}{\partial x} A^{-1} \tag{3.2}$$

and $x \mapsto A(x)$ is a $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function as in (2.4).

Let $\psi \in \mathrm{Lip}(\partial W \to D)$ be a simple closed rectifiable curve in D. Then

$$\Sigma(\psi, D) = \{\varphi \in \mathrm{Lip}(\overline{W} \to D) \ : \ \varphi(\partial W) = \psi(\partial W)\}$$

is nonempty since D is simply connected. Define a functional $\mathcal{S}$ on $\Sigma(\psi, D)$ by

$$\mathcal{S}\varphi = \int_{\overline{W}} \left| A(\varphi)\frac{\partial \varphi}{\partial u_1} \wedge \frac{\partial \varphi}{\partial u_2}\right|. \tag{3.3}$$

From Proposition 2.2 of [11] and the fact that $|A^{-1}(x)|$ is uniformly bounded for x in any compact subset of D, for each compact $F \subset D$, there exists $\delta > 0$ such that

$$\mathcal{S}\varphi \ge \delta \tag{3.4}$$

for all $\varphi \in \Sigma(\psi, D)$ such that $\varphi(\overline{W}) \subset F$. Let $\varphi_t = x(t, \varphi)$. Then $y_i(t) = \frac{\partial \varphi_t}{\partial u_i}$, $i = 1, 2$, are solutions of the linear variational equation of (1.1)

$$y'(t) = \frac{\partial f}{\partial x}(\varphi_t(u))\, y(t) \tag{3.5}$$

and $z(t) = \frac{\partial \varphi_t}{\partial u_1} \wedge \frac{\partial \varphi_t}{\partial u_2}$ a solution of the second compound equation of (3.5) (see [15], [17]),

$$z'(t) = \frac{\partial f^{[2]}}{\partial x}(\varphi_t(u))\, z(t). \tag{3.6}$$

Straightforward differentiation shows that $w(t) = A(\varphi_t)\, \frac{\partial \varphi_t}{\partial u_1} \wedge \frac{\partial \varphi_t}{\partial u_2}$ satisfies the differential equation $w'(t) = B(\varphi_t(u))\, w(t)$ with B given in (3.2). Suppose $\bar{q}_2 < 0$. Let $2\epsilon_0 = -\bar{q}_2 > 0$. Then there exists $T > 0$ such that, for $t > T$ and $x_0 \in K$,

$$\int_0^t \mu(B(x(s,x_0)))ds \le -\epsilon_0\, t.$$

From a property of Lozinskiĭ measure,

$$\begin{aligned}
\mathcal{S}\varphi_t &= \int_{\overline{W}} \left| A(\varphi_t)\frac{\partial \varphi_t}{\partial u_1} \wedge \frac{\partial \varphi_t}{\partial u_2}\right| \\
&\le \int_{\overline{W}} \left| A(\varphi)\frac{\partial \varphi}{\partial u_1} \wedge \frac{\partial \varphi}{\partial u_2}\right| \exp\left(\int_0^t \mu(B(\varphi_s(u))\, ds\right) \\
&\le \mathcal{S}\varphi \exp(-\epsilon_0\, t).
\end{aligned}$$

(see [3, p. 41]). Therefore, $\mathcal{S}\varphi_t \to 0$ as $t \to \infty$. This contradicts (3.4) if ψ is invariant with respect to (1.1) since in this case, $\varphi_t \in \Sigma(\psi, D)$ and $\varphi_t(\overline{W}) \subset K$ for all sufficiently large t. We thus have established the following result.

THEOREM 3.1. *Under assumptions* (H_1) *and* (H_2), *if*

$$\overline{q}_2 < 0, \tag{3.7}$$

then no simple closed rectifiable curve in D can be invariant with respect to (1.1). *In particular,* (3.7) *is a Bendixson criterion for* (1.1).

Remarks. (1) When $A = I$ and $|\cdot|$ is chosen as the euclidean norm, the corresponding $\overline{q}_2$ is related to a quantity q_2 defined by Temam ([23, p. 277]) in the context of evolution equations in an infinite-dimensional Hilbert space. Temam defines q_2 over a compact invariant set, whereas $\overline{q}_2$ is defined over a compact absorbing set. Suppose K is a compact absorbing set in D. Then its omega limit set $F = \omega(K)$ is the maximal compact invariant set in D, which is usually called the *global attractor* of (1.1) in D. Suppose q_2 is defined over F. Then it is proved in [23, Chap. V, Prop. 2.1 and Thm. 3.3] that $q_2 < 0$ implies that the Hausdorff dimension of F is less than two. By a result of Smith [22, Thm. 5], E contains no simple closed piecewise smooth invariant curves. In particular, (1.1) has no nonconstant periodic solutions. This shows that $q_2 < 0$ is a Bendixson criterion. Since the global attractor E is not necessarily preserved under a local C^1 perturbation of (1.1) at a nonwandering point, the criterion $q_2 < 0$ may not be robust under such perturbations.

(2) Condition (3.7) is clearly implied by (2.4).

Let $x_0 \in D$ be a nonwandering point such that $f(x_0) \neq 0$. Then, for each sufficiently small neighbourhood U of x_0, there exists $t_1 > 0$ such that $x(t_1, U) \cap U = \emptyset$, and $x(t, U) \cap U \neq \emptyset$ for some $t > t_1$. The following quantities are then well defined.

$$\tau(U;\ x_0) = \inf\{\, t > 0 \ :\ x(t, U) \cap U \neq \emptyset, \\ \text{and } \exists\ t_1 < t \text{ such that } x(t_1, U) \cap U = \emptyset\,\}$$

and

$$\tau(x_0) = \sup\{\, \tau(U;\ x_0) \ :\ U \text{ is a sufficiently small neighbourhood of } x_0\}; \tag{3.8}$$

when x_0 is an equilibrium, $\tau(x_0)$ is defined to be zero. We call $\tau(x_0)$ the *minimum return time* at the nonwandering point x_0. It follows from the continuous dependence on initial conditions that x_0 is an equilibrium if and only if $\tau(x_0) = 0$. In fact, the following is true.

LEMMA 3.2. *Let x_0 be nonwandering. A solution $x(t, x_0)$ to* (1.1) *is periodic if and only if $\tau(x_0)$ is finite, in which case $\tau(x_0)$ is the minimum period.*

Proof. From the definition, if $\tau(x_0) < \infty$, there exist sequences $x_k \to x_0$, $t_k \to \tau(x_0)$ $(k \to \infty)$ such that $x(t_k, x_{2k}) = x_{2k+1}$, which implies that $x(t, x_0)$ is a periodic solution of period $\tau(x_0)$. Thus the least period T satisfies $0 \leq T \leq \tau(x_0)$. Conversely, if $x(t, x_0)$ is a periodic solution of period T, then $\tau(U, x_0) \leq T$ for all sufficiently small neighbourhoods U of x_0 since $x(T, x_0) = x_0 \in U$. Thus $\tau(x_0) \leq T$. The lemma follows from these two observations. □

PROPOSITION 3.3. *Suppose $\tau(x_0) = +\infty$. Then the condition $\overline{q}_2 < 0$ is robust under C^1 local perturbations of f at x_0.*

Proof. Let $\delta = -\overline{q}_2 > 0$. Since K is absorbing, there exists $T > 1$ such that $x(t, K) \subset K$ if $t > T$ and

$$\int_{t_2}^{t_1} \mu\,(B(x(s, x_1)))\, ds \leq -\frac{\delta(t_1 - t_2)}{2} \tag{3.9}$$

for all $t_1, t_2 \geq 0$ such that $t_1 - t_2 > T$ and all $x_1 \in K$. The assumption $\tau(x_0) = +\infty$ implies that $f(x_0) \neq 0$ and $\tau(U; x_0) > T$ for all sufficiently small neighbourhoods U

of x_0. Let Π be a $n-1$-dimensional transversal to the vector $f(x_0)$ at x_0 and U_1 be a sufficiently small ball in Π centered at x_0. Consider the flow box

$$\Sigma = \{ x(t, U_1) \ : \ -\alpha \le t \le \alpha \}$$

generated by the evolution of the ball $U_1 \subset \Pi$ along the solutions of (1.1) for a small time interval $[-\alpha, \alpha]$ (see Figure 1). Let $\Gamma_+ = x(\alpha, U_1)$ and $\Gamma_- = x(-\alpha, U_1)$. By taking the ball $U_1 \subset \Pi$ and $\alpha > 0$ sufficiently small, we can ensure that all solutions of (1.1) starting in Σ leave Σ and that $\tau(\Sigma; x_0) > T$. As a consequence, each trajectory starting at Γ_+ leaves $\overline{\Sigma}$ and returns to Γ_-, if it ever returns, at a time greater than T.

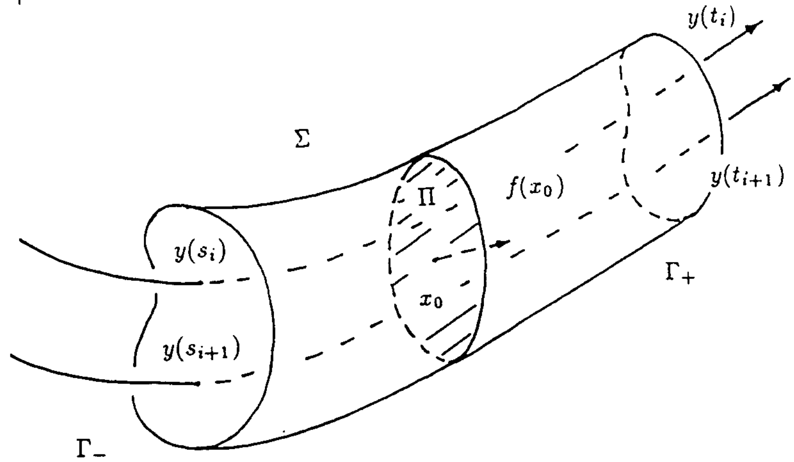

FIG. 1.

Let g be a C^1 local ϵ-perturbation of f at x_0 such that $\operatorname{supp}(f-g) \subset \Sigma$. Consider the differential equation (2.1). K is also absorbing for (2.1) if Σ is sufficiently small since f and g agree on $D \backslash \Sigma$. Let B^f and B^g denote the matrix B defined in (3.2) and $\overline{q}_2(f)$ and $\overline{q}_2(g)$ the quantity $\overline{q}_2$ in (3.1) for f and g, respectively. If the trajectory of a solution $y(t, y_0)$ to (2.1) does not intersect Σ after a certain time, then it coincides with the trajectory of a solution to (1.1) for sufficiently large t. There exists $\bar{t} > 0$ such that no solution of (1.1) and (2.1) remains in $\overline{\Sigma}$ for a time interval greater than $\bar{t}$. For such a solution, it follows from (3.9) that

$$\frac{1}{t} \int_0^t \mu\left(B^g(y(s, y_0))\right) ds \le -\frac{\delta}{4}.$$

Suppose the trajectory of $y(t, y_0)$ intersects Σ infinitely often. We may assume that $y_0 \in \Gamma_+ \cap K$. Let $t_0 = 0$ and

$$T < s_1 < t_1 < s_2 < t_2 < \cdots < s_n < t_n < s_{n+1} < \cdots$$

be a sequence such that

(i) s_i and t_i are the successive time $y(t, y_0)$ intersects Γ_- and Γ_+, respectively, when it returns to Σ,

(ii) $y(t, y_0) \in \Sigma$, $s_i \le t \le t_i$, for each $i \ge 1$,

(iii) $y(t, y_0) \notin \Sigma$, $t_i < t < s_{i+1}$ for each $i \ge 0$.

Then we have

(iv) $t_i - s_i \le \bar{t}$ for each $i \ge 1$,

(v) $s_{i+1} - t_i > T$ for each $i \ge 0$,

(vi) $y(t, y_0)$ coincides with the solution $x(t, y_i)$ of (1.1) for $t_i < t < s_{i+1}$, where $y_i = y(t_i, y_0)$ for each $i \ge 0$ (see Figure 1).

Since $|f-g|_{C^1}<\epsilon$, we may choose ϵ sufficiently small that

$$|\mu(B^f(x)) - \mu(B^g(y))| < \frac{\delta}{4\bar{t}}$$

for x, y in Σ. Therefore, for each $i \geq 0$,

(3.10)

$$\int_{t_i}^{t_{i+1}} \mu(B^g(y(s,y_0)))\,ds = \int_{t_i}^{t_{i+1}} \mu(B^f(x(s,y_i)))\,ds + \int_{t_i}^{t_{i+1}} \left[\mu(B^f(x(s,y_i))) - \mu(B^g(y(s,y_0)))\right] ds$$

$$\leq -\frac{\delta}{2}(t_{i+1}-t_i) + (t_{i+1}-s_{i+1})\frac{\delta}{4\bar{t}}$$

$$\leq -\frac{\delta}{2}(t_{i+1}-t_i) + \frac{\delta}{4} \leq -\frac{\delta}{4}(t_{i+1}-t_i)$$

since $t_{i+1}-t_i \geq T > 1$. Thus for all sufficiently large t, $t_n < t \leq t_{n+1}$ for some n, and

$$\frac{1}{t}\int_0^t \mu(B^g(y(s,y_0)))\,ds = \frac{1}{t}\int_0^{t_n}\mu(B^g) + \frac{1}{t}\int_{t_n}^t \mu(B^g)$$

$$= \frac{1}{t}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\mu(B^g) + \frac{1}{t}\int_{t_n}^t \mu(B^g)$$

$$\leq -\frac{\delta}{4}\frac{1}{t}\sum_{i=0}^{n-1}(t_{i+1}-t_i) + \frac{1}{t}\int_{t_n}^t \mu(B^g)$$

$$\leq -\frac{\delta}{4}\frac{t_n}{t} + \frac{1}{t}\int_{t_n}^t \mu(B^g).$$

If $t-t_n > T$, then, as in (3.10), $\frac{1}{t}\int_{t_n}^t \mu(B^g) < -\frac{\delta}{4}\frac{t-t_n}{t}$. Therefore, in this case,

$$\frac{1}{t}\int_0^t \mu(B^g) \leq -\frac{\delta}{4}.$$

If $t-t_n \leq T$, then $\frac{t-t_n}{t} \leq \frac{T}{t}$ and thus $\frac{t_n}{t} \geq 1-\frac{T}{t} > \frac{1}{2}$ when t is sufficiently large. Hence, in this case,

$$\frac{1}{t}\int_0^t \mu(B^g) < -\frac{\delta}{4}\frac{t_n}{t} + \frac{t-t_n}{t}\max_{x\in K}\mu(B^g(x)) < -\frac{\delta}{16}.$$

Therefore, for sufficiently large t and for $y_0 \in K$,

$$\frac{1}{t}\int_0^t \mu(B^g(y(s,y_0)))\,ds < -\frac{\delta}{16},$$

which leads to $\bar{q}_2(g) < 0$, completing the proof of the lemma. □

By Theorem 2.4, the results established in Theorem 3.1 and Proposition 3.3 imply that the global stability of the unique equilibrium $\bar{x}$ is equivalent to its local stability under condition (3.7). The following result is in the spirit of Proposition 2.4 in [12]; it deals with the asymptotic behaviour of solutions to (1.1) near an equilibrium under condition (3.7) when multiple equilibria are allowed.

PROPOSITION 3.4. *Under assumptions* (H_1) *and* (H_2), *if* $\bar{q}_2 < 0$, *then the dimension of the stable manifold of any equilibrium is at least* $(n-1)$; *if an equilibrium is not isolated, then its stable manifold has dimension* $(n-1)$ *and it has a centre manifold of dimension one which contains all nearby equilibria.*

Proof. Observe that at an equilibrium x_1, $\bar{q}_2 < 0$ implies

$$\mu\left(A\frac{\partial f^{[2]}}{\partial x}A^{-1}\right) < 0$$

since $f(x_1) = 0$ implies $A_{f(x_1)}(x_1) = 0$. This is inequality (8) in [12], and the rest of the proof is the same as the corresponding part in the proof of Proposition 2.4 in [12]. $\square$

THEOREM 3.5. *Under assumptions* (H_1), (H_2), *and* (H_3), *the unique equilibrium* $\bar{x}$ *is globally stable in* D *if* $\bar{q}_2 < 0$.

Proof. From Theorems 2.4 and 3.1 and Proposition 3.3, it remains to prove the local stability of $\bar{x}$. Assume the contrary. Then $\bar{x}$ is both the omega limit point and alpha limit point of a homoclinic trajectory, which gives rise to a simple closed rectifiable curve γ whose existence is precluded by $\bar{q}_2 < 0$ from Theorem 3.1. The proof for the rectifiability of γ is the same as that given in the proof of Corollary 2.6 in [12]. The key to that proof was the local structure of solutions to (1.1) near equilibria established in Proposition 3.4. $\square$

Remark. In the presence of multiple equilibria, it was proved in [12] and [22] that every nonempty alpha and and omega limit set is a single equilibrium under any of the Bendixson criteria (2.4), (2.6), and (2.7). Results of this type are called *autonomous convergence theorems.* The main ingredients in the proof given in [12] are the C^1 robustness of the Bendixson criterion, a result like Proposition 3.4, and the center manifold theorem (see [4]). Therefore, the same result holds under assumptions (H_1) and (H_2) and our weaker condition $\bar{q}_2 < 0$.

4. Example: An epidemiological model.

Let S, E, I, and R denote the susceptible, exposed, infectious, and recovered fractions in a population. A one-population *SEIRS* model for the spread of an infectious disease in the population is described by the following system of differential equations:

$$\begin{aligned} S' &= -\lambda SI + \nu - \nu S + \delta R, \\ E' &= \lambda IS - (\epsilon + \nu)E, \\ I' &= \epsilon E - (\gamma + \nu)I, \\ R' &= \gamma I - (\delta + \nu)R. \end{aligned} \tag{4.1}$$

Individuals are susceptible, then exposed (in the latent period), then infectious, then recovered with temporary immunity, and then susceptible again when the immunity is lost. The parameter δ describes the rate that the recovered population loses immunity, ϵ is the rate that the exposed population becomes infectious, and γ denotes the rate that the infectious population becomes recovered. There is no disease-related death. The natural death rate and birth rate are assumed to be equal (denoted by ν), and thus $S + E + I + R = 1$ for all time. All parameters are nonnegative. The case $\nu = 0$ corresponds to no death and no birth. If $\delta = 0$, infected individuals recover with permanent immunity; that is, once recovered, they will not become susceptible again. Note that ϵ is assumed to be positive since we consider an infectious disease. We also assume that $\nu + \delta > 0$; otherwise, the model (4.1) is not interesting in that all the population will eventually be recovered and there will be no susceptibles.

The model (4.1) has been extensively studied in the literature; see [14] and its references. It is known that the qualitative behaviour of (4.1) is determined by the contact number $\sigma = \lambda\epsilon/(\epsilon+\nu)(\gamma+\nu)$, which satisfies a threshold condition. If $\sigma \leq 1$, the disease-free equilibrium $P_0 = (1,0,0,0)$ is the only equilibrium and is globally asymptotically stable in the feasible region $\Gamma = \{(S,E,I,R) \in \mathbf{R}^4_+ \ : \ S+E+I+R = 1\}$; namely, the disease dies out. If $\sigma > 1$, then P_0 loses its stability and a unique endemic equilibrium P^* emerges in the interior of Γ and is locally asymptotically stable. It has been conjectured (see [14]) that P^* is globally asymptotically stable in the interior of Γ when $\sigma > 1$ such that the disease remains endemic and approaches a unique endemic equilibrium for all initial configurations.

This conjecture was proved to be true for the case $\delta = 0$ in [13]. A crucial part of the proof is that, when $\delta = 0$, (4.1) can be reduced to a three-dimensional competitive system. Since this property of (4.1) may not be preserved for $\delta > 0$, the method in [13] does not apply to the case $\delta > 0$. In this section, we apply the theory developed in previous sections to show that this conjecture is also true for small δ.

Using $R = 1 - S - E - I$, we can reduce (4.1) to the following three-dimensional system:

$$\begin{aligned} S' &= -\lambda SI + \nu - \nu S + \delta(1 - S - E - I), \\ E' &= \lambda SI - (\epsilon + \nu)E, \\ I' &= \epsilon E - (\gamma + \nu)I, \end{aligned} \tag{4.2}$$

and transform the simplex $\Gamma \subset \mathbf{R}^4_+$ to the following convex region in $\mathbf{R}^3_+$:

$$T = \{(S,E,I) \in \mathbf{R}^3_+ \ : \ 0 \leq S+E+I \leq 1\}.$$

The disease-free equilibrium P_0 becomes $(1,0,0)$ and the endemic equilibrium P^* becomes an interior equilibrium in T. For simplicity of notation, we will denote these two equilibria of (4.2) by P_0 and P^*. The following result can be proved in the same way as in §3 of [14].

PROPOSITION 4.1. *If $\sigma \leq 1$, P_0 is globally asymptotically stable in T. If $\sigma > 1$, P_0 is unstable and the trajectories sufficiently close to P_0 leave P_0 except those on the S-axis which approach P_0 along this axis.*

In the following, we apply Theorem 3.5 to (4.2) to show that P^* is globally stable in the interior of T when $\sigma > 1$.

To show the existence of a compact set which is absorbing in the interior of T is equivalent to proving that (4.2) is uniformly persistent, which means that there exists $c > 0$ such that every solution $(S(t),E(t),I(t))$ of (4.2) with $(S(0),E(0),I(0))$ in the interior of T satisfies

$$\liminf_{t\to\infty} |(S(t),\ E(t),\ I(t))| \ \geq\ c$$

(cf. [2]). Uniform persistence may also be defined for discrete dynamical systems or iterated maps (see [9]). It can be proved that (4.2) is uniformly persistent if and only if the time-one map associated with (4.2) is uniformly persisitent in the sense of [9].

A compact invariant set $F \subset T$ of (4.2) is said to be *isolated* if there is a neighbourhood $N \subset T$ of F such that F is the maximal invariant subset of N. The *stable set*

F^s of F is the set of $P \in T$ such that the omega limit set $\omega(P) \subset F$. These concepts can be similarly defined for the time-one map associated with (4.2) (see [9]). Using Theorem 4.1 of [9], we can prove the following result.

PROPOSITION 4.2. *System* (4.2) *is uniformly persistent in* T *when* $\sigma > 1$.

Proof. We show that, when $\sigma > 1$, the time-one map associated with (4.2) satisfies the conditions of Theorem 4.1 of [9], namely, (i) the maximal compact invariant set M in the boundary of T is isolated and (ii) the stable set M^s of M is contained in the boundary of T. It can be shown that the time-one map of (4.2) satisfies (i) and (ii) if (4.2) does. Since $M = \{P_0\}$, Proposition 4.1 implies that, when $\sigma > 1$, M^s is contained in the S-axis and thus in the boundary of T. It also implies that M^s is isolated in T. Therefore, the proposition follows from Theorem 4.1 of [9]. □

THEOREM 4.3. *Assume that* $\sigma > 1$. *Then there exists* $\bar{\delta} > 0$ *such that the unique interior equilibrium* P^* *is globally stable in the interior of* T *when* $\delta \le \bar{\delta}$.

Proof. By Proposition 4.2, when $\sigma > 1$, there exists a compact set K in the interior of T which is absorbing for (4.2). The proof of the theorem consists of choosing a suitable vector norm $|\cdot|$ in $\mathbf{R}^3$ and a 3×3 matrix-valued function $A(x)$ so that the quantity $\bar{q}_2$ defined in (3.1) is negative. We set A as the following diagonal matrix:

$$A(S, E, I) = \operatorname{diag}\left(1, \frac{E}{I}, \frac{E}{I}\right). \tag{4.3}$$

Then A is C^1 and nonsingular in the interior of T. Let f denote the vector field of (4.2). Then

$$A_f A^{-1} = \operatorname{diag}\left(0, \frac{I}{E}\left(\frac{E}{I}\right)_f, \frac{I}{E}\left(\frac{E}{I}\right)_f\right).$$

The second compound matrix $J^{[2]}$ of the Jacobian matrix $J = \frac{\partial f}{\partial x}$ can be calculated as follows:

$$J^{[2]} = \begin{bmatrix} -\lambda I - \delta - \epsilon - 2\nu & \lambda S & \lambda S + \delta \\ \epsilon & -\lambda I - \delta - \gamma - 2\nu & -\delta \\ 0 & \lambda I & -\epsilon - \gamma - 2\nu \end{bmatrix}$$

(see the appendix of [12]). Therefore, the matrix $B = A_f A^{-1} + A J^{[2]} A^{-1}$ can be written in the following block form:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{4.4}$$

with

$$B_{11} = -\lambda I - \delta - \epsilon - 2\nu, \quad B_{12} = \left(\frac{\lambda S I}{E}, \frac{(\lambda S + \delta) I}{E}\right), \quad B_{21} = \left(\frac{\epsilon E}{I}, 0\right)^*,$$

$$B_{22} = \begin{bmatrix} \frac{I}{E}\left(\frac{E}{I}\right)_f - \lambda I - \delta - \gamma - 2\nu & -\delta \\ \lambda I & \frac{I}{E}\left(\frac{E}{I}\right)_f - \epsilon - \gamma - 2\nu \end{bmatrix}.$$

The vector norm $|\cdot|$ in $\mathbf{R}^3 \cong \mathbf{R}^{\binom{3}{2}}$ is chosen as

$$|(u, v, w)| = \sup\{|u|, |v| + |w|\}. \tag{4.5}$$

The Lozinskiĭ measure $\mu(B)$ with respect to $|\cdot|$ can be estimated as follows (see [16] or [18]):

$$\mu(B) \le \sup\{g_1, g_2\}, \tag{4.6}$$

where

$$g_1 = B_{11} + |B_{12}| = -\lambda I - \delta - \epsilon - 2\nu + \frac{(\lambda S + \delta)I}{E}, \tag{4.7}$$

$$g_2 = \mu_1(B_{22}) + |B_{21}| \le \frac{I}{E}\left(\frac{E}{I}\right)_f - \delta - \gamma - 2\nu + \frac{\epsilon E}{I} \tag{4.8}$$

if $\delta \le \epsilon/2$. Note that $\mu_1(B_{22})$ is the Lozinskiĭ measure of the 2×2 matrix B_{22} with respect to the l_1 norm in $\mathbf{R}^2$, $|B_{12}|$ and $|B_{21}|$ are the operator norms of B_{12} and B_{21} when they are regarded as mappings from $\mathbf{R}^2$ to $\mathbf{R}$ and from $\mathbf{R}$ to $\mathbf{R}^2$, respectively, and $\mathbf{R}^2$ is endowed with the l_1 norm. Also note that since B_{11} is a scalar, its Lozinskiĭ measure with respect to any vector norm in $\mathbf{R}^1$ is equal to B_{11}. A solution $(S(t), E(t), I(t))$ to (4.2) with $(S(0), E(0), I(0))$ in the absorbing set K exists for all $t > 0$. From the equations in (4.2), we find

$$\frac{I}{E}\left(\frac{E}{I}\right)_f = \frac{E'}{E} - \frac{I'}{I} \tag{4.9}$$

and

$$\frac{\lambda SI}{E} = \frac{E'}{E} + \epsilon + \nu, \tag{4.10}$$

$$\frac{\epsilon E}{I} = \frac{I'}{I} + \gamma + \nu. \tag{4.11}$$

Relations (4.6)–(4.11) imply

$$\mu(B) \le \frac{E'}{E} - \delta - \nu + \sup\left\{\frac{\delta}{E} - \lambda I,\ 0\right\}.$$

Since (4.2) is uniformly persistent when $\sigma > 1$, there exist $c > 0$ and $T > 0$ such that $t > T$ implies

$$E(t) \ge c, \qquad I(t) \ge c, \qquad \text{and} \qquad \frac{1}{t}\log E(t) < \frac{\delta + \nu}{2}$$

for all $(S(0), E(0), I(0)) \in K$. Set $\bar{\delta} = \min\{\epsilon/2,\ \lambda c^2\}$. Then $t > T$ and $\delta < \bar{\delta}$ imply $\delta/E - \lambda I \le 0$, and thus

$$\frac{1}{t}\int_0^t \mu(B)\,dt < \log E(t) - (\delta + \nu) < -\frac{1}{2}(\delta + \nu)$$

for all $(S(0), E(0), I(0)) \in K$, which in turn implies that $\bar{q}_2 < 0$, completing the proof of Theorem 4.3. □

Acknowledgments. The final draft of this paper was completed when one of the authors (M. Y. Li) was visiting the Centre de Recherches Mathématiques (CRM) at the Université de Montréal as a NSERC postdoctoral fellow. He is grateful for the hospitality of the CRM and the support of the NSERC Postdoctoral Fellowship. The authors would like to thank the referees for their valuable suggestions that helped to improve the original manuscript.

REFERENCES

[1] G. J. BUTLER, S. B. HSU, AND P. WALTMAN, *Coexistence of competing predators in a chemostat*, J. Math. Biol., 17 (1983), pp. 133–151.

[2] G. J. BUTLER AND P. WALTMAN, *Persistence in dynamical systems*, Proc. Amer. Math. Soc., 96 (1986), pp. 425–430.

[3] W. A. COPPEL, *Stability and Asymptotic Behavior of Differential Equations*, Heath, Boston, 1965.

[4] J. GUCKENHEIMER AND P. HOLMES, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Appl. Math. Sci. 42, Springer-Verlag, New York, 1983.

[5] J. K. HALE, *Ordinary Differential Equations*, John Wiley, New York, 1969.

[6] P. HARTMAN, *Ordinary Differential Equations*, John Wiley, New York, 1965.

[7] M. W. HIRSCH, *Systems of differential equations that are competitive or cooperative* VI: *A local C^r closing lemma for 3-dimensional systems*, Ergodic Theory Dynamical Systems, 11 (1991), pp. 443–454.

[8] ———, *Systems of differential equations which are competitive or cooperative* I: *Limit sets*, SIAM J. Math. Anal., 13 (1982), pp. 167–179.

[9] J. HOFBAUER AND J. W.-H. SO, *Uniform persistence and repellors for maps*, Proc. Amer. Math. Soc., 107 (1989), pp. 1137–1142.

[10] J. P. LASALLE, *The Stability of Dynamical Systems*, Regional Conference Series in Applied Mathematics 25, Society for Industrial and Applied Mathematics, Philadephia, 1976.

[11] Y. LI AND J. S. MULDOWNEY, *On Bendixson's criterion*, J. Differential Equations, 106 (1994), pp. 27–39.

[12] Y. M. LI AND J. S. MULDOWNEY, *On R. A. Smith's autonomous convergence theorem*, Rocky Mountain J. Math., 25 (1995), pp. 365–379.

[13] ———, *Global Stability for the SEIR model in epidimiology*, Math. Biosci., 125 (1995), pp. 155–164.

[14] W-M. LIU, H. W. HETHCOTE, AND S. A. LEVIN, *Dynamical behavior of epidemiological models with nonlinear incidence rates*, J. Math. Biol., 25 (1987), pp. 359–380.

[15] D. LONDON, *On derivations arising in differential equations*, Linear and Multilinear Algebra, 4 (1976), pp. 179–189.

[16] R. H. MARTIN, JR., *Logarithmic norms and projections applied to linear differential systems*, J. Math. Anal. Appl., 45 (1974), pp. 432–454.

[17] J. S. MULDOWNEY, *Compound matrices and ordinary differential equations*, Rocky Mountain J. Math., 20 (1990), pp. 857–872.

[18] ———, *Dichotomies and asymptotic behaviour for linear differential systems*, Trans. Amer. Math. Soc., 283 (1984), pp. 465–484.

[19] C. C. PUGH, *An improved closing lemma and the general density theorem*, Amer. J. Math., 89 (1967), pp. 1010–1021.

[20] C. C. PUGH AND C. ROBINSON, *The C^1 closing lemma including Hamiltonians*, Ergodic Theory Dynamical Systems, 3 (1983), pp. 261–313.

[21] H. L. SMITH, *Systems of ordinary differential equations which generate an order preserving flow*, SIAM Rev., 30 (1988), pp. 87–113.

[22] R. A. SMITH, *Some applications of Hausdorff dimension inequalities for ordinary differential equations*, Proc. Roy. Soc. Edinburgh Sect. A, 104 (1986), pp. 235–259.

[23] R. TEMAM, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Appl. Math. Sci. 68, Springer-Verlag, New York, 1988.