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Bifurcation analysis in a neutral differential equation ☆

Ying Qu [a], Michael Yi Li [b], Junjie Wei [a,*]

[a] Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China
[b] Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, T6G 2G1, Canada

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ABSTRACT

The dynamics of a neural network model in neutral form is investigated. We prove that a sequence of Hopf bifurcations occurs at the origin as the delay increases. The direction of the Hopf bifurcations and the stability of the bifurcating periodic solutions are determined by using normal form method and center manifold theory. Global existence of periodic solutions is established using a global Hopf bifurcation result of Krawcewicz et al. and a Bendixson's criterion for higher dimensional ordinary differential equations due to Li and Muldowney.

1. Introduction

We are concerned with the following neutral differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t)+px(t-\tau)\big]=-ax(t)+b\tanh x(t-\tau),\quad t\geqslant 0,\tag{E}$$

where a, b, p, τ are real numbers with τ, $a>0$ and $|p|<1$. With each solution $x(t)$ of Eq. (E) we assume the initial condition:

$$x(t)=\phi(t),\quad t\in[-\tau,0],\ \phi\in C\big([-\tau,0],\mathbb{R}\big).$$

Delay differential equations of various types that contain (E) as a special case have been proposed by many authors for the study of the dynamical characteristics of neural networks of Hopfield type (see [5,8,21–23] and the references cited therein). A majority of results on Eq. (E) deal with stability, oscillatory, non-oscillatory as well as asymptotic behaviors of solutions such as global attractability of zero. An earlier result of El-Morshedy and Gopalsamy [5] proved, under the assumption $0<-pe^{a\tau}<1$ and $ap+b<0$ that solutions of (E) oscillate about the zero if either one of the following conditions is satisfied:

(i) $-(ap+b)\tau e^{a\tau+1}>1+pe^{a\tau}(1-\frac{(ap+b)\tau e^{a\tau}}{1+pe^{a\tau}})$,
(ii) $-(ap+b)\tau e^{a\tau+1}>(1+pe^{a\tau})^2$.

A question of mathematical and biological interest is whether stable and sustained oscillations are possible for Eq. (E). In the present paper, we provide a detailed analysis of this question. By applying the local Hopf bifurcation theory (see [12]

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* Corresponding author.
E-mail address: weijj@hit.edu.cn (J. Wei).

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and also [25,26,31,33]), we investigate the existence of stable periodic oscillations for Eq. (E). More specially, we prove that, the equilibrium $x = 0$ loses its stability as τ increases, and a sequence of Hopf bifurcations occurs at the origin. Whereafter, based on the normal form and center manifold theory due to [3,14], by using the method introduced in Wang and Wei [25], we derive a sufficient condition for determining the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions. Furthermore, the existence of periodic solutions for τ far away from the Hopf bifurcation values is also established, by using a global Hopf bifurcation result due to Krawcewicz, Wu and Xia [16] (also see [34,36]). A key step in establishing the global extension of the local Hopf branch at the first critical value $\tau = \tau_0$ is to verify that Eq. (E) has no non-constant periodic solutions of period 4τ. This is accomplished by applying a higher dimensional Bendixson's criterion for ordinary differential equations given by Li and Muldowney [18].

For the neutral differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t) + px(t-\tau)\big] = f(x_t), \tag{1.1}$$

it can be interpreted as a combination of the difference equation

$$x(t) + px(t-\tau) = 0 \tag{1.2}$$

and a delay differential equation. The most of theory on delay differential equations can be paralleled to this type of equation, including the decomposition of phase space, formal adjoint equation, representation of the solution operator and so on. However, the solution operator $T(t)$ of the linearization of Eq. (1.1) is the sum of an operator involving the solution operator of Eq. (1.2) and a completely continuous operator, which implies $T(t)$ is not compact any more when time exceeding delay. This is also the crucial role which the difference equation (1.2) plays in the dynamical behaviors of Eq. (1.1).

The assumption $|p| < 1$ is to ensure that the zero solution of Eq. (1.2) is asymptotically stable. Therefore, the essential spectrum radius of $T(t)$ can be estimated since it is an α-contraction (the spectrum of $T(t)$ is completely determined by its infinitesimal generator). This allows to restrict Eq. (1.1) on an invariant manifold to an ordinary differential equation that might totally determine the local dynamics of original equation. Such a reduction will simply Eq. (1.1) essentially and has been adopted in many applications especially in the case of investigating Hopf bifurcation problems. However, it is not the case if $|p| > 1$ as it is toughly to measure the essential spectrum radius of $T(t)$. This indicates one cannot tell where the solution near the equilibrium goes in general even if all the spectrum of infinitesimal generator is confirmed. For example, all roots of the characteristic equation with negative real parts will not be leading to the asymptotical stability of the equilibrium. Likewise, the restricted equation on the invariant manifold may not have the decisive effect on the original equation. Based on the above mentioned facts, we assume that $|p| < 1$ in Eq. (E).

When $p = 0$, Eq. (E) becomes a retarded type differential equation, the generalization of which is in the following form:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\mu x(t) + f\big(x(t-\tau)\big). \tag{1.3}$$

Eq. (1.3) has been studied widely. For example, Walther [24], and Liz and Röst [19,20] have investigated the structure of tractor under certain conditions, Wei [28] has studied the bifurcation of Eq. (1.3).

We would like to mention that, as far as we know, there are a few articles on the global existence of periodic solutions for neutral differential equations, we refer to Krawcewicz, Wu and Xia [16,34,36] and Wei and Ruan [31]. Recently, several interesting articles on the stability, bifurcation theory and numerical solutions of neutral differential equations, and the fundamental theory of the neutral type differential equations and inclusions, have been published, we refer to [1,9,10,15, 25–27,2] and [11,13,17], respectively. The present paper is the first to study the global existence of periodic solutions of neutral differential equations by combining the global Hopf bifurcation theory of neutral equations due to Krawcewicz, Wu and Xia and the higher dimensional Bendixson's criterion for ordinary differential equations due to Li and Muldowney. On the study of global existence of periodic solutions of delay differential equations by using the combination of the global Hopf bifurcation theory of delay equations due to Wu [35] and a higher dimensional Bendixson's criterion for ordinary differential equations given by Li and Muldowney [18], we refer to [6,28–30,32].

The rest of this paper is organized as follows: in Section 2, taking b and τ as parameters, we give the analysis of stability and bifurcations at equilibria. Section 3 is devoted to establish the direction and stability of Hopf bifurcation. Finally, a global Hopf bifurcation is established and some numerical simulations are carried out to illustrate the analytic results in Section 4.

2. Stability and bifurcation analysis

In this section, we shall investigate the stability and bifurcations of the equilibria by taking b and τ as bifurcation parameters.

Obviously, $x = 0$ is always an equilibrium of Eq. (E). The linearization of Eq. (E) at $x = 0$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t) + px(t-\tau)\big] = -ax(t) + bx(t-\tau),$$

whose characteristic equation is

$$\Delta(\lambda) = \lambda\big(1 + pe^{-\lambda\tau}\big) + \big(a - be^{-\lambda\tau}\big) = 0. \tag{2.1}$$

Case 1. Choose b as parameter.

For convenience, we give two claims at first.

Claim 2.1. *All the roots of Eq.* (2.1) *with $b = 0$ have negative real parts for any $\tau \geqslant 0$.*

Proof. When $b = 0$ and $\tau = 0$, Eq. (2.1) becomes $\lambda(1 + p) + a = 0$, and the only root is $\lambda = \frac{-a}{1+p} < 0$.
$\mathrm{i}v_0$ ($v_0 > 0$) is a zero of $\Delta(\lambda)$ with $b = 0$ if and only if v_0 solves

$$\mathrm{i}v_0(1 + p\cos v_0\tau - \mathrm{i}p\sin v_0\tau) + a = 0.$$

Separating the real and imaginary parts gives that $1 + p\cos v_0\tau = 0$, which contradicts with $|p| < 1$. Then the conclusion follows Wei and Ruan [31] and the proof is complete. □

Claim 2.2. *Eq.* (2.1) *has at least one positive root when $b > a$.*

In fact, the conclusion follows from $\Delta(0, \tau) = a - b < 0$ and $\lim_{\lambda\to\infty} \Delta(\lambda, \tau) = \infty$.
When $b \neq 0$, $\mathrm{i}v(v > 0)$ is a root of Eq. (2.1) if and only if v satisfies

$$\begin{aligned} &b\sin v\tau = -v(1 + p\cos v\tau),\\ &b\cos v\tau = a + pv\sin v\tau. \end{aligned} \tag{2.2}$$

It follows that

$$pv + a\sin v\tau + v\cos v\tau = 0,$$

and

$$b^2 = v^2\left(1 - p^2\right) + a^2.$$

Denote $G(v) = pv + a\sin v\tau + v\cos v\tau$. It is straightforward that

$$G\left(\frac{(2m-1)\pi}{\tau}\right) = (p-1)\frac{(2m-1)\pi}{\tau} < 0, \qquad G\left(\frac{2m\pi}{\tau}\right) = (p+1)\frac{2m\pi}{\tau} > 0, \quad m \in \mathbb{Z}^+$$

and

$$\begin{aligned} &G'(v) = p + (a\tau + 1)\cos v\tau - v\tau\sin v\tau,\\ &G''(v) = -\left(a\tau^2 + 2\tau\right)\sin v\tau - v\tau^2\cos v\tau. \end{aligned}$$

Then $G''(v) = 0$ if and only if $\tan v\tau = -\frac{v\tau}{a\tau+2}$. Now, we can distinguish two cases. First, $b > 0$. For one thing, the zeros of $G(v)$ should be restricted on the intervals $(\frac{(2m-1)\pi}{\tau}, \frac{2m\pi}{\tau})$, $m \in \mathbb{Z}^+$ by (2.2). For anther, $G''(v)$ has only one zero on $(\frac{(2m-1)\pi}{\tau}, \frac{2m\pi}{\tau})$, say $v_m^* \in (\frac{(2m-\frac{1}{2})\pi}{\tau}, \frac{2m\pi}{\tau})$. This, combined with $G''(\frac{(2m-1)\pi}{\tau}) > 0$ and $G''(\frac{2m\pi}{\tau}) < 0$, gives

$$G''(v)\begin{cases} > 0, & v \in (\frac{(2m-1)\pi}{\tau}, v_m^*),\\ < 0, & v \in (v_m^*, \frac{2m\pi}{\tau}). \end{cases}$$

Then, using the information $G'(\frac{(2m-1)\pi}{\tau}) < 0$ and $G'(\frac{2m\pi}{\tau}) > 0$, one can easily show that there exists a unique value of v, denoted by v_m^+, satisfying (2.2) on the interval $(\frac{(2m-1)\pi}{\tau}, \frac{2m\pi}{\tau})$, $m \in \mathbb{Z}^+$. Second, $b < 0$. With the similar process as above, it is obtained that the only value of v, denoted by v_m^-, solves (2.2) on each interval $(\frac{2(m-1)\pi}{\tau}, \frac{(2m-1)\pi}{\tau})$, $m \in \mathbb{Z}^+$.
Define

$$b_m^+ = \sqrt{v_m^{+2}\left(1 - p^2\right) + a^2} \quad \text{and} \quad b_m^- = -\sqrt{v_m^{-2}\left(1 - p^2\right) + a^2}, \quad m \in \mathbb{Z}^+. \tag{2.3}$$

$b_m^\pm$ make sense since $|p| < 1$, and b_m^+ and b_m^- are all dependent on τ. Hence, $\pm\mathrm{i}v_m^+(\pm\mathrm{i}v_m^-)$ are a pair of purely imaginary roots of Eq. (2.1) with $b = b_m^+(b_m^-)$.
Let $\lambda(b)$ be the root of Eq. (2.1) satisfying $\lambda(b_m^\pm) = \mathrm{i}v_m^\pm$. By substitution $\lambda(b)$ into Eq. (2.1), it follows that

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}b}\right)^{-1} = e^{\lambda\tau} + p - \lambda\tau p + b\tau.$$

This implies that

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}b}\right)^{-1}\Bigg|_{b=b_m^\pm} = \frac{a(b_m^\pm + pa) + b_m^\pm\tau(a^2 + v_m^{\pm2})}{a^2 + v_m^{\pm2}},$$

where $\cos v_m^{\pm}\tau = \frac{ab - p v_m^{\pm 2}}{a^2 + v_m^{\pm 2}}$ is used. Thus, we have the transversal condition

$$\operatorname{Sign}\operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}b}\right)\bigg|_{b=b_m^{\pm}} = \begin{cases} 1, & b = b_m^{+}, \\ -1, & b = b_m^{-}, \end{cases} \quad m \in \mathbb{Z}^{+}. \tag{2.4}$$

Accordingly, a Hopf bifurcation at $x=0$ occurs when $b = b_m^{\pm}$.

Moreover, it is fulfilled that $b_{m+1}^{-} < b_m^{-} < 0 < b_m^{+} < b_{m+1}^{+}$, following the fact that $0 < v_1^{-} < v_1^{+} < v_2^{-} < v_2^{+} < \cdots < v_m^{-} < v_m^{+} < \cdots$, $m \in \mathbb{Z}^{+}$.

Summarizing the discussion above and applying Claims 1 and 2, we have the following.

Lemma 2.1. *Assume that $a > 0$ and $|p| < 1$. Then there exist a sequence values of b defined by* (2.3) *such that all the roots of Eq.* (2.1) *have negative real parts when $b \in (b_1^{-}, a)$, and Eq.* (2.1) *has at least one root with positive real part when $b \in (-\infty, b_1^{-}) \cup (a, +\infty)$, a pair of purely imaginary roots when $b = b_m^{\pm}$ ($m \in \mathbb{Z}^{+}$).*

Lemma 2.2. *Eq.* (E) *undergoes a pitchfork bifurcation at $x=0$ when $b=a$.*

Proof. Let $F(x) = -ax + b\tanh x$. Then we have $F(0) = 0$, $F(\pm\infty) = \mp\infty$ and $F(-x) = -F(x)$. On the other hand, from $F'(x) = -a + \frac{4b}{(e^x + e^{-x})^2}$, we have $F'(0) = b - a > 0$, and $F''(x) = \frac{-8b(e^x - e^{-x})}{(e^x + e^{-x})^3} < 0$. Thus, Eq. (E) has exactly a pair of non-zero equilibria $x = \pm c$ $(c \in \mathbb{R}^{+})$ besides zero as $b > a$, and only one steady-state $x = 0$ when $b \leqslant a$. Accordingly, pitchfork bifurcation at $x=0$ described as Fig. 1 occurs. This completes the proof. $\square$

The conclusion of Lemma 2.2 has been obtained in El-Morshedy and Gopalsamy [5, Theorem 2.3]. From the above lemma, it is known that there are two fixed points $x = \pm c$ $(c > 0)$ of (E) bifurcating from zero when $b > a$. In the following, we will investigate the stability of $x = c$ ($x = -c$ has similar results).

The linearization of Eq. (E) at $x = c$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[x(t) + px(t-\tau)\big] = -ax(t) + bqx(t-\tau),$$

whose characteristic equation is

$$\Delta_c(\lambda,\tau) = \lambda\big(1 + pe^{-\lambda\tau}\big) + \big(a - qbe^{-\lambda\tau}\big) = 0, \tag{2.5}$$

where $q = \frac{4}{(e^c + e^{-c})^2} \in (0,1)$. In particular,

$$\Delta_c(\lambda,0) = \lambda(1+p) + (a - qb) = 0,$$

which implies

$$\lambda = \frac{qb - a}{1+p} = \frac{F'(c)}{1+p} < 0,$$

where $F'(x)$ is given in the proof of Lemma 2.2.

Eq. (2.5) has no zero root. In fact, if zero is a root of Eq. (2.5), then $qb - a = F'(c) = 0$, which is a contradiction.

In order to prove that $x = c$ is asymptotically stable for all $\tau \in \mathbb{R}^{+}$, what we need is to verify that Eq. (2.5) has no purely imaginary roots. Let $\mathrm{i}\eta$ $(\eta > 0)$ be the root of $\Delta_c(\lambda,\tau)$, then η solves

$$a\cos\eta\tau - \eta\sin\eta\tau = bq,$$

$$\eta\cos\eta\tau + a\sin\eta\tau = -\eta p.$$

This leads to

$$\eta^2 = \frac{b^2q^2 - a^2}{1-p^2} = \frac{(bq+a)F'(c)}{1-p^2} < 0.$$

The assertion follows.

Lemma 2.3. *The zero solution of Eq.* (E) *is asymptotically stable when $b = a$.*

Proof. Assume $b = a$. It is known from (2.1) that this characteristic equation has a simple zero root. Particularly, all the other roots except $\lambda = 0$ have negative real parts. In order to investigate the stability of $x = 0$ for Eq. (E), what we need is to employ the center manifold theory and normal form method (see [3,7,12]). Next, we will use the method presented in

Wang and Wei [26] based on the method of computing normal forms for FDEs with parameters introduced by Faria et al. [7], to carry out the study of stability of the zero solution $x=0$.

Following the same algorithms as that in [26], let $\Lambda=\{0\}$ and respectively $B=0$. Clearly, the non-resonance conditions relative to Λ are satisfied. Therefore, there exists a 1-dimensional ODE which governs the dynamics of Eq. (E) near the origin (see [3] or [26]).

Introducing a new parameter $b=a+\mu$, $\mu\in\mathbb{R}$, Eq. (E) is written as

$$\frac{\mathrm{d}}{\mathrm{d}t}[Dx_t]=L_0x_t+(L_\mu-L_0)x_t+F(x_t,\mu), \tag{2.6}$$

where, for any $\phi\in C:=C([-\tau,0],\mathbb{R})$,

$$D\phi=\phi(0)+p\phi(-\tau),\qquad L_\mu(\phi)=-a\phi(0)+(a+\mu)\phi(-\tau)$$

and

$$F(\phi,\mu)=-\frac{1}{3}(a+\mu)\phi^3(-\tau)+\frac{2}{15}(a+\mu)\phi^5(-\tau)+\cdots.$$

Choosing

$$\mu_0(\theta)=\begin{cases}p, & \theta=-\tau,\\ 0, & \theta\in(-\tau,0]\end{cases}\quad\text{and}\quad \eta_0(\mu,\theta)=\begin{cases}-(a+\mu), & \theta=-\tau,\\ 0, & \theta\in(-\tau,0),\\ -a, & \theta=0,\end{cases}$$

thus we obtain

$$D\phi=\phi(0)-\int_{-\tau}^{0}\mathrm{d}\mu_0(\theta)\phi(\theta)\quad\text{and}\quad L_0\phi=\int_{-\tau}^{0}\mathrm{d}\eta_0(\theta)\phi(\theta),$$

where $\eta_0(\theta)=\eta_0(0,\theta)$.

Using the formal adjoint theory for NFDEs (see [12]), we decompose C by Λ as $C=P\oplus Q$, where $P=\mathrm{span}\{\Phi(\theta)\}$ with $\Phi(\theta)=1$ is the center space for $\frac{\mathrm{d}}{\mathrm{d}t}[Dx_t]=L_0x_t$. Choose a basis Ψ for the adjoint space P^*, such that $\langle\Psi,\Phi\rangle=1$, where $\langle\cdot,\cdot\rangle$ is the bilinear form on $C^*\times C$ defined by

$$\langle\psi,\phi\rangle=\psi(0)\phi(0)-\int_{-\tau}^{0}\mathrm{d}\left[\int_{\alpha=0}^{\theta}\psi(\theta-\alpha)\,\mathrm{d}\mu_0(\alpha)\right]\phi(\theta)-\int_{-\tau}^{0}\int_{\theta=0}^{s}\psi(\theta-s)\,\mathrm{d}\eta_0(s)\phi(\theta)\,\mathrm{d}\theta,$$

with $C^*:=C([0,\tau],\mathbb{R}^*)$. Thus $\Psi(s)=\frac{1}{1+p+\tau a}$.

Taking the enlarged phase space,

$$BC=\Big\{\phi\colon [-\tau,0]\to\mathbb{C},\ \phi\text{ is continuous on }[-\tau,0)\text{ and }\lim_{\theta\to 0}\phi(\theta)\text{ exists}\Big\},$$

we obtain the abstract ODE in the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}x_t=Ax_t+X_0G(x_t,\mu), \tag{2.7}$$

where for any $\phi\in C([-\tau,0],\mathbb{R})$, $\mu\in\mathbb{R}$,

$$\begin{aligned}A\phi&=\phi(\theta)'+X_0\big[L_0\phi-D\phi'\big],\\ G(\phi,\mu)&=(L_\mu-L_0)\phi+F(\phi,\mu)\\ &=\mu\phi(-\tau)-\frac{1}{3}(a+\mu)\phi^3(-\tau)+\frac{2}{15}(a+\mu)\phi^5(-\tau)+\cdots,\end{aligned}$$

and $X_0=X_0(\theta)$ is given by

$$X_0(\theta)=\begin{cases}I, & \theta=0,\\ 0, & \theta\in[-\tau,0).\end{cases}$$

Consider the projection

$$\pi:BC\mapsto P,\qquad \pi(\phi+X_0\alpha)=\Phi\big[(\Psi,\phi)+\psi(0)\alpha\big],$$

which leads to the decomposition $BC=P\oplus\mathrm{Ker}\,\pi$. Then, using the decomposition $x_t=\Phi x(t)+y$, $x(t)\in\mathbb{C}$, $y=y(\theta)\in Q^1$, we decompose (2.7) as

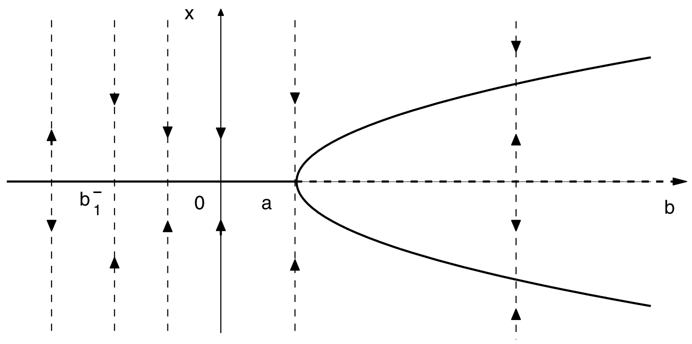

Fig. 1. Pitchfork bifurcation at $x = 0$.

$$\begin{aligned} \dot{x} &= Bx + \Psi(0)G(\Phi x + y, \mu), \\ \dot{y} &= A_{Q^1} y + (I - \pi)X_0 G(\Phi x + y, \mu). \end{aligned} \tag{2.8}$$

Note that

$$\begin{aligned} &\Psi(0)G(\Phi x + y, \mu) \\ &= \frac{1}{1+p+\tau a}\left\{\mu\big(x + y(-\tau)\big) - \frac{1}{3}(a+\mu)\big(x+y(-\tau)\big)^3 + \frac{2}{15}(a+\mu)\big(x+y(-\tau)\big)^5 + \cdots\right\}. \end{aligned} \tag{2.9}$$

Therefore, the locally invariant manifold for Eq. (E) tangent to P at zero satisfies $y(\theta) = 0$ and the flow on this manifold is given by the following 1-dimensional ODE

$$\dot{x} = \frac{1}{1+p+\tau a}\left\{\mu x - \frac{1}{3}(\mu + a)x^3 + \frac{2}{15}(\mu + a)x^5 + \cdots\right\}. \tag{2.10}$$

Clearly, the zero solution of Eq. (2.10) is asymptotically stable when $\mu = 0$, thus the zero solution of Eq. (E) is asymptotically stable when $b = a$. The proof is complete. $\square$

Applying Lemmas 2.1, 2.2 and 2.3, we have the following results.

Theorem 2.4. *Assume that $a > 0$ and $|p| < 1$. Then*

(i) *The zero solution of Eq.* (E) *is asymptotically stable when $b_1^- < b \leqslant a$, and unstable when $b \in (-\infty, b_1^-(\tau)) \cup (a, \infty)$. Moreover, $x = 0$ is globally asymptotically stable when $b^2 < a^2(1-p^2)$ (see* [5, Theorem 3.1]*).*
(ii) *Eq.* (E) *undergoes a pitchfork bifurcation at $x = 0$ when $b = a$. More precisely, a pair of new equilibria with opposite sign bifurcate from zero and they are both asymptotically stable for $\tau > 0$ when $b > a$.*
(iii) *Eq.* (E) *undergoes a Hopf bifurcation at $x = 0$ when $b = b_m^{\pm}(\tau)$, $m \in \mathbb{Z}^+$.*

The conclusions of Theorem 2.4 are illustrated in Fig. 1.

Case 2. Regard τ as parameter.

First of all, we know that the root of Eq. (2.1) with $\tau = 0$ satisfies that $\lambda = \frac{b-a}{1+p} > 0$ when $b > a$, and $\lambda = \frac{b-a}{1+p} < 0$ when $b < a$.

Let $\mathrm{i}\omega(\omega > 0)$ be a root of Eq. (2.1), then it follows that

$$\sin\omega\tau = -\frac{\omega(pa+b)}{a^2+\omega^2} \quad \text{and} \quad \cos\omega\tau = \frac{ab - \omega^2 p}{a^2+\omega^2}. \tag{2.11}$$

This leads to $\omega^2 = \frac{b^2-a^2}{1-p^2}$. $\omega_0 \overset{def}{=} \sqrt{\frac{b^2-a^2}{1-p^2}}$ makes sense when $|b| > a$. Define

$$\tau_j \overset{def}{=} \begin{cases} \frac{1}{\omega_0}\left(\arccos\frac{ab-\omega_0^2 p}{a^2+\omega_0^2} + 2j\pi\right), & b < -a, \\ \frac{1}{\omega_0}\left(-\arccos\frac{ab-\omega_0^2 p}{a^2+\omega_0^2} + 2(j+1)\pi\right), & b > a, \end{cases} \qquad j = 0, 1, 2, \ldots. \tag{2.12}$$

Then $\mathrm{i}\omega_0$ is a purely imaginary root of Eq. (2.1) with $\tau = \tau_j$ defined by (2.12).

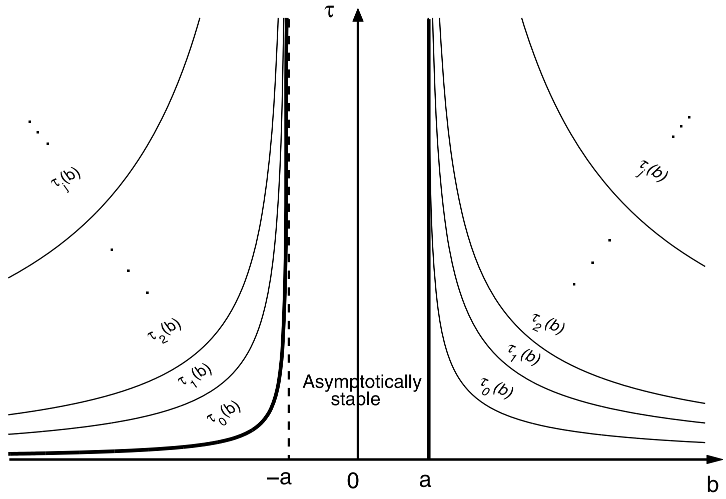

Fig. 2. Bifurcation set on b–τ plane.

Let $\lambda(\tau) = \alpha(\tau) + \mathrm{i}\varrho(\tau)$ be the root of (2.1), satisfying

$$\alpha(\tau_j) = 0 \quad \text{and} \quad \varrho(\tau_j) = \omega_0.$$

Differentiating both sides of (2.1) gives that

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = \frac{e^{\lambda\tau} + p}{\lambda(\lambda p - b)} - \frac{\tau}{\lambda}.$$

Therefore,

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\bigg|_{\tau=\tau_j} = \frac{1-p^2}{\omega_0^2 p^2 + b^2} > 0.$$

This implies that $\alpha'(\tau_j) > 0$, $j = 0, 1, 2, \ldots$.

Summarizing the discussions above, one can obtain the following.

Lemma 2.5.

(i) *If $|b| < a$, then all roots of Eq.* (2.1) *have negative real parts.*
(ii) *If $|b| > a$, then there exist a sequence values of τ defined by* (2.12) *such that Eq.* (2.1) *has a pair of purely imaginary roots $\pm\mathrm{i}\omega_0$ when $\tau = \tau_j$. Additionally, if $b < -a$, then all roots of Eq.* (2.1) *have negative real parts when $\tau \in [0, \tau_0)$, all roots of Eq.* (2.1)*, except $\pm\mathrm{i}\omega_0$, have negative real parts when $\tau = \tau_0$, and Eq.* (2.1) *has at least a pair of roots with positive real parts when $\tau > \tau_0$; if $b > a$, then Eq.* (2.1) *has at least one positive root.*

Spectral properties in Lemma 2.5 immediately lead to the dynamics near the origin described by the following theorem.

Theorem 2.6. *For* (E)*, the following hold.*

(i) *If $b > a$, then for all $\tau > 0$, $x = 0$ is always unstable*;
(ii) *If $|b| < a$, then $x = 0$ is asymptotically stable for all $\tau \in \mathbb{R}^+$*;
(iii) *If $b < -a$, then $x = 0$ is asymptotically stable when $0 < \tau < \tau_0 = \tau_0(b)$, and unstable when $\tau > \tau_0$*;
(iv) *If $|b| > a$, then* (E) *undergoes a Hopf bifurcation at $x = 0$ when $\tau = \tau_j$, $j = 0, 1, 2, \ldots$.*

The conclusions are illustrated in the following bifurcation set on the (b, τ)-plane (see Fig. 2). Here, $\tau_0(b), \tau_1(b), \tau_2(b), \ldots, \tau_j(b), \ldots$ are Hopf bifurcation curves and $b = a$ is pitchfork bifurcation curve. When

$$(b, \tau) \in D := \{(b, \tau) \mid b < -a,\ 0 \leqslant \tau \leqslant \tau_0(b)\} \cup \{(b, \tau) \mid -a \leqslant b \leqslant a,\ \tau \geqslant 0\},$$

$x = 0$ is asymptotically stable, and when $(b, \tau) \in \{(b, \tau) \mid D^c,\ \tau \geqslant 0\}$, the zero solution is unstable, where D^c denotes the complement of D on the set $\{(b, \tau) \in \mathbb{R}^2 \mid \tau \geqslant 0\}$.

Remark 2.1. Denote the curves $\tau = \tau_0(b)$ and $b = a$ by l_1 and l_2, respectively. Clearly, l_1 and l_2 are two parts of the boundary of D. Theorem 2.4(i) shows that the zero solution of Eq. (E) is asymptotically stable when $(b, \tau) \in l_2$. We shall prove that the zero solution of Eq. (E) is also asymptotically stable when $(b, \tau) \in l_1$ in the next section (see Remark 3.1).

Remark 2.2. Proceeding as in Case 2, p or a can also be chosen as bifurcation parameters to obtain the similar results on stability and Hopf bifurcation of (E) at $x = 0$. However, b is the only parameter to make (E) undergoes pitchfork bifurcation apart from Hopf bifurcation.

3. Properties of Hopf bifurcation

Theorem 2.4(iii) and Theorem 2.6(iv) in the previous section give the sufficient conditions for (E) to undergo Hopf bifurcations with b and τ as bifurcation parameters. In this section, we shall investigate the direction of Hopf bifurcation and stability of the bifurcating periodic solutions, following the same algorithms as Wang and Wei's recent work and using a computation process similar to that in [25] (see also [27]). We should mention that we will choose τ as bifurcation parameter and the similar results follow when choosing other coefficients as bifurcation parameters.

Let $y(t) = x(\tau t)$, then (E) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[y(t) + py(t-1)\big] = -a\tau y(t) + b\tau \tanh y(t-1), \tag{E$_0$}$$

whose characteristic equation around $y = 0$ is

$$(a\tau + z)e^z + (zp - b\tau) = 0. \tag{3.1}$$

Comparing (3.1) with (2.1), it is found that $z = \lambda\tau$ for $\tau \neq 0$. Therefore, combining this fact with Lemma 2.5, one has

Lemma 3.1. *Assume $|b| > a$.*

(i) *If $\tau = \tau_j$, $j = 0, 1, 2, \ldots,$ then* (3.1) *has a pair of purely imaginary roots $\pm \mathrm{i}\omega_0\tau_j$, where τ_j and ω_0 are defined by* (2.12);
(ii) *Let $z(\tau) = \gamma(\tau) + \mathrm{i}\zeta(\tau)$ be the root of* (3.1), *satisfying*

$$\gamma(\tau_j) = 0 \quad \text{and} \quad \zeta(\tau_j) = \omega_0\tau_j,$$

then

$$\gamma'(\tau_j) = \tau_j\alpha'(\tau_j) > 0, \quad j = 0, 1, 2, \ldots; \tag{3.2}$$

(iii) (3.1) *has at least a pair of roots with positive real parts when $\tau = \tau_j$ for $j \geqslant 1$, and it has a positive root when $b > a$ and $\tau = \tau_0$. All roots of* (3.1) *with $\tau = \tau_0$, except $\pm\mathrm{i}\omega_0\tau_0$, have negative real parts when $b < -a$.*

For convenience, denote $\tau = \tau_j + \nu$. Then we know that Eq. (E$_0$) undergoes a Hopf bifurcation at the origin when $\nu = 0$. For $\phi \in C([-1, 0], \mathbb{R})$, let

$$D(\phi) = \phi(0) + p\phi(-1), \qquad L(\nu, \phi) = -a(\tau_j + \nu)\phi(0) + b(\tau_j + \nu)\phi(-1)$$

and

$$F(\nu, \phi) = -\frac{1}{3}b(\tau_j + \nu)\phi^3(-1) + \frac{2}{15}b(\tau_j + \nu)\phi^5(-1) + \cdots. \tag{3.3}$$

By the Riesz Representation Theorem, there exist functions $\eta(\theta)$ and $\mu(\theta)$ such that

$$D(\phi) = \phi(0) - \int_{-1}^{0} \mathrm{d}\mu(\theta)\phi(\theta), \qquad L(\nu, \phi) = \int_{-1}^{0} \mathrm{d}\eta(\theta)\phi(\theta).$$

In fact, we can choose

$$\mu(\theta) = \begin{cases} p, & \theta = -1, \\ 0, & \theta \in (-1, 0] \end{cases} \quad \text{and} \quad \eta(\theta) = \begin{cases} -b(\tau_j + \nu), & \theta = -1, \\ 0, & \theta \in (-1, 0), \\ -a(\tau_j + \nu), & \theta = 0. \end{cases}$$

Define

$$A(\nu)\phi = \begin{cases} \mathrm{d}\phi(\theta)/\mathrm{d}\theta, & \theta \in [-1, 0), \\ \phi'(0) - D\phi'(\theta) + L\phi, & \theta = 0 \end{cases}$$

and

$$R(\nu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ F(\nu, \phi), & \theta = 0. \end{cases}$$

Then (E_0) can be written as

$$\dot{y}_t = A(\nu) y_t + R(\nu) y_t. \tag{3.4}$$

Clearly, Eq. (3.4) is an abstract ODE on the phase space BC' (see [26]) of Eq. (E_0), where

$$BC' = \Big\{\phi\colon [-1, 0] \to \mathbb{C},\ \phi \text{ is continuous on } [-1, 0) \text{ and } \lim_{\theta\to 0} \phi(\theta) \text{ exists}\Big\}.$$

The adjoint operator $\tilde{A}^*$ is defined by $\tilde{A}^*\psi = -\frac{\mathrm{d}\psi}{\mathrm{d}s}$ with domain

$$D\big(\tilde{A}^*\big) = \left\{\psi \in C' \overset{def}{=} C\big([0, 1], \mathbb{R}\big)\colon \frac{\mathrm{d}\psi}{\mathrm{d}s} \in C';\ D\frac{\mathrm{d}\psi}{\mathrm{d}s} = -L\psi\right\}.$$

For $(\psi, \varphi) \in C' \times C$, define a bilinear form:

$$(\psi, \phi) = \bar{\psi}(0)\phi(0) - \int_{-1}^{0} \mathrm{d}\left[\int_{\alpha=0}^{\theta} \bar{\psi}(\theta - \alpha)\,\mathrm{d}\mu(\alpha)\right]\phi(\theta) - \int_{-1}^{0}\int_{\theta=0}^{s} \bar{\psi}(\theta - s)\,\mathrm{d}\eta(s)\phi(\theta)\,\mathrm{d}\theta.$$

It is not difficult to verify that $q(\theta) = e^{\mathrm{i}\omega_0\tau_j\theta}$ $(\theta \in [-1, 0])$ and $q^*(s) = \bar{l}e^{\mathrm{i}\omega_0\tau_j s}$ $(s \in [0, 1])$ are the eigenvectors of $A(0)$ and A^* corresponding to the eigenvalues $\mathrm{i}\omega_0\tau_j$ and $-\mathrm{i}\omega_0\tau_j$, respectively, where

$$l = \frac{1}{1 + pe^{-\mathrm{i}\omega_0\tau_j} + b\tau_j e^{-\mathrm{i}\omega_0\tau_j}}$$

and $(q^*, q) = 1$.

Now we compute the center manifold C_0 at $\nu = 0$. Define

$$z(t) = (q^*, y_t), \qquad W(t, \theta) = y_t(\theta) - 2\,\mathrm{Re}\big\{z(t)q(\theta)\big\},$$

then we have

$$\dot{z}(t) = \mathrm{i}\omega_0\tau_j z + \bar{q}^*(0)F(0, y_t). \tag{3.5}$$

Eq. (3.5) can be written in the abbreviated form as

$$\dot{z}(t) = \mathrm{i}\omega_0\tau_j z + g(z, \bar{z}) \tag{3.6}$$

with

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.7}$$

Noting that $y_t(\theta) = W(t, \theta) + z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta)$, we have

$$y_t(-1) = ze^{-\mathrm{i}\omega_0\tau_j} + \bar{z}e^{\mathrm{i}\omega_0\tau_j} + W_{20}(-1)\frac{z^2}{2} + W_{11}(-1)z\bar{z} + W_{02}(-1)\frac{\bar{z}^2}{2} + \cdots.$$

Therefore, from (3.3), we have

$$\begin{aligned} F(0, y_t) &= -\frac{1}{3}b\tau_j\left[ze^{-\mathrm{i}\omega_0\tau_j} + \bar{z}e^{\mathrm{i}\omega_0\tau_j} + W_{20}(-1)\frac{z^2}{2} + W_{11}(-1)z\bar{z} + W_{02}(-1)\frac{\bar{z}^2}{2} + \cdots\right]^3 + \cdots \\ &\overset{def}{=} F_{z^2}\frac{z^2}{2} + F_{z\bar{z}}z\bar{z} + F_{\bar{z}^2}\frac{\bar{z}^2}{2} + F_{z^2\bar{z}}\frac{z^2\bar{z}}{2} + \cdots. \end{aligned}$$

Substituting the expression of $F(0, y_t)$ into (3.5) and comparing its coefficients with that of (3.6) gives that

$$g_{20} = g_{02} = g_{11} = 0,$$

and

$$g_{21} = \bar{q}^*(0) \cdot F_{z^2\bar{z}} = -2b\tau_j l e^{-\mathrm{i}\omega_0\tau_j}.$$

It is well known that the coefficient $c_1(0)$ of third degree term of Poincaré normal form of Eq. (3.5) is given by (see [14])

$$c_1(0) = \frac{\mathrm{i}}{2\omega_0\tau_j}\left(g_{20}g_{11} - 2\|g_{11}\|^2 - \frac{1}{3}\|g_{02}\|^2\right) + \frac{g_{21}}{2}. \tag{3.8}$$

Consequently,

$$c_1(0) = -lb\tau_j e^{-\mathrm{i}\omega_0\tau_j} = -\frac{b\tau_j}{e^{\mathrm{i}\omega_0\tau_j} + p + b\tau_j}. \tag{3.9}$$

From (2.1) we know that

$$e^{\mathrm{i}\omega_0\tau_j} = \frac{b - \mathrm{i}p\omega_0}{a + \mathrm{i}\omega_0} = \frac{ab - p\omega_0^2 - \mathrm{i}\omega_0(b + ap)}{a^2 + \omega_0^2}.$$

Substituting this into (3.9) yields that

$$c_1(0) = -\frac{1}{\Delta}\left[b\tau_j\left(a^2 + \omega_0^2\right)\left(b\tau_j\left(a^2 + \omega_0^2\right) + ab + a^2p\right) + \mathrm{i}\omega_0 b\tau_j\left(a^2 + \omega_0^2\right)(b + ap)\right]$$

and hence

$$\operatorname{Re} c_1(0) = -\frac{1}{\Delta}b\tau_j\left(a^2 + \omega_0^2\right)\left[b\tau_j\left(a^2 + \omega_0^2\right) + ab + a^2p\right], \tag{3.10}$$

where

$$\Delta = \left(b\tau_j\left(a^2 + \omega_0^2\right) + ab + a^2p\right)^2 + \omega_0^2(b + ap)^2.$$

Notice that $|b| > a$ and $|p| < 1$, it follows that $\operatorname{Sign}(ab + a^2p) = \operatorname{Sign} b$. Hence, from (3.10), we obtain

$$\operatorname{Re} c_1(0) < 0.$$

Therefore, from $\gamma'(\tau_j) > 0$ as well as

$$\mu_2 = -\frac{\operatorname{Re} c_1(0)}{\gamma'(\tau_j)} \quad \text{and} \quad \beta_2 = 2\operatorname{Re} c_1(0)$$

we have, respectively,

$$\mu_2 > 0 \quad \text{and} \quad \beta_2 < 0.$$

Summarizing the above analysis, we have the theorem as follows.

Theorem 3.2. *Assume $|b| > a$. Then the direction of the Hopf bifurcation at the origin when $\tau = \tau_j$ $(j = 0, 1, \ldots)$ is forward, that is the bifurcating periodic solutions exist for $\tau > \tau_j$ $(j = 0, 1, \ldots)$ and close to τ_j. And the bifurcating periodic solutions are asymptotically stable at the first bifurcation value τ_0 when $b < -a$.*

Remark 3.1. The zero solution of Eq. (E_0) (or Eq. (E)) is asymptotically stable when $(b, \tau) \in l_1$, where l_1 is defined in Remark 2.1.

In fact, we have known that the normal form of the restriction of Eq. (E) with $\tau = \tau_0$ on the center manifold is given by

$$\dot{z}(t) = \mathrm{i}\omega_0\tau_j z + c_1(0)z^2\bar{z} + \cdots \tag{3.11}$$

and $\operatorname{Re} c_1(0) < 0$. It is not difficult to obtain that the zero solution of Eq. (3.11) is asymptotically stable via Liapunov's second method, and hence the zero solution of Eq. (E) does.

4. Global Hopf bifurcation analysis

Our objective in this section is to obtain the global continuation of periodic solutions bifurcating from the point $(0, \tau_j)$, $j = 0, 1, 2, \ldots$ for Eq. (E) by using a global Hopf bifurcation theorem given by Krawcewicz et al. [16]. For the reader's convenience, we copy Eq. (E_0), which is equivalent to (E), as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[y(t) + py(t-1)\right] = -a\tau\, y(t) + b\tau \tanh y(t-1). \tag{E_0}$$

We have known that Eq. (E_0) undergoes a local Hopf bifurcation at the origin when $\tau = \tau_j$ $(j = 0, 1, \ldots)$ and the bifurcation is supercritical. Now we begin to show that each bifurcation branch can be continued with respect to the parameter τ

under certain conditions. To bring out the ideas in the results of subsequent part, it is convenient to introduce the following notations:

$$
\begin{aligned}
&X = C\big([-1,0],\mathbb{R}\big),\\
&\Sigma = \mathrm{Cl}\big\{(y,\tau,T)\colon\ y \text{ is a T-periodic solution of } (\mathrm{E}_0)\big\} \subset X\times\mathbb{R}_+\times\mathbb{R}_+,\\
&N = \big\{(\hat{y},\tau,T)\colon\ a\hat{y} = b\tanh\hat{y}\big\}.
\end{aligned}
$$

Denote $C(0,\tau_j,2\pi/(\tau_j\omega_0))$ the connected component of $(0,\tau_j,2\pi/(\tau_j\omega_0))$ in Σ, where $\omega_0=\sqrt{\frac{b^2-a^2}{1-p^2}}$ and τ_j $(j=0,1,2,\ldots)$ are defined by (2.12).

Lemma 4.1. *If $p\in(-\frac{1}{2},0)$, then all periodic solutions of (E_0) are uniformly bounded. Precisely, if $x(t)$ is a periodic solution of (E_0), then $x(t)\in[-\frac{|b|}{a(1+2p)},\frac{|b|}{a(1+2p)}]$.*

Proof. Let $x(t)$ be a periodic solution of (E_0). Then there exist t_1 and t_2 such that

$$
\begin{aligned}
&x(t_1)+px(t_1-1)=\max_{t\in\mathbb{R}}\big[x(t)+px(t-1)\big],\\
&x(t_2)+px(t_2-1)=\min_{t\in\mathbb{R}}\big[x(t)+px(t-1)\big],
\end{aligned}
\tag{4.1}
$$

together with T^0 and T_0 such that

$$
x\big(T^0\big)=\max_{t\in\mathbb{R}^+}x(t)\quad\text{and}\quad x(T_0)=\min_{t\in\mathbb{R}^+}x(t).
$$

Employing the way used by Krawcewicz, Wu and Xia [16, p. 211] or Wei and Ruan [31], one can obtain that for $p\in(-1,0)$,

$$
\frac{x(t_2)+px(t_2-1)}{1+p}\leqslant x(t)\leqslant\frac{x(t_1)+px(t_1-1)}{1+p}.
\tag{4.2}
$$

By (4.1), we have from (E_0) that

$$
x(t_i)=\frac{b}{a}\tanh x(t_i-1)\in\left[-\frac{|b|}{a},\frac{|b|}{a}\right],\quad i=1,2,
\tag{4.3}
$$

and

$$
x(t_1)+px(t_1-1)\geqslant x\big(T^0\big)+px\big(T^0-1\big).
\tag{4.4}
$$

Hence, (4.2), (4.3) and (4.4) yield that

$$
\begin{aligned}
x\big(T^0\big)+px\big(T^0-1\big)&\leqslant x(t_1)+px(t_1-1)\\
&\leqslant\frac{|b|}{a}+\frac{p}{1+p}\big[x(t_2)+px(t_2-1)\big]\\
&\leqslant\frac{|b|}{a}+\frac{p}{1+p}\left[-\frac{|b|}{a}+px(t_2-1)\right].
\end{aligned}
$$

Noting the meaning of T^0, we arrive at

$$
(1+p)x\big(T^0\big)\leqslant\frac{|b|}{a(1+p)}+\frac{p^2}{1+p}x\big(T^0\big).
$$

Hence,

$$
x\big(T^0\big)\leqslant\frac{|b|}{a(1+2p)}.
$$

Similarly, one can prove that

$$
x(T_0)\geqslant-\frac{|b|}{a(1+2p)}.
$$

Thus the proof is complete. □

Lemma 4.2. *If $b \in (-\sqrt{2}a, -a)$, then there exists a $p_0 \in (-\frac{1}{2}, 0)$ such that Eq.* (E_0) *has no periodic non-constant solution of period* 4 *when $p \in (p_0, 0]$.*

Proof. Let $y(t)$ be a periodic solution to Eq. (E_0) of period 4. Set

$$u_j(t) = y\big(t-(j-1)\big), \quad j = 1,2,3,4.$$

Then $u(t) = (u_1(t), u_2(t), u_3(t), u_4(t))$ is a periodic solution to the following system of ordinary differential equations:

$$\begin{aligned}
\dot{u}_1(t) + p\dot{u}_2(t) &= -a\tau u_1(t) + b\tau \tanh u_2(t) \overset{def}{=} f_1,\\
\dot{u}_2(t) + p\dot{u}_3(t) &= -a\tau u_2(t) + b\tau \tanh u_3(t) \overset{def}{=} f_2,\\
\dot{u}_3(t) + p\dot{u}_4(t) &= -a\tau u_3(t) + b\tau \tanh u_4(t) \overset{def}{=} f_3,\\
\dot{u}_4(t) + p\dot{u}_1(t) &= -a\tau u_4(t) + b\tau \tanh u_1(t) \overset{def}{=} f_4,
\end{aligned} \tag{4.5}$$

which is equivalent to

$$\begin{aligned}
\dot{u}_1(t) &= \frac{1}{1-p^4}\big(f_1 - pf_2 + p^2 f_3 - p^3 f_4\big),\\
\dot{u}_2(t) &= \frac{1}{1-p^4}\big(f_2 - pf_3 + p^2 f_4 - p^3 f_1\big),\\
\dot{u}_3(t) &= \frac{1}{1-p^4}\big(f_3 - pf_4 + p^2 f_1 - p^3 f_2\big),\\
\dot{u}_4(t) &= \frac{1}{1-p^4}\big(f_4 - pf_1 + p^2 f_2 - p^3 f_3\big),
\end{aligned} \tag{4.6}$$

where $\cdot$ denotes $\frac{\mathrm{d}}{\mathrm{d}t}$. Denote

$$G = \left\{u \in \mathbb{R}^4\colon\ u_j \in \left[-\frac{|b|}{a(1+2p)}, \frac{|b|}{a(1+2p)}\right],\ j = 1,2,3,4\right\}.$$

Lemma 4.1 shows that the periodic solutions of Eq. (E_0) belong to the region G. To rule out the 4-periodic solution to Eq. (E_0), it suffices to prove the nonexistence of non-constant periodic solutions of (4.6) in the region G. To do the latter, we use a general Bendixson's criterion in higher dimensions developed by Li and Muldowney [18]. More specifically, we shall apply Corollary 3.5 in [18]. The Jacobian matrix $J = J(u)$ of (4.6), for $u \in \mathbb{R}^4$, is

$$J(u) = \frac{\tau}{1-p^4}\begin{pmatrix}
-a - bp^3F(u_1) & ap + bF(u_2) & -ap^2 - bpF(u_3) & ap^3 + bp^2F(u_4)\\
ap^3 + bp^2F(u_1) & -a - bp^3F(u_2) & ap + bF(u_3) & -ap^2 - bpF(u_4)\\
-ap^2 - bpF(u_1) & ap^3 + bp^2F(u_2) & -a - bp^3F(u_3) & ap + bF(u_4)\\
ap + bF(u_1) & -ap^2 - bpF(u_2) & ap^3 + bp^2F(u_3) & -a - bp^3F(u_4)
\end{pmatrix},$$

where $F(\upsilon) = \frac{4}{(e^{\upsilon}+e^{-\upsilon})^2} \in (0,1]$.

The second additive compound matrix $J^{[2]}(u)$ of $J(u)$ is

$$J^{[2]}(u) = \frac{\tau}{1-p^4}\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix},$$

where

$$A_{11} = \begin{pmatrix}
-2(a + bp^3(F_1+F_2)) & ap + bF_3 & -ap^2 - bpF_4\\
ap^3 + bp^2F_2 & -2(a + bp^3(F_1+F_3)) & ap + bF_4\\
-ap^2 - bpF_2 & ap^3 + bp^2F_3 & -2(a + bp^3(F_1+F_4))
\end{pmatrix},$$

$$A_{12} = \begin{pmatrix}
ap^2 + bpF_3 & -ap^3 - bp^2F_4 & 0\\
ap + bF_2 & 0 & -ap^3n - bp^2F_4\\
0 & ap + bF_2 & -ap^2 - bpF_3
\end{pmatrix},$$

$$A_{21} = \begin{pmatrix}
ap^2 + bpF_1 & ap^3n + bp^2F_1 & 0\\
-ap - bF_1 & 0 & ap^3 + bp^2F_1\\
0 & -ap - bF_1 & -ap^2 - bpF_1
\end{pmatrix},$$

$$A_{21} = \begin{pmatrix}
-2(a + bp^3(F_2+F_3)) & ap + bF_4 & ap^2 + bpF_4\\
ap^3 + bp^2F_3 & -2(a + bp^3(F_2+F_4)) & ap + bF_3\\
ap^2 + bpF_2 & ap^3 + bp^2F_2 & -2(a + bp^3(F_3+F_4))
\end{pmatrix}.$$

Here $F_i = F(u_i)$, $i = 1,2,3,4$.

Choose a vector norm in $\mathbb{R}^6$

$$\left|(x_1,x_2,x_3,x_4,x_5,x_6)\right| = \max\left\{\sqrt{2}|x_1|, |x_2|, \sqrt{2}|x_3|, \sqrt{2}|x_4|, |x_5|, \sqrt{2}|x_6|\right\}.$$

Then the Lozinskii measure $\mu(J^{[2]}(u))$ of $J^{[2]}(u)$ with respect to this norm is (see [4])

$$\mu\left(J^{[2]}(u)\right) = \frac{\tau}{1-p^4}\max E,$$

where E is a set with 6 elements as follows:

$$E = \left\{\begin{array}{l} -2[a+bp^3(F_1+F_2)] + (\sqrt{2}-p)|ap+bF_3| - p(1-\sqrt{2}p)|ap+bF_4|, \\ -2[a+bp^3(F_1+F_3)] + \frac{\sqrt{2}}{2}(1+p^2)(|ap+bF_2|+|ap+bF_4|), \\ -2[a+bp^3(F_1+F_4)] + (\sqrt{2}-p)|ap+bF_2| - p(1-\sqrt{2}p)|ap+bF_3|, \\ -2[a+bp^3(F_2+F_3)] + (\sqrt{2}-p)|ap+bF_4| - p(1-\sqrt{2}p)|ap+bF_1|, \\ -2[a+bp^3(F_2+F_4)] + \frac{\sqrt{2}}{2}(1+p^2)(|ap+bF_1|+|ap+bF_3n|), \\ -2[a+bp^3(F_3+F_4)] + (\sqrt{2}-p)|ap+bF_1| - p(1-\sqrt{2}p)|ap+bF_2|\} \end{array}\right\}.$$

By Corollary 3.5 in [18], system (4.6) has no periodic orbits in the region G if $\mu(J^{[2]}(u)) < 0$ for all $u \in G$.

In fact, when $b < -a$, using the fact that $F_i = F(u_i) \in (0,1]$, one can acquire the inequality as below,

$$\mu\left(J^{[2]}(u)\right) < \frac{\tau}{1-p^4}\left\{-2a + (ap+b)\left[2p - \sqrt{2}\left(1+p^2\right)\right]\right\}.$$

Denote

$$H(p) = -2a + (ap+b)\left[2p - \sqrt{2}\left(1+p^2\right)\right].$$

It is easy to see that

$$H(0) = -2a - \sqrt{2}b < 0, \qquad H\left(-\frac{1}{2}\right) = \left(-\frac{3}{2} + \frac{5}{8}\sqrt{2}\right)a - \left(1 + \frac{5}{4}\sqrt{2}\right)b > 0$$

and $H'(p) < 0$ for $p \in (-1,0]$. Thus, there exists unique zero p_0 of $H(p)$ on $(-\frac{1}{2},0)$ such that $\mu(J^{[2]}(u)) < 0$ when $p \in (p_0,0]$, which completes the proof. □

Lemma 4.3. *The following hold*:

(i) *Eq.* (E_0) *has no periodic non-constant solution of period* 1;
(ii) *If either* $b < -a$ *and* $p \in (-1,0]$ *or* $b > a$ *and* $p \in (-a/b,0]$ *is satisfied, then Eq.* (E_0) *has no periodic non-constant solution of period* 2 (*see also* [16, Lemma 6.3]).

Proof. The assumption that Eq. (E_0) has no non-constant periodic solution of period 1 is equivalent to the fact that Eq. (E) with $\tau = 0$ has no non-constant periodic solution. It is well known that the first order autonomous ODE has no non-constant periodic solutions. Eq. (E) with $\tau = 0$ is the first order autonomous ODE, and the proof of (i) is complete.

As in the proof of Lemma 4.2, let $u(t)$ be a periodic solution of (E_0) of period 2, then $u_1(t) = u(t)$ and $u_2(t) = u(t-1)$ are periodic solutions of the system of ordinary differential equations

$$\dot{u}_1(t) + p\dot{u}_2(t) = -a\tau u_1(t) + b\tau\tanh u_2(t),$$
$$\dot{u}_2(t) + p\dot{u}_1(t) = -a\tau u_2(t) + b\tau\tanh u_1(t),$$

which is equivalent to

$$\begin{aligned} \dot{u}_1(t) &= \frac{\tau}{1-p^2}\left[-a\left(u_1(t) - pu_2(t)\right) + b\left(\tanh u_2(t) - p\tanh u_1(t)\right)\right] \stackrel{def}{=} P(u_1,u_2), \\ \dot{u}_2(t) &= \frac{\tau}{1-p^2}\left[-a\left(u_2(t) - pu_1(t)\right) + b\left(\tanh u_1(t) - p\tanh u_2(t)\right)\right] \stackrel{def}{=} Q(u_1,u_2). \end{aligned} \tag{4.7}$$

Then

$$\frac{\partial P}{\partial u_1} + \frac{\partial Q}{\partial u_2} = -\frac{2\tau}{1-p^2}\left[a + 2bp\left(\frac{1}{(e^{u_1}+e^{-u_1})^2} + \frac{1}{(e^{u_2}+e^{-u_2})^2}\right)\right] < 0$$

when either $b < -a$ and $p \in (-1,0]$ or $b > a$ and $p \in (-a/b,0]$ holds. Thus the classical Bendixson's negative criterion implies that (4.7) has no non-constant periodic solution and thus the proof of lemma is complete. □

Lemma 4.4. *If $b > a$ and $p \in (-1, 0]$, then $\tau_0\omega_0 \in (\frac{3\pi}{2}, 2\pi)$. If $b < -a$ and $p \in (\frac{a}{b}, 0]$, then $\tau_0\omega_0 \in (\frac{\pi}{2}, \pi)$. Here τ_0 and ω_0 are defined as in* (2.12)*.*

Proof. From (2.11), we have that $\sin\omega_0\tau_0 = -\frac{\omega_0(pa+b)}{a^2+\omega_0^2}$ and $\cos\omega_0\tau_0 = \frac{ab-\omega_0^2 p}{a^2+\omega_0^2}$. $b > a$ and $p \in (-1, 0]$ imply that $\tau_0\omega_0 \in (\frac{3\pi}{2}, 2\pi)$.

Now we verify the second conclusion. Using (2.11) again, we have that if $b < -a$, then $\sin\omega_0\tau_0 > 0$ and thus $\tau_0\omega_0 \in (0, \pi)$. From $\cos\omega_0\tau_0 = \frac{ab-\omega_0^2 p}{a^2+\omega_0^2}$ and $\omega_0^2 = \frac{b^2-a^2}{1-p^2}$, it follows that

$$ab - \omega_0^2 p = \frac{1}{1-p^2}\big[ab - abp^2 - \big(b^2 - a^2\big)p\big].$$

Denote

$$g(p) \stackrel{def}{=} ab - abp^2 - \big(b^2 - a^2\big)p.$$

One can obtain that $g(p) < 0$ when $p \in (\frac{a}{b}, 0]$. This implies that $\cos\omega_0\tau_0 < 0$ and completes the proof. $\square$

Up to now, we have prepared sufficiently to state the following global Hopf bifurcation results.

Theorem 4.5. *Let τ_j $(j = 0, 1, 2, \ldots)$ be defined by* (2.12)*.*

(i) *If $|b| > a$ and $p \in (-\frac{1}{2}, 0]$, then for each $\tau > \tau_j$ $(j \geqslant 1)$, Eq.* (E_0) *has at least one periodic solution.*
(ii) *If $b \in (-\sqrt{2}a, -a)$, then there exists a $p_0 \in (-\frac{1}{2}, 0)$ such that, for $p \in (p_0, 0]$, Eq.* (E_0) *has at least one periodic solutions for $\tau > \tau_j$ $(j \geqslant 0)$ and at least two periodic solutions for $\tau > \tau_j$ $(j \geqslant 1)$.*
(iii) *If $b > a$ and $p \in (-\frac{a}{b}, 0] \cap (-\frac{1}{2}, 0]$, then Eq.* ($\mathrm{E}_0$) *has at least one periodic solution for $\tau > \tau_j$ $(j \geqslant 0)$ and at least two periodic solutions for $\tau > \tau_j$ $(j \geqslant 1)$.*

Proof. First note that for any τ_j, the stationary points $(0, \tau_j, \frac{2\pi}{\tau_j\omega_0})$ of (E_0) are nonsingular and isolated centers (see Krawcewicz, Wu and Xia [16]) under the assumption $|b| > a$, then the hypothesis (H_2) in [16] is satisfied. By (3.2), there exist $\epsilon > 0$, $\delta > 0$ and a smooth curve $\lambda : (\tau_j - \delta, \tau_j + \delta) \to \mathbb{C}$, such that

$$\Delta_0\big(\lambda(\tau)\big) = \Delta_{0(0,\tau,T)}\big(\lambda(\tau)\big) = 0, \qquad \big|\lambda(\tau) - \mathrm{i}\tau_j\omega_0\big| < \epsilon,$$

for all $\tau \in [\tau_j - \delta, \tau_j + \delta]$, where Δ_0 is defined as (3.1), and

$$\lambda(\tau_j) = \mathrm{i}\tau_j\omega_0, \qquad \left.\frac{\mathrm{d}\,\mathrm{Re}(\lambda(\tau))}{\mathrm{d}\tau}\right|_{\tau=\tau_j} > 0.$$

Denote $p_j = \frac{2\pi}{\tau_j\omega_0}$ and let

$$\Omega_\epsilon = \big\{(0, q)\colon\ 0 < u < \epsilon,\ |q - p_j| < \epsilon\big\}.$$

Clearly, if $|\tau - \tau_j| \leqslant \delta$ and $(u, q) \in \Omega_\epsilon$ such that $\Delta_{0(0,\tau,T)}(u + \frac{2\pi\mathrm{i}}{q}) = 0$, then $\tau = \tau_j$, $u = 0$ and $q = p_j$. Moreover, putting

$$H^{\pm}\left(0, \tau_j, \frac{2\pi}{\tau_j\omega_0}\right)(u, q) = \Delta_{0(0,\tau_j\pm\delta,T)}\left(u + \mathrm{i}\frac{2\pi}{q}\right),$$

we have the crossing number

$$\gamma_1\left(0, \tau_j, \frac{2\pi}{\tau_j\omega_0}\right) = \deg_B\left(H^-\left(0, \tau_j, \frac{2\pi}{\tau_j\omega_0}\right),\ \Omega_\epsilon\right) - \deg_B\left(H^+\left(0, \tau_j, \frac{2\pi}{\tau_j\omega_0}\right),\ \Omega_\epsilon\right) = -1.$$

By the local Hopf bifurcation theorem for NFDE [16, Theorem 5.6], we conclude that the connected component $C(0, \tau_j, \frac{2\pi}{\tau_j\omega_0})$ through $(0, \tau_j, \frac{2\pi}{\tau_j\omega_0})$ in Σ is nonempty. Meanwhile, we have

$$\sum_{(\hat{y},\tau,T)\in C(0,\tau_j,\frac{2\pi}{\tau_j\omega_0})} \gamma_1(\hat{y}, \tau, T) < 0$$

and thus $C(0, \tau_j, \frac{2\pi}{\tau_j\omega_0})$ is unbounded by the global Hopf bifurcation theorem given by Krawcewicz, Wu and Xia [16, Theorem 5.14].

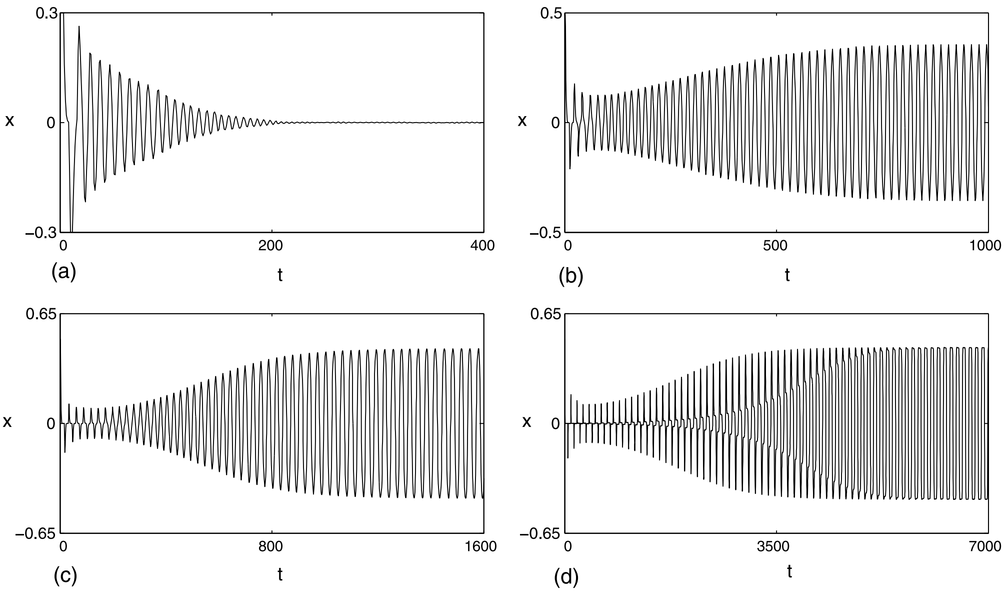

Fig. 3. Matlab simulations of (E) with $a = 3$, $b = -3.2$, $p = -0.1$, where (a) with $\tau = 2.4 < \tau_0 = 2.4568$, (b) with $\tau = 5 \in (\tau_0, \tau_1)$, (c) with $\tau = 10 \in (\tau_1, \tau_2)$ and (d) with sufficiently large $\tau = 50$.

Lemma 4.1 implies that the projection of $C(0, \tau_j, \frac{2\pi}{\tau_j\omega_0})$ onto the y-space is bounded. Meanwhile, Eq. (E) with $\tau = 0$ has no periodic non-constant solutions since it is the first order autonomous ordinary differential equation. Therefore, the projection of $C(0, \tau_j, \frac{2\pi}{\tau_j\omega_0})$ onto the τ-space is bounded below.

By the definition of τ_j given in (2.12), we know that

$$2\pi < \tau_j\omega_0 < 2(j+1)\pi, \quad j \geqslant 1,$$

under the assumptions that $|b| > a$ and $p \in (-1, 0]$, which implies

$$\frac{1}{j+1} < \frac{2\pi}{\tau_j\omega_0} < 1.$$

Applying Lemma 4.3(i), one has that $\frac{1}{j+1} < T < 1$ if $(x, \tau, T) \in C(0, \tau_j, 2\pi/(\tau_j\omega_0))$ for $j \geqslant 1$, when $p \in (-\frac{1}{2}, 0]$ and $|b| > a$. This and Lemma 4.1 show that in order for $C(0, \tau_j, 2\pi/(\tau_j\omega_0))$ to be unbounded, its projection onto the τ-space must be unbounded. Consequently, the projection of $C(0, \tau_j, 2\pi/(\tau_j\omega_0))$ onto the τ-space include $[\tau_j, \infty)$ for $j \geqslant 1$ if $p \in (-\frac{1}{2}, 0]$ and $|b| > a$. The conclusion of (i) follows.

For (ii), $b \in (-\sqrt{2}a, -a)$ implies that $b > -\sqrt{2}a > -2a$ and thus $\frac{a}{b} < -\frac{1}{2}$. This leads to $(-\frac{1}{2}, 0] \subseteq (\frac{a}{b}, 0]$ when $b \in (-\sqrt{2}a, -a)$. Consequently, we know by Lemma 4.4 that when $b \in (-\sqrt{2}a, -a)$ and $p \in (-\frac{1}{2}, 0]$,

$$\frac{\pi}{2} < \tau_0\omega_0 < \pi,$$

which implies

$$2 < \frac{2\pi}{\tau_0\omega_0} < 4.$$

Applying Lemmas 4.2 and 4.3(ii), we have that $2 < T < 4$ when

$$(x, \tau, T) \in C\big(0, \tau_0, 2\pi/(\tau_0\omega_0)\big),$$

when $b \in (-2\sqrt{a}, -a)$ and $p \in (p_0, 0]$. This and Lemma 4.1 show that in order for $C(0, \tau_0, 2\pi/(\tau_0\omega_0))$ to be unbounded, its projection onto the τ-space must be unbounded. Thus, the projection of $C(0, \tau_0, 2\pi/(\tau_0\omega_0))$ onto the τ − space include $[\tau_0, \infty)$. Besides, the number of periodic solutions will be explained in the consequent remark. The proof of (ii) is complete. □

In the same way, one can prove the result in (iii), we omit the proof here.

Remark 4.1. From the proof of the Theorem 4.5, we know that the first global Hopf branch contains periodic solutions of period between 2 and 4. They are the slowly oscillating periodic solutions. The j-th branches, for $j \geqslant 1$, contain fast-oscillating periodic solutions, since the periods are less than 1.

Next we carry out some numerical simulations for Eq. (E).

Assume that $a = 3, b = -3.2 \in (-3\sqrt{2}, -3)$ and $p = -0.1$. From Lemma 4.3, it is obtained that the zeros of $H(p)$ on $(-\frac{1}{2}, 0)$ is $p_0 \approx -0.1229$, thus $p \in (p_0, 0)$. Moreover, we compute that $\omega_0 \approx 1.1192$ and $\tau_j \approx 2.4568 + 5.6140j$ $(j = 0, 1, 2, \ldots)$. Accordingly, it is known that $x = 0$ is asymptotically stable for $\tau \in (0, \tau_0)$ and unstable for $\tau > \tau_0$, and Hopf bifurcation at $x = 0$ occurs when $\tau = \tau_0$. By Theorem 3.1, the direction of the Hopf bifurcation at $\tau = \tau_0$ is forward, and the bifurcating periodic solutions are asymptotically stable. Furthermore, according to Theorem 4.5, Eq. (E) with this set of parameters has at least one periodic solution when $\tau > \tau_0$.

Notice also that, under this data the precondition $0 < -pe^{a\tau} < 1$ appeared in H. El-Morshedy and K. Gopalsamy [5] for the oscillatory of (E) implies $0 < \tau < 0.7675$. However, out of this range, all the theorem in [5] for judging oscillation of (E) cannot work, therefore, we cannot know the existence of oscillatory solutions. Here, from our analysis as well as simulations (see Fig. 3), it is obviously that there exist periodic oscillatory solutions when $\tau \geqslant 2.4568$.

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