JOURNAL OF DIFFERENTIAL EQUATIONS **124**, 425–448 (1996)
ARTICLE NO. 0018

Phase Asymptotic Semiflows, Poincaré's Condition, and the Existence of Stable Limit Cycles

MICHAEL Y. LI

Centre de Recherches Mathématiques, Université de Montréal, C.P. 6128-A Montréal, Québec, Canada H3C 3J7

AND

JAMES S. MULDOWNEY

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

Received February 1, 1994; revised September 1, 1994

A concept of phase asymptotic semiflow is defined. It is shown that any Lagrange stable orbit at which the semiflow is phase asymptotic limits to a stable periodic orbit. A Lagrange stable solution of a C^1 differential equation is considered. When the second compound of the variational equation with respect to this solution is uniformly asymptotically stable and the omega limit set contains no equilibrium, then the semiflow is phase asymptotic at the orbit of the solution and the omega limit set is a stable periodic orbit. Analogous results are obtained for discrete semiflows and periodic differential equations.

1. INTRODUCTION

The classical Poincaré-Bendixson theorem [10] states that a positive Lagrange stable orbit in a two dimensional continuous semiflow has as its limit set a periodic orbit provided the limit set contains no equilibrium. Some results for higher dimensional systems which are in this spirit are due to Fiedler and Mallet-Paret [8], Hirsch [12], Mallet-Paret and H. L. Smith [18], Pliss [26], Sell [29], H. L. Smith [30], and R. A. Smith [31, 32, 34]. All give stability conditions for an orbit to have a limit cycle. In [26] it is shown that a type of orbital asymptotic stability with asymptotic phase of a Poincaré stable orbit implies that the orbit is periodic and a wide variety of applications is given. In [29] a similar type of asymptotic phase stability is shown to imply that an orbit which is also Lagrange and uniform Lyapunov stable limits to a periodic orbit. The papers [12, 30] demonstrate that 3-dimensional order-preserving flows have the classical

0022-0396/96 $12.00

Poincaré-Bendixson property. A higher dimensional theory is developed in [31, 32, 34] where conditions are formulated in terms of guiding functions which guarantee that the positive Lagrange stable motions in the semiflow behave in a sufficiently 2-dimensional fashion that a Poincaré-Bendixson result can be obtained. The versatility of the theory is demonstrated by an analysis of the feedback control equation and of a fourth order scalar equation; a detailed study of the delayed Goodwin equations which model certain biochemical reactions is given in [34]. Other work on special high order systems which exhibit the Poincaré-Bendixson behaviour may be found in [8] and [18] where the scalar reaction diffusion equation and monotone cyclic feedback systems respectively are shown to have this property.

In §2 of the present paper we consider semiflows which satisfy a strong asymptotic phase condition with respect to certain orbits and show that these limit to periodic orbits. The result pertains to both discrete and continuous systems. Section 3 develops a preliminary technical result concerning dichotomies for linear nonautonomous equations. The results of §2, 3 are applied to autonomous differential equations in $\mathbb{R}^n$ in Section 4. A condition is imposed on orbits of these equations which when applied to a periodic orbit with $n=2$ reduces to the Poincaré condition ([5, p. 85; 10, p. 220, 11, p. 256]) for the orbital asymptotic stability of the orbit. It is shown that, when an orbit satisfies this generalized Poincaré condition and the omega limit set contains no equilibrium, the orbit has the asymptotic phase property of Section 2 and therefore limits to a periodic orbit. In §5 analogous results are formulated for discrete systems and applied to nonautonomous periodic differential equations.

Throughout this paper terms such as *stable*, *uniformly stable*, *asymptotically stable and uniformly asymptotically stable* as they pertain to solutions of differential equations and of recursions are used in the usual sense as in, for example, [5, Chap. III]. Some concepts of stability for semiflows are discussed in the next section.

2. Phase Asymptotic Semiflows

Let $\{X, d\}$ be a metric space and $\mathbb{T}=\mathbb{R}_+$ or $\mathbb{T}=\mathbb{Z}_+$. A map φ from $\mathbb{T}\times X$ to X is a semiflow on X if

(i) $\varphi(0, x)=x,\ x\in X$

(ii) $\varphi(t+s, x)=\varphi(t, \varphi(s, x)),\ t, s\in\mathbb{T},\ x\in X$

(iii) the map $(t, x)\mapsto\varphi(t, x)$ is continuous.

Definitions. (a) For any $x\in X$, the *positive orbit* of x is $\Gamma_+(x)=\bigcup_{t\geqslant 0}\varphi(t, x)$ and the *omega limit set* is $\Omega(x)=\bigcap_{s\geqslant 0} cl \bigcup_{t\geqslant s}\varphi(t, x)$ where *cl* denotes the topological closure. $\Gamma_+(x)$ is *periodic with period* ω if $\varphi(t+\omega, x) = \varphi(t, x)$ for some $\omega > 0$ in $\mathbb{T}$.

(b) The semiflow φ is *positive Lagrange stable* at x if $cl\,\Gamma_+(x)$ is compact.

(c) The semiflow φ is *positive Lyapunov stable* at $S \subset X$ if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that $x_0 \in S$, $t \in \mathbb{T}$, $x \in X$ and $d[x_0, x] < \delta$ implies $d[\varphi(t, x_0), \varphi(t, x)] < \varepsilon$. When S is an orbit Γ_+, this is the usual concept of uniform Lyapunov stability of Γ_+.

(d) The semiflow φ is *positive phase asymptotic* at $S \subset X$ provided there exist $\rho, \eta > 0$ such that, for each $x_0 \in S$, there is a real-valued phase function $x \mapsto h(x)$ with $\eta > |h(x)| \in \mathbb{T}$ and $d[x_0, x] < \rho$ implies

$$\lim_{t \to \infty} d[\varphi(t + h(x), x), \varphi(t, x_0)] = 0. \tag{2.1}$$

Throughout this paper only behaviour for $t \to \infty$ is considered and the qualification *positive* is usually omitted from the descriptions of these topics.

The concept of phase asymptotic flow is not the subject of [1] 1.3.29 or [24] Chapter V 9.01 where a positively asymptotic orbit $\Gamma(x)$ is one such that $\Gamma_+(x) \cap \Omega(x)$ is empty. The use of the word *asymptotic* here is to indicate that certain orbits attract nearby orbits as in the case of asymptotic stability. It is to be noted that, while ρ, η are independent of $x_0 \in S$ in (2.1), the phase function $h(\cdot)$ in general depends on x_0 but this dependence is suppressed in the notation.

When S is a periodic orbit of a differential equation satisfying the Poincaré stability condition, the proofs in the well-known textbooks [4], [5], [10] of asymptotic orbital stability with asymptotic phase of S in fact show the stronger conclusion that the semiflow is positive Lyapunov stable and phase asymptotic at S. Indeed it is shown in these works that the rate of convergence in (2.1) is exponential. The existence of bounded phase functions $h(\cdot)$ follows from the fact that, if $h(x)$ satisfies (2.1), then so also does $h(x) + \omega$ where ω is any period of S.

THEOREM 2.1. *Suppose the semiflow* φ *is Lagrange stable at* x_*. *Then the statements* (a), (b), (c) *are equivalent. The phrases in parentheses may be either all included or all excluded.*

(a) *The semiflow is phase asymptotic* (*and Lyapunov stable*) *at* $\Gamma_+(x_*)$.

(b) *The semiflow is phase asymptotic* (*and Lyapunov stable*) *at some* $x_0 \in \Omega(x_*)$.

(c) $\Omega(x_*)$ *is a periodic orbit and the semiflow is phase asymptotic* (*and Lyapunov stable*) *at* $cl\,\Gamma_+(x_*)$.

This result, in particular, establishes the absence of chaotic behaviour near a Lagrange stable orbit which is phase asymptotic.

Theorem 2.1 generalizes results of Sell [29] and Pliss [26]. A detailed discussion of Sell's work is contained in Cronin [2, Chap. 6] and in Saperstone [27, Chap. III]. In [29, Theorem 1] it is shown that a positive Lagrange stable motion which is asymptotically stable has as its omega limit set an asymptotically stable periodic orbit. Asymptotic stability in the sense of Sell [29] is called phase asymptotic stability by Cronin [2]. In the terms used here it requires that the semiflow be Lyapunov stable and phase asymptotic at $\Gamma_+(x_*)$ without the requirement that the phase functions be bounded. However, for orbits which are Lagrange stable, the existence of bounded phase functions is implied by the hypotheses since the omega limit set is a periodic orbit which is asymptotically stable. The proof in [29] uses properties of minimal sets of almost periodic motions. Theorem 2.1 also generalizes Theorem 1.6 of Pliss [26] where a condition is given that a positive Poisson stable motion $\varphi(t, x_0)$ be periodic. There is no assumption of Lyapunov stability but rather a strong form of the condition that the semiflow be phase asymptotic at x_0. This requires a uniformity with respect to x of the convergence in (2.1) and further that the bound η on the phase $h(x)$ can be made arbitrarily small by choosing ρ sufficiently small. Since the motion is Poisson stable, $x_0 \in \Omega(x_0)$ and Pliss' result is implied by the statement that (b) implies (c) in Theorem 2.1 with the parenthetic phrases excluded. The proof in [26] uses Brouwer's fixed point theorem.

Proof of Theorem 2.1. We first prove that (a) implies (b). Suppose the semiflow is phase asymptotic at $\Gamma_+(x_*)$ and $x_0 \in \Omega(x_*)$. Choose $x_1 \in \Gamma_+(x_*)$ such that $d[x_0, x_1] < \rho/2$ and therefore $d[x, x_1] < \rho$ when $d[x, x_0] < \rho/2$. Thus $d[x, x_0] < \rho/2$ implies

$$\lim_{t \to \infty} d[\varphi(t + h(x), x), \varphi(t, x_1)] = 0,$$

$$\lim_{t \to \infty} d[\varphi(t + h(x_0), x_0), \varphi(t, x_1)] = 0$$

and hence $\lim_{t \to \infty} d[\varphi(t + h(x) - h(x_0), x),\ \varphi(t, x_0)] = 0$. Thus, with the phase function $x \mapsto h(x) - h(x_0)$, the semiflow is phase asymptotic at x_0 with ρ, η replaced by $\rho/2$, 2η since $|h(x) - h(x_0)| < 2\eta$. Indeed, since x_0 is arbitrary, we have proved that the semiflow is phase asymptotic at $\Omega(x_*)$. A similar argument shows that the semiflow is Lyapunov stable at $x_0 \in \Omega(x_*)$ if it has this property at $\Gamma_+(x_*)$.

To prove that (b) implies (c), we begin by proving that, if φ is phase asymptotic at $x_0 \in \Omega(x_*)$, then $\Omega(x_*)$ is a periodic orbit at which the semiflow is phase asymptotic. Since $x_0 \in \Omega(x_*)$, there exist $x_1, x_2 \in \Gamma_+(x_*)$ with

$d[x_i, x_0] < \rho$, $i = 1, 2$ and $\varphi(t_2, x_1) = x_2$, $t_2 \geqslant 2\eta$. Thus $\lim_{t\to\infty} d[\varphi(t + h(x_i), x_i), \varphi(t, x_0)] = 0$ and hence $\lim_{t\to\infty} d[\varphi(t + h(x_2), x_2), \varphi(t + h(x_1), x_1)] = 0$ which in turn implies $\lim_{t\to\infty} d[\varphi(t + t_2 + h(x_2), x_1), \varphi(t + h(x_1), x_1)] = 0$ or equivalently $\lim_{t\to\infty} d[\varphi(t + \omega, x_1), \varphi(t, x_1)] = 0$, where $\omega = t_2 + h(x_2) - h(x_1) > 0$ since $t_2 \geqslant 2\eta$. Now choose a sequence $t_n \in \mathbb{T}$, $t_n \to \infty$, such that $x_n = \varphi(t_n, x_1) \to x_0$ $(n \to \infty)$. It follows that $\lim_{n\to\infty} d[\varphi(\omega, x_n), x_n] = \lim_{n\to\infty} d[\varphi(t_n + \omega, x_1), \varphi(t_n, x_1)] = 0$ and therefore $\varphi(\omega, x_0) = x_0$: the semiflow is periodic at x_0 and $\Gamma_+(x_0) \subset \Omega(x_*)$ since $\Omega(x_*)$ is invariant ([35, Lemma 1.1]). Moreover, since $\Gamma_+(x_0)$ is compact and attracts all orbits which intersect the ball $B(x_0, \rho)$, including $\Gamma_+(x_*)$, it follows that $\Omega(x_*) \subset \Gamma_+(x_0)$ and hence $\Omega(x_*) = \Gamma_+(x_0)$.

Next we show that the semiflow is phase asymptotic at $\Omega(x_*)$. This is clear if $\Omega(x_*) = \{x_0\}$. In the nonequilibrium case, let the periodic orbit $\Gamma_+(x_0) = \Omega(x_*)$ have least period $\omega > 0$. Then, since φ is uniformly continuous on $[0, \omega] \times \Gamma_+(x_0)$, there exists ρ_1, $0 < \rho_1 \leqslant \rho$, such that $x_1 \in \Gamma_+(x_0)$ and $d[x_1, x] < \rho_1$ implies $d[\varphi(t_1, x), x_0] < \rho$, where $\varphi(t_1, x_1) = x_0$, $t_1 \in [0, \omega)$. Since φ is phase asymptotic at x_0,

$$\begin{aligned}
&\lim_{t\to\infty} d[\varphi(t + h(\varphi(t_1, x)), x), \varphi(t, x_1)] \\
&\quad= \lim_{t\to\infty} d[\varphi(t + t_1 + h(\varphi(t_1, x)), x), \varphi(t + t_1, x_1)] \\
&\quad= \lim_{t\to\infty} d[\varphi(t + h(\varphi(t_1, x)), \varphi(t_1, x)), \varphi(t, \varphi(t_1, x_1))] \\
&\quad= \lim_{t\to\infty} d[\varphi(t + h(\varphi(t_1, x)), \varphi(t_1, x)), \varphi(t, x_0)] \\
&\quad= 0.
\end{aligned}$$

Thus φ is phase asymptotic at the periodic orbit $\Gamma_+(x_0) = \Omega(x_*)$; the phase function associated with $x_1 \in \Gamma(x_0)$ is $x \mapsto \tilde{h}(x) = h(\varphi(t_1, x))$ where $\varphi(t_1, x_1) = x_0$ and h is the phase function associated with x_0. The phase function $\tilde{h}$ associated with x_1 has the same bound as the phase function h associated with x_0 and the bound is therefore independent of $x_1 \in \Omega(x_*)$ as required.

To complete the proof that φ is phase asymptotic at $cl\, \Gamma_+(x_*)$, let ρ_1, $\tilde{h}$, x_1 be as in the preceding paragraph. There exists $y_0 = \varphi(t_0, x_*) \in \Gamma_+(x_*)$ such that $d[\varphi(t, y_0), \Omega(x_*)] < \rho_1/2$ if $t \in \mathbb{T}$. Thus, if $y_1 \in \Gamma_+(y_0) \subset \Gamma_+(x_*)$, then $d[x_1, y_1] < \rho_1/2$ by some $x_1 \in \Omega(x_*)$. It follows that $d[y_1, x] < \rho_1/2$ implies $d[x_1, x] < \rho_1$ so that, since φ is phase asymptotic at $\Omega(x_*)$ and $x_1 \in \Omega(x_*)$,

$$\lim_{t\to\infty} d[\varphi(t + \tilde{h}(x), x), \varphi(t, x_1)] = 0$$

$$\text{and} \qquad \lim_{t\to\infty} d[\varphi(t + \tilde{h}(y_1), y_1), \varphi(t, x_1)] = 0$$

and therefore

$$\lim_{t\to\infty} d[\varphi(t+\tilde{h}(x)-\tilde{h}(y_1), x), \varphi(t, y_1)]$$

$$= \lim_{t\to\infty} d[\varphi(t+\tilde{h}(x), x), \varphi(t+\tilde{h}(y_1), y_1)] = 0.$$

This shows that φ is phase asymptotic at $\Gamma_+(y_0) \subset \Gamma_+(x_*)$. The phase function associated with $y_1 \in \Gamma_+(y_0)$ is $x \mapsto \tilde{h}(x) - \tilde{h}(y_1)$ and has a bound independent of y_1 since the bound on $\tilde{h}(\cdot)$ is independent of its associated point $x_1 \in \Omega(x_*)$. As in the case of the periodic orbit $\Gamma_+(x_0) = \Omega(x_*)$, the fact that φ is phase asymptotic at y_0 may be used to establish that it also has this property at $\Gamma_+(x_*) \backslash \Gamma_+(y_0) = \{\varphi(t, x_*): t \in [0, t_0]\}$ hence at $\Gamma_+(x_*)$. Combined with the result of the preceding paragraph, this shows that φ is phase asymptotic at $cl\, \Gamma_+(x_*) = \Gamma_+(x_*) \cup \Omega(x_*)$ completing the proof that (b) implies (c). The parenthetic assertion on Lyapunov stability is proved similarly.

The fact that (c) implies (a) is obvious, concluding the proof of Theorem 2.1. ∎

3. A Linear Result

Let A denote the $n \times n$ matrix for which the entry in the i-row and j-column is a_i^j. Then $A^{(2)}$, $A^{[2]}$ denote $N \times N$ matrices, $N = \binom{n}{2}$, the second multiplicative compound and second additive compound of A respectively, which are defined as follows. For any integer $i = 1, ..., N$, let $(i) = (i_1, i_2)$ be the i-th member in the lexicographic ordering of integer pairs (i_1, i_2) such that $1 \leqslant i_1 < i_2 \leqslant n$. Then the entry in the i-row and j-column of $A^{(2)}$ is $a_{i_1 i_2}^{j_1 j_2}$, the minor of A determined by the rows i_1, i_2 and the columns j_1, j_2. The entry in the i-row and j-column of $A^{[2]}$ is

$$\begin{array}{ll} a_{i_1}^{i_1} + a_{i_2}^{i_2}, & \text{if } (j) = (i) \\ (-1)^{r+s}\, a_{i_r}^{j_s}, & \text{if exactly one entry } i_r \text{ of } (i) \text{ does not occur in } (j) \\ & \text{and } j_s \text{ does not occur in } (i) \\ 0, & \text{if neither entry from } (i) \text{ occurs in } (j). \end{array}$$

The compounds have the properties

$$(AB)^{(2)} = A^{(2)} B^{(2)}, \qquad (A+B)^{[2]} = A^{[2]} + B^{[2]}$$

and

$$D(I + hA)^{(2)}\,|_{h=0} = A^{[2]},$$

where D denotes differentiation with respect to h. Consequently, if $Y(\cdot)$ is a fundamental matrix for a linear system

$$\dot{y} = A(t)\, y, \tag{3.1}$$

with $A(\cdot)$ a continuous real $n \times n$ matrix-valued function, then $Z(\cdot) = Y^{(2)}(\cdot)$ is a fundamental matrix for the system

$$\dot{z} = A^{[2]}(t)\, z. \tag{3.2}$$

From this it follows that $z(\cdot) = y_1(\cdot) \wedge y_2(\cdot)$, where $\wedge$ denotes the exterior product, is a solution of (3.2) whenever $y_1(\cdot)$, $y_2(\cdot)$ are solutions of (3.1).

If $\lambda_1, ..., \lambda_n$ are the eigenvalues of A, then $\lambda_i \lambda_j$ and $\lambda_i + \lambda_j$, $1 \leqslant i < j \leqslant n$, are the eigenvalues of $A^{(2)}$ and $A^{[2]}$ respectively. The numbers $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n \geqslant 0$ are called the *singular values* of A if $\sigma_1^2, ..., \sigma_n^2$ are the eigenvalues of A^*A. It follows that the singular values of $A^{(2)}$ are $\sigma_i \sigma_j$, $1 \leqslant i < j \leqslant n$. Therefore,

$$|A| = \sigma_1, \qquad |A^{(2)}| = \sigma_1 \sigma_2 \tag{3.3}$$

where $|\cdot|$ denotes the norm on $\mathbb{R}^n$ or $\mathbb{R}^N$ defined by $|x| = (x^*x)^{1/2}$ and the matrix norm which it induces.

For a more detailed discussion of these and other compound matrices and their applications, the reader is referred to [9, 14, 19, 21, 23, 28].

In the following proposition, it is required by condition (a) that the equation (3.2) be uniformly asymptotically stable. The solution space of (3.1) is required by (b) to have a 1-dimensional strongly stable subspace and (c) specifies that solutions have bounded growth. Under these circumstances (3.1) has a dichotomy which splits its solution space into the strongly stable subspace and a complementary $(n-1)$-dimensional subspace which is uniformly asymptotically stable. This is closely related to a result in [21] on the dimension of the set of solutions $y(\cdot)$ of (3.1) which satisfy $\lim_{t \to \infty} y(t) = 0$.

PROPOSITION 3.1. *Suppose that the conditions* (a), (b), (c) *are satisfied*:

(a) *There exist constants* $K, \alpha > 0$ *such that*

$$|z(t)| < K\, |z(s)|\, e^{-\alpha(t-s)}, \qquad 0 \leqslant s \leqslant t$$

for each solution $z(\cdot)$ *of* (3.2).

(b) *There is a constant* $L > 1$ *and a nonzero solution* $y_1(\cdot)$ *of* (3.1) *such that*

$$|y_1(t)| \leqslant L\, |y_1(s)|, \qquad 0 \leqslant s, t.$$

(c) *There exist constants* M, β *such that*

$$|y(t)| \leqslant M\,|y(s)|\,e^{\beta(t-s)}, \qquad 0 \leqslant s \leqslant t$$

for each solution $y(\cdot)$ *of* (3.1).

Then, if $Y(\cdot)$ *is a fundamental matrix of* (3.1), *there exist supplementary projections* P_1, P_2 *on* $\mathbb{R}^n$, $rkP_1 = 1$, $rkP_2 = n-1$ *and a constant* C *such that*

$$|Y(t)\,P_1\,Y^{-1}(s)| \leqslant C, \qquad 0 \leqslant s, t$$

and

$$|Y(t)\,P_2\,Y^{-1}(s)| \leqslant Ce^{-\alpha(t-s)}, \qquad 0 \leqslant s \leqslant t.$$

In particular, (3.1) *is uniformly stable.*

Proof. A fundamental matrix for (3.2) is $Z(\cdot) = Y^{(2)}(\cdot)$ and therefore $Z(t)\,Z^{-1}(s) = (Y(t)\,Y^{-1}(s))^{(2)}$. Thus the conditions (a), (b), (c) imply from (2.3) that $\sigma_1\sigma_2(s, t) \leqslant Ke^{-\alpha(t-s)}$, $\sigma_1(s, t) \leqslant Me^{\beta(t-s)}$, $0 \leqslant s \leqslant t$ and $1/L \leqslant \sigma_1(s, t)$, $0 \leqslant s, t$, where $\sigma_1(s, t) \geqslant \sigma_2(s, t) \geqslant \cdots \geqslant \sigma_n(s, t) > 0$ are the singular values of $Y(t)Y^{-1}(s)$. It follows that $\sigma_2(s, t) \leqslant LKe^{-\alpha(t-s)}$, $0 \leqslant s \leqslant t$. Let $\delta > 0$ and choose T sufficiently large that $t = s + T$ implies

$$\sigma_1\sigma_2(s, t) < \delta, \qquad \sigma_2(s, t) < \delta, \tag{3.4}$$

for each $s \geqslant 0$. Consider the two solution subspaces $\mathscr{Y}_1$, $\mathscr{Y}_2$ for (3.1) defined by $\mathscr{Y}_1 = \text{span}\{y_1(\cdot)\}$, $\mathscr{Y}_2 = \text{span}\{y_2(\cdot), ..., y_n(\cdot)\}$, where $y_i(\cdot)$ is a solution of (3.1) with $y_i(s)$ an eigenvector of $Y^{*-1}(s)\,Y^*(t)\,Y(t)\,Y^{-1}(s)$ corresponding to the eigenvalue $\sigma_i(t, s)^2$, $i = 2, ..., n$. Then $y(\cdot) \in \mathscr{Y}_2$ implies $|y(t)| \leqslant |y(s)|\,\sigma_2(t, s)$, $t = s + T$, and, if T is chosen sufficiently large to allow $\delta < 1/L$ in (3.4), then $\mathscr{Y}_1$, $\mathscr{Y}_2$ are supplementary subspaces, from (b). Then also $y(\cdot) \in \mathscr{Y}_2$ implies

$$0 < 1/L - \delta \leqslant \left|\frac{y_1(t)}{|y_1(s)|} - \frac{y(t)}{|y(s)|}\right| \leqslant \left|\frac{y_1(s)}{|y_1(s)|} - \frac{y(s)}{|y(s)|}\right| \sigma_1(s, t)$$

if $t = s + T$ and the angular separation, $\inf |(y_1(s)/|y_1(s)|) - (y(s)/|y(s)|)|$, $y(\cdot) \in \mathscr{Y}_2$, between the spaces of initial values $\{y(s)\colon y(\cdot) \in \mathscr{Y}_i\}$, $i = 1, 2$, is at least $(1 - L\delta)/L\sigma_1(s, t)$. Therefore the supplementary projections $P_i(s)$, $i = 1, 2$, on $\mathbb{R}^n$ onto these initial value subspaces satisfy (cf. [7, p. 156])

$$|P_i(s)| \leqslant \gamma\sigma_1(s, t), \qquad i = 1, 2 \tag{3.5}$$

where $\gamma = (2L/1 - L\delta)$. The space $\mathscr{Y}_1$ is independent of (s, t) while $\mathscr{Y}_2$ is not necessarily so. If $s_0 \geqslant 0$ and $s_k = s_0 + kT$, $k = 0, 1, ...$, let $\mathscr{Y}_{2,k}$ denote the space $\mathscr{Y}_2$ corresponding to $(s, t) = (s_k, s_{k+1})$. Let $y(\cdot) = y_{1,k}(\cdot) + y_{2,k}(\cdot)$ be any solution of (3.1) $y_{1,k}(\cdot) \in \mathscr{Y}_1$, $y_{2,k}(\cdot) \in \mathscr{Y}_{2,k}$, $k = 0, 1, 2, \ldots$. Then

$$y_{i,k}(s_k) = P_i(s_k)\, y(s_k) = P_i(s_k)\, y_{1,k-1}(s_k) + P_i(s_k)\, y_{2,k-1}(s_k),$$

$i = 1, 2$ so that, for $k = 1, 2, \ldots,$

$$y_{1,k}(s_k) = y_{1,k-1}(s_k) + P_1(s_k)\, y_{2,k-1}(s_k) \tag{3.6}$$

$$y_{2,k}(s_k) = P_2(s_k)\, y_{2,k-1}(s_k) \tag{3.7}$$

Now, from (3.5) and the definition of $\mathscr{Y}_{2,k-1}$,

$$\begin{aligned} |P_i(s_k)\, y_{2,k-1}(s_k)| &\leqslant \gamma\sigma_1(s_k, s_{k+1})\, |y_{2,k-1}(s_k)| \\ &\leqslant \gamma\sigma_1(s_k, s_{k+1})\, \sigma_2(s_{k-1}, s_k)\, |y_{2,k-1}(s_{k-1})|, \end{aligned}$$

$i = 1, 2$, $k = 0, 1, \ldots$. Similarly $|y_{2,0}(s_0)| = |P_2(s_0)\, y(s_0)| \leqslant \gamma\sigma_1(s_0, s_1)\, |y(s_0)|$. Hence, by induction, we find

$$\begin{aligned} |P_i(s_k)\, y_{2,k-1}(s_k)| &\leqslant \gamma^{k+1}\sigma_1(s_k, s_{k+1}) \prod_{j=1}^{k} \sigma_1\sigma_2(s_{j-1}, s_j)\, |y(s_0)| \\ &\leqslant \gamma^{k+1}\delta^k M e^{\beta T}\, |y(s_0)|, \end{aligned}$$

from (c) and (3.4). Since $y_{1,k}(\cdot) = c_k y_1(\cdot)$ for some constant c_k, (3.6) implies $c_k = c_{k-1} + \Delta_k$, where $|\Delta_k| \leqslant \gamma^{k+1}\delta^k M e^{\beta T}\, |y(s_0)|/|y_1(s_k)|$, $k = 1, 2, \ldots$ and $|c_0| \leqslant \gamma M e^{\beta T}\, |y(s_0)|/|y_1(s_0)|$. Therefore $|c_k| \leqslant (\gamma M e^{\beta T}/m_1(1-\gamma\delta))\, |y(s_0)|$, if T is chosen so that $\gamma\delta < 1$, and $|y_{1,k}(s_k)| \leqslant (\gamma M_1 M e^{\beta T}/m_1(1-\gamma\delta))\, |y(s_0)|$ with $m_1 = \inf |y_1(s)|$, $M_1 = \sup |y_1(s)|$, $s \geqslant 0$. Also $|y_{2,k}(s_k)| \leqslant \gamma M e^{\beta T}\, |y(s_0)|$. This gives

$$|y(s_k)| \leqslant |y_{1,k}(s_k)| + |y_{2,k}(s_k)| \leqslant \gamma M e^{\beta T} \left[\frac{M_1}{m_1(1-\gamma\delta)} + 1 \right] |y(s_0)|$$

and, since $|y(t)| \leqslant M e^{\beta T}\, |y(s_k)|$, $s_k \leqslant t \leqslant s_{k+1}$ from (c), we find

$$|y(t)| \leqslant H\, |y(s_0)|, \qquad 0 \leqslant s_0 \leqslant t \tag{3.8}$$

where $H = \gamma M^2 e^{2\beta T}[M_1/m_1(1-\gamma\delta)) + 1]$. We conclude that (3.1) is uniformly stable since H is independent of s_0.

From (a) and the uniform stability of (3.1) now established, by a result in [21] and [23], there exists a $(n-1)$-dimensional subspace of solutions $y(\cdot)$ of (3.1) such that $\lim_{t \to \infty} y(t) = 0$. Let $\mathscr{Y}_2$ now denote this subspace and $\mathscr{Y}_1 = span\{y_1(\cdot)\}$ as before. From the theory of dichotomies, [6, Chap. 2], Proposition 3.1 will be established if we can demonstrate:

I The subspace $\mathscr{Y}_2$ is uniformly asymptotically stable.

II If $y(\cdot) \in \mathscr{Y}_2$ is nonzero, the angle $\theta(t)$ between $y_1(t)$ and $y(t)$ is bounded away from 0 uniformly with respect to $t \geqslant 0$ and $y(\cdot) \in \mathscr{Y}_2$.

Since $z(\cdot) = y_1(\cdot) \wedge y(\cdot)$ is a solution of (3.2) and $|z(t)| = |\sin \theta(t)| \, |y_1(t)| \, |y(t)|$ ([35], page 254), (a) implies $|\sin \theta(t)| \, |y_1(t)| \, |y(t)| \leqslant K \, |\sin \theta(s)| \, |y_1(s)| \, |y(s)| \, e^{-\alpha(t-s)}$ so that

$$|y(t)| \leqslant KL \, |y(s)| \, e^{-\alpha(t-s)} / |\sin \theta(t)|$$

and II implies I. To prove II, it is sufficient to show that $2 \, |\sin 1/2\theta(t)| = |(y_1(t)/|y_1(t)|) - (y(t)/|y(t)|)|$ is bounded away from 0. Choose $t > s$ sufficiently large that $|y(t)| < 1/2L \, |y(s)|$; then (3.8) implies

$$0 < \frac{1}{2} L < L - \frac{|y(t)|}{|y(s)|} \leqslant \left| \frac{y_1(t)}{|y_1(s)|} - \frac{y(t)}{|y(s)|} \right| \leqslant H \left| \frac{y_1(s)}{|y_1(s)|} - \frac{y(s)}{|y(s)|} \right|$$

so that $0 < (L/2H) < 2 \, |\sin 1/2\theta(s)|$ completing the proof of Proposition 3.1. ∎

4. The Poincaré Condition and Existence of Limit Cycles

Let D be an open subset of $\mathbb{R}^n$ and $x \mapsto f(x)$ a C^1 function from D to $\mathbb{R}^n$. We consider a solution $x = \varphi(\cdot)$ of the autonomous differential equation

$$\dot{x} = f(x) \tag{4.1}$$

such that $\varphi(t)$ exists for all $t \geqslant 0$. Let $\Gamma_+ = \{\varphi(t) \colon t \geqslant 0\}$. The variational equation of (4.1) at Γ_+ is

$$\dot{y} = \frac{\partial f}{\partial x} (\varphi(t)) \, y, \tag{4.2}$$

where $(\partial f/\partial x)(x)$ is the Jacobian matrix of f at $x \in D$. It governs the evolution of infinitesimal oriented line segments y along Γ_+ subject to the dynamics of (1.1). The equation which governs the evolution of infinitesimal oriented 2-dimensional areas z is

$$\dot{z} = \frac{\partial f^{[2]}}{\partial x} (\varphi(t)) \, z, \tag{4.3}$$

where $(\partial f^{[2]}/\partial x)(x)$ is the second additive compound of $(\partial f/\partial x)(x)$ discussed in Section 3.

When $n=2$, the Poincaré stability condition ([5, 10, 11]) states that a periodic solution $\varphi(\cdot)$ of (4.1) with least period $\omega>0$ is asymptotically orbitally stable with asymptotic phase if $\int_0^\omega \operatorname{div} f(\varphi(t))\,dt<0$ or, equivalently, if the Liouville equation $\dot{y}=\operatorname{div} f(\varphi(t))\,y$ is uniformly asymptotically stable in the sense of Liapunov. In [23], it was shown that the same conclusion holds with $n>2$ if (4.3) is uniformly asymptotically stable. Since (4.3) is the Liouville equation when $n=2$, this provides an extension to higher dimensions of the Poincaré stability condition for periodic orbits.

In this section we investigate the implications of the Poincaré condition, uniform asymptotic stability of (4.3), when periodicity of the solution $\varphi(\cdot)$ is replaced by *positive Lagrange stability* of Γ_+ ($cl\,\Gamma_+$ is a compact subset of D). When $n=2$, the Poincaré–Bendixson theory ensures that, if the omega limit set Ω of Γ_+ contains no equilibrium, then Ω is a periodic orbit. Continuity considerations show that, if the Liouville equation is uniformly asymptotically stable, then the periodic orbit satisfies the Poincaré condition so that it is asymptotically orbitally stable and thus, together with Γ_+, attracts all nearby orbits. When $n>2$, the Poincaré-Bendixson theory is no longer generally applicable. However, by showing that the semiflow corresponding to (4.1) is phase asymptotic at $x_0\in\Omega$, we deduce from Theorem 2.1 that Ω is still a stable periodic orbit which attracts all nearly orbits. We also discuss the implications for an equilibrium $x_0\in\Omega$; in this case it can no longer be concluded that Ω attracts all nearly orbits.

We recall that the semiflow of (4.1) is defined locally by $\varphi(t,x_0)=x(t)$, where $x(\cdot)$ is the solution such that $x(0)=x_0$ and satisfies the requirements set out in Section 2 if $x(t)$ exists for all $t\geqslant 0$ as it does when $x_0\in\Gamma_+$.

THEOREM 4.1. *Suppose Γ_+ is positive Lagrange stable with omega limit set Ω and that* (4.3) *is uniformly asymptotically stable.*

(a) *If Ω contains no equilibrium, then it is a periodic orbit and there exist positive constants H, γ, ρ such that*

$$|\varphi(t+h,x)-\varphi(t,x_0)|\leqslant H\,|x-x_0|\,e^{-\gamma t}$$

for some real h, if $x_0\in\Omega$ and $|x-x_0|<\rho$.

(b) *If Ω contains an equilibrium x_0, then either x_0 is asymptotically stable or it has a $(n-1)$-dimensional stable manifold together with a* 1-*dimensional centre manifold or a* 1-*dimensional unstable manifold.*

(c) *If Ω contains an equilibrium x_0 and* (4.2) *is uniformly stable, then $\Omega=\{x_0\}$.*

When Ω contains no equilibrium, the uniform asymptotic stability of (4.3) will be seen to imply the uniform stability of (4.2). This is not the case when there is an equilibrium in Ω as illustrated by Example 4.3 with $0 < \lambda < 1$.

In [3] Cronin gives conditions on a C^3 function f which ensure that Lagrange stable solutions of (4.1) are phase asymptotically stable (asymptotically stable in the sense of Sell [29]) and therefore limit to a phase asymptotically stable periodic solution. While it is clear that this result does not imply Theorem 4.1, we are unable to determine whether the converse statement is true: Do Cronin's conditions imply that the second compound equation (4.3) is uniformly asymptotically stable? Several of her conditions are technical restrictions on the spectrum of the Jacobian matrix $(\partial f/\partial x)$ in D which are difficult to verify. Examples of readily verifiable conditions for the uniform asymptotic stability of (4.3) are given by the formula (4.4).

Let $x \mapsto \mu(x)$ be defined by any of the expressions

$$\text{(i)} \quad \mu = \sup \left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q \neq r, s} \left(\left| \frac{\partial f_q}{\partial x_r} \right| + \left| \frac{\partial f_q}{\partial x_s} \right| \right) : 1 \leqslant r < s \leqslant n \right\}$$

$$\text{(ii)} \quad \mu = \sup \left\{ \frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q \neq r, s} \left(\left| \frac{\partial f_r}{\partial x_q} \right| + \left| \frac{\partial f_s}{\partial x_q} \right| \right) : 1 \leqslant r < s \leqslant n \right\}$$

$$\text{(iii)} \quad \mu = \lambda_1 + \lambda_2$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ are the eigenvalues of $1/2((\partial f/\partial x)^* + (\partial f/\partial x))$. If there exist $T, \alpha > 0$ such that, for one of these functions,

$$\int_s^t (\mu \circ \varphi) \leqslant -\alpha(t-s), \qquad \text{if} \quad t - s \geqslant T \tag{4.4}$$

and if Γ_+ is positive Lagrange stable, then (4.3) is uniformly asymptotically stable; this follows by consideration of $|z| = \sup_i |z_i|$, $\sum_i |z_i|$, $(z^*z)^{1/2}$ respectively as a Lyapunov function for (4.3). We therefore have the following corollary. When $n = 2$, each of the expressions (i), (ii), (iii) gives $\mu = (\partial f_1/\partial x_1) + (\partial f_2/\partial x_2) = \operatorname{div} f$ and (4.4) is then the Poincaré condition.

COROLLARY 4.2. *If* (4.4) *is satisfied by a solution* $\varphi(\cdot)$ *of* (4.1) *whose orbit is positive Lagrange stable, then the conclusion of Theorem* 4.1 *holds for the omega limit set* Ω *of* $\varphi(\cdot)$.

Even when $n = 2$, application of the Poincaré condition to establish orbital stability may involve considerable subtlety as, for example, in the case the stability of a limit cycle in the Liénard equation, [5, p. 86]. For example, it was shown in [15] that, if one of the expressions (i), (ii), (iii) is negative throughout a convex open set D, then the only nonwandering points are equilibria so that there are no nontrivial periodic solutions and Theorem 4.1(a) would hold only vacuously. It was also shown that any omega limit set is a single equilibrium with a stable manifold of dimension $n-1$ at least. Theorem 4.1 and Corollary 4.2 can thus be regarded as extending this discussion to situations where μ is negative only in an averaged sense along an orbit.

Before proving Theorem 4.1, we discuss some examples.

EXAMPLE 4.3. When the equation (4.1) is

$$\dot{x}_1 = \lambda x_1, \qquad \dot{x}_2 = -x_2$$

with $n=2$, $\varphi(t) = (0, e^{-t})$ and $\Omega = \{(0,0)\}$, then (4.3) is the Liouville equation $\dot{z} = (\lambda - 1) z$ which is uniformly asymptotically stable if $\lambda - 1 < 0$. All three possibilities of Theorem 4.1(b) are exhibited here. If $\lambda < 0$, the equilibrium $\Omega = \{(0, 0\}$ is asymptotically stable; if $0 \leqslant \lambda < 1$ there is a 1-dimensional stable manifold together with a 1-dimensional centre manifold when $\lambda = 0$ and a 1-dimensional unstable manifold when $0 < \lambda < 1$.

The preceding example shows that, in contrast with Theorem 4.1(a), when Ω contains an equilibrium the semiflow is not necessarily phase asymptotic at any point of $cl\, \Gamma_+$. In the example, the orbit Γ_+ is in the stable manifold of its omega limit set but this need not be the case: Γ_+ could be in a centre manifold of Ω as in Example 4.4. The example also shows that, when Ω contains an equilibrium, (4.2) may be unstable.

EXAMPLE 4.4. Let (4.1) be the equation

$$\dot{x}_1 = -x_1^2, \qquad \dot{x}_2 = -x_2$$

with $n=2$, $\varphi(t) = ((t+1)^{-1}, 0)$, $t > -1$, and $\Omega = \{(0,0)\}$. Here equation (4.3) is $\dot{z} = -(2(t+1)^{-1} + 1) z$ which is uniformly asymptotically stable. In this case Γ_+ is in the centre manifold $\{0\} \times \mathbb{R}$ of Ω. While the semiflow is phase asymptotic at each point of Γ_+, this is not the case at $cl\, \Gamma_+$ since $(0, 0)$ is unstable. Any positive orbit in the stable manifold $\mathbb{R} \times \{0\}$ of $(0, 0)$ also satisfies the conditions of Theorem 4.1(b): in this case the semiflow is not phase asymptotic at any point of the orbit.

In Example 4.4, if we consider the semiflow restricted to the right half-plane $x_1 \geqslant 0$, it is phase asymptotic at each positive orbit and each such orbit limits to the now stable periodic orbit $\Omega = \{(0,0)\}$, as asserted by Theorem 2.1 with $X = \{(x_1, x_2)\colon x_1 \geqslant 0\}$.

EXAMPLE 4.5. Consider the 3-dimensional system

$$\dot{x} = y\omega(z) + (1 - x^2 - \alpha y^2)\, x, \qquad \dot{y} = -x\omega(z) + (1 - x^2 - \alpha y^2)\, y,$$
$$\dot{z} = \beta(x + y) - \gamma z \tag{4.5}$$

where $z \mapsto \omega(z) \neq 0$ is C^1 and $1 \leqslant \alpha \leqslant 2$, $\gamma > 0$, β are real constants. The stable manifold of the equilibrium (0, 0, 0) is the z-axis $\{(0, 0)\} \times \mathbb{R}$. There is a 2-dimensional unstable manifold which, when $\beta = 0$, is the (x, y)-plane $\mathbb{R}^2 \times \{0\}$. In this case it is an easy exercise, using the Poincaré-Bendixson Theorem and the Poincaré stability condition, that there is a unique non-constant periodic orbit which attracts all orbits not in the stable manifold of (0, 0, 0). Here we will investigate values of the parameters for which this phenomenon persists if $\beta \neq 0$. With

$$E = x^2 + y^2, \qquad F = |z|,$$

we find

$$D_+ E = 2(x^2 + y^2)(1 - x^2 - \alpha y^2), \qquad D_+ F \leqslant \beta\, |x + y| - \gamma\, |z|.$$

where D_+ denotes the derivative from the right with respect to t. Therefore $D_+ E > 0$, if $0 < x^2 + \alpha y^2 < 1$; $D_+ E < 0$, if $1 < x^2 + \alpha y^2$; $D_+ F < 0$, if $\gamma\, |z| > \sqrt{2}\, \beta (x^2 + y^2)^{12}$. Thus all orbits except those in the stable manifold $\{(0, 0)\} \times \mathbb{R}$ ultimately enter and remain in the toroidal region

$$D_0 = \left\{ (x, y, z) \colon \frac{1}{\alpha} \leqslant x^2 + y^2 \leqslant 1,\, \gamma\, |z| \leqslant \sqrt{2}\, \beta (x^2 + y^2)^{1/2} \right\}$$

which contains no equilibrium if $\omega(z) \neq 0$. We consider an arbitrary solution $\varphi(\cdot)$ of (4.5), $\varphi(t) = (x(t), y(t), z(t))$, whose positive orbit is in D_0. From §3 or, more explicitly, [15, Fig. 1], the equation corresponding to (4.3) in this case is

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 2 - 4(x^2 + \alpha y^2) & -x\omega'(z) & -y\omega'(z) \\ \beta & 1 - \gamma - (3x^2 + \alpha y^2) & \omega(z) - 2\alpha xy \\ -\beta & -\omega(z) - 2xy & 1 - \gamma - (x^2 + 3\alpha y^2) \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \tag{4.6}$$

$(x, y, z) = \varphi(t)$. We consider the Liapunov function $G = \sup\{|u|, (v^2 + w^2)^{12}\}$. Then, since $\varphi(t) \in D_0$,

$$\begin{aligned} D_+ |u| &\leqslant (2 - 4(x^2 + \alpha y^2))\, |u| + |\omega'(z)|\, (|x|\, |v| + |y|\, |w|) \\ &\leqslant (2 + |\omega'(z)| - 4/\alpha)\, |u|, \qquad \text{if} \quad (v^2 + w^2)^{1/2} \leqslant |u|; \end{aligned}$$

$$
\begin{aligned}
D_+(v^2+w^2)^{1/2} = {} & (v^2+w^2)^{-1/2}\, v[\beta u + (1-\gamma-3x^2-\alpha y^2)\, v \\
& + (\omega(z) - 2\alpha xy)\, w] \\
& + (v^2+w^2)^{-1/2}\, w[-\beta u - (\omega(z)+2xy)\, v \\
& + (1-\gamma-x^2-3\alpha y^2)\, w] \\
\leqslant {} & (\sqrt{2}\,|\beta| + \tfrac{1}{2}(\alpha+3) - (\gamma+1/\alpha))(v^2+w^2)^{1/2}, \\
& \text{if} \quad |u| \leqslant (v^2+w^2)^{1/2}.
\end{aligned}
$$

We conclude that (4.6) is uniformly asymptotically stable if, for $(x, y, z) \in D_0$,

$$
2 + |\omega'(z)| < 4/\alpha, \qquad \sqrt{2}\,|\beta| + \tfrac{1}{2}(\alpha+3) < \gamma + 1/\alpha. \tag{4.7}
$$

Theorem 4.1(a) then shows that (4.7) implies each orbit of (4.1), other than those in the z-axis, together with neighbouring orbits tends to an omega limit cycle Ω in D_0. The subset of $D = \{(x, y, z): (x, y) \neq (0, 0)\}$ attracted to a given limit cycle Ω is open and does not intersect the subset attracted to any other limit cycle. Therefore, since D is connected, there is a unique limit cycle which is the global attractor in D. The uniqueness of the limit cycle could also be deduced directly from (4.7). This is a higher dimensional Dulac condition for D_0 (cf. [14, Sect. 3]) and, in the spirit of a result of Lloyd [17] for nonsimply connected regions in the plane, implies that D_0 contains at most one periodic orbit. In summary, if (4.7) is satisfied, the semiflow is phase asymptotic at every compact subset of D with a unique limit cycle in D_0.

The following proof is an adaptation of one given for periodic orbits in [4, p. 323] and [5, p. 82].

Proof of Theorem 4.1. If $Y(\cdot)$ is a fundamental matrix for (4.2), then $Z(\cdot)$ is a fundamental matrix for (4.3). Furthermore, if $x = \varphi(s)$,

$$
Y(t)\, Y^{-1}(s) = \frac{\partial \varphi}{\partial x}(t-s, x), \qquad Z(t)\, Z^{-1}(s) = \frac{\partial \varphi^{(2)}}{\varphi x}(t-s, x). \tag{4.8}
$$

The uniform asymptotic stability of (4.3), the generalized Poincaré Condition, is therefore equivalent to the existence of constants $K, \alpha > 0$ such that

$$
\left| \frac{\partial \varphi^{(2)}}{\partial x}(t, x) \right| \leqslant K e^{-\alpha t} \tag{4.9}
$$

if $x \in \Gamma_+$ and $t \geqslant 0$. Since K, α are independent of x, t, it follows that (4.9) is satisfied if $x \in cl\,\Gamma_+$ and $t \geqslant 0$. Therefore any orbit in the omega limit set Ω also satisfies the stability conditions imposed on Γ_+ and we may assume without loss of generality that $\varphi(t) = \varphi(t, x_0)$, $x_0 \in \Omega$ and thus $\Gamma_+ \subset \Omega$.

From (4.8) and (4.9), $|z(t)| \leqslant K\,|z(s)|\,e^{-\alpha(t-s)}$, $0 \leqslant s \leqslant t$, for all solutions $z(\cdot)$ of (4.3). Observe that $y_1(\cdot) = \dot{\varphi}(\cdot)$ is a solution of (4.2). Since $\dot{\varphi}(t) = f(\varphi(t))$ and Γ_+ is positive Lagrange stable, $|y_1(t)| = |\dot{\varphi}(t)|$ is bounded, $0 \leqslant t < \infty$. Furthermore, if Ω contains no equilibrium, $1/|y_1(t)|$ is also bounded. Therefore there exists a constant $L > 1$ such that $|y_1(t)| \leqslant L\,|y_1(s)|$, $0 \leqslant s, t$. Lagrange stability of Γ_+ also implies that $|(\partial f/\partial x)(\varphi(t))|$ is bounded so that there exist constants M, β such that $|y(t)| \leqslant M\,|y(s)|\,e^{\beta(t-s)}$, $0 \leqslant s \leqslant t$ for all solutions $y(\cdot)$ of (4.2). The conditions of Theorem 4.1(a) therefore imply that all the hypotheses of Proposition 3.1 are satisfied when $A(t) = (\partial f/\partial x)(\varphi(t))$. Therefore (4.2) is uniformly stable and there exist supplementary projections P_1, P_2 on $\mathbb{R}^n$, $rkP_1 = 1$, $rkP_2 = n-1$ and a constant $C > 0$ such that $Y(t) = (\partial\varphi/\partial x)(t, x_0)$ satisfies

$$
\begin{aligned}
&|Y(t)\,P_1\,Y^{-1}(s)| \leqslant C, && 0 \leqslant s, t;\\
&|Y(t)\,P_2\,Y^{-1}(s)| \leqslant Ce^{-\alpha(t-s)}, && 0 \leqslant s \leqslant t.
\end{aligned}
\tag{4.10}
$$

Substituting $x = z + \varphi(t)$, we find that (4.1) is equivalent to

$$
\dot{z} = \frac{\partial f}{\partial x}(\varphi(t))\,z + F(t, z), \qquad F(t, z) = f(z + \varphi(t)) - f(\varphi(t)) - \frac{\partial f}{\partial x}(\varphi(t))\,z.
$$

Since f is of class C^1 and $cl\,\Gamma_+$ is compact, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|z_1| \leqslant \delta$, $|z_2| \leqslant \delta$ implies

$$
|F(t, z_2) - F(t, z_1)| \leqslant \varepsilon\,|z_2 - z_1|. \tag{4.11}
$$

If $0 < \gamma < \alpha$, consider the Banach space

$$
\mathscr{B}_\gamma = \{z \in C([0, \infty) \to \mathbb{R}^n) \colon \|z\| < \infty\},
$$

where $\|z\| = \sup_{t \geqslant 0} e^{\gamma t}\,|z(t)|$. If $z \in \mathscr{B}_\gamma$, $\|z\| \leqslant \delta$ and $\xi \in \mathbb{R}^n$, let $\mathscr{T}_\xi z$ be defined by

$$
\begin{aligned}
\mathscr{T}_\xi z(t) = Y(t)P_2\xi &+ \int_0^t Y(t)\,P_2 Y^{-1}(s)\,F(s, z(s))\,ds\\
&- \int_t^\infty Y(t)\,P_1 Y^{-1}(s)\,F(s, z(s))\,ds.
\end{aligned}
\tag{4.12}
$$

Now, from (4.10), (4.11) and (4.12)

$$\begin{aligned}|\mathcal{T}_\xi z(t)| &\leqslant C\left[|P_2\xi|\, e^{-\alpha t} + \varepsilon\, \|z\| \int_0^t e^{-\alpha(t-s)}\, e^{-\gamma s}\, ds + \int_t^\infty e^{-\gamma s}\, ds\right] \\ &\leqslant C\left[|P_2\xi|\, e^{-\alpha t} + \varepsilon\, \|z\| \left(\frac{e^{-\gamma t}}{\alpha-\gamma} + \frac{e^{-\gamma t}}{\gamma}\right)\right] \\ &= C\,|P_2\xi|\, e^{-\alpha t} + \theta\, \|z\|\, e^{-\gamma t}, \qquad \theta = C\varepsilon \frac{\alpha}{\gamma(\alpha-\gamma)}. \end{aligned} \tag{4.13}$$

Choose ε and ξ so that $0 < \theta < 1$ and $C\,|P_2\xi| \leqslant (1-\theta)\delta$ and therefore

$$e^{\gamma t}\, |\mathcal{T}_\xi z(t)| \leqslant (1-\theta)\, \delta + \theta\delta = \delta.$$

Let $z, z_i \in \mathcal{B}_\gamma$, $\|z\| \leqslant \delta$, $\|_z i\| \leqslant \delta$, $i = 1, 2$, and $\xi \in \mathbb{R}^n$, $|\xi| \leqslant (1-\theta)\, \delta / C\,|P_2|)$. Then $\mathcal{T}_\xi z \in \mathcal{B}_\gamma$ and $\|\mathcal{T}_\xi z\| \leqslant \delta$. A similar estimate shows $\|\mathcal{T}_\xi z_2 - \mathcal{T}_\xi z_1\| \leqslant \theta\, \|z_2 - z_1\|$ so that $\mathcal{T}_\xi$ is a uniform contraction on $\mathcal{B}_\gamma$ with respect to the domain specified for ξ; see Hale [10, p. 6]. It follows that $\mathcal{T}_\xi$ has a unique fixed point $z(\cdot, \xi) \in \mathcal{B}_\gamma$ which is continuous in ξ. From (4.12) $z(t, 0) = 0$, $z(t, \xi) = z(t, \xi_2)$ where $\xi_2 = P_2\xi$ and the function

$$(t, \xi_2) \mapsto x(t, \xi_2) := \varphi(t) + z(t, \xi_2) = \varphi(t, x_0 + z(0, \xi_2)) \tag{4.14}$$

is a solution of (4.1). Note that $z(t, \xi_2)$, originally defined only for $t \geqslant 0$, is defined by (4.14) and continuous on a neighbourhood of $(t, \xi_2) = (0, 0)$. The preceding discussion shows that, for $|t|$, $|s|$, $|\xi_2|$, $|\eta_2|$ close to zero,

$$x(t, \xi_2) - x(s, \eta_2) = f(x_0)(t-s) + (\xi_2 - \eta_2) + o(1)\, |t-s| + o(1)\, |\xi_2 - \eta_2|. \tag{4.15}$$

Since $P_1 f(x_0) = f(x_0)$, $P_2 f(x_0) = 0$, and $P_1 + P_2 = I$, from Proposition 3.1, this implies that the continuous map (4.14) is one-to-one on a neighbourhood U of $(0, 0)$. By the Invariance of Domain Theorem, [15, p. 50], $x(U)$ is a neighbourhood of $x(0, 0) = x_0$. Thus there exists $\rho > 0$ such that $|x - x_0| < \rho$ implies $x = x(s, \xi_2)$ where $(s, \xi_2) \in U$. Then $\varphi(t-s, x) = \varphi(t-s, \varphi(s, x_0 + z(0, \xi_2))) = \varphi(t, x_0 + z(0, \xi_2)) = \varphi(t) + z(t, \xi_2)$ so that, with $h(x) = -s$, $\lim_{t\to\infty} |\varphi(t + h(x), x) - \varphi(t)| = \lim_{t\to\infty} |z(t, \xi_2)| = 0$ since $z(\cdot, \xi_2) \in \mathcal{B}_\gamma$. This shows that the semiflow is phase asymptotic at x_0 and therefore, from Theorem 2.1(b), Ω is a periodic orbit. To complete the proof of Part (a), let $|(t, \xi_2)| = (t^2 + |\xi_2|^2)^{12}$. The function $(t, \xi_2) \mapsto x(t, \xi_2)$ is bi-lipschitzian in a neighbourhood of $(0, 0)$, from (4.15), so that

$|\xi_2| \leqslant |(s, \xi_2) - (0, 0)| \leqslant N\,|x - x_0|$, where N is a lipschitz constant associated with the inverse of this function. Now (4.12) and (4.13) imply

$$|\varphi(t+h, x) - \varphi(t, x_0)| = |z(t, \xi_2)| \leqslant \frac{C}{1-\theta}\,|\xi_2|\, e^{-\gamma t}$$

$$\leqslant \frac{CN}{1-\theta}\,|x - x_0|\, e^{-\gamma t} \leqslant H\,|x - x_0|\, e^{-\gamma t}$$

as asserted. The constants H, ρ may be chosen independent of $x_0 \in \Omega$ since Ω is compact and $x \mapsto \varphi(t, x)$ is a diffeomorphism for each t.

To prove Theorem 4.1(b), suppose $x_0 \in \Omega$ is an equilibrium. Since (4.9) is satisfied with $x = x_0$, all of the eigenvalues λ_i of the Jacobian matrix $(\partial f/\partial x)(x_0)$ satisfy $Re\,\lambda_i < 0$ with at most one exception. This establishes the asserted stability character of the semiflow at x_0. Moreover, $x_0 \in \Omega$, $f(x_0) = 0$ implies $\liminf_{t \to \infty} |\dot{\varphi}(t)| = \liminf_{t \to \infty} |f(\varphi(t))| = 0$ for any solution $\varphi(\cdot)$ whose omega limit set contains x_0. Then $\lim_{t \to \infty} |f(\varphi(t))| = \lim_{t \to \infty} |\dot{\varphi}(t)| = 0$, if (4.2) is uniformly stable, since $\dot{\varphi}(\cdot)$ is a solution of (4.2). Therefore $f(x_0) = 0$ for each $x_0 \in \Omega$. If there is more than one point in Ω, then with each $x_0 \in \Omega$ there is associated a $(n-1)$-dimensional stable manifold and a 1-dimensional centre manifold which contains a continuum of equilibria in Ω, each with a stable manifold of dimension $n-1$. The Centre Manifold Theorem, [13, p. 48], implies that every orbit which intersects a neighbourhood of x_0 is asymptotic to an orbit in the centre manifold. Therefore $\lim_{t \to \infty} \varphi(t) = x_0$ so that, in fact, $\Omega = \{x_0\}$ when (4.2) is uniformly stable. ∎

5. Discrete Semiflows, Periodic Systems and Massera's Theorem

Let $(t, x) \mapsto f(t, x)$ be a continuous function from $\mathbb{R} \times U$ to $\mathbb{R}^n$ where $U \subset \mathbb{R}^n$ is open. Suppose $\omega > 0$, $f(t+\omega, x) = f(t, x)$, if $(t, x) \in \mathbb{R} \times U$, and that solutions of

$$\dot{x} = f(t, x) \tag{5.1}$$

are uniquely determined by initial conditions. In [18, Theorem 1] Massera shows that, when $n = 1$, the existence of a bounded solution ($U = \mathbb{R}$) implies the existence of a ω-periodic solution. This is clearly related to the Poincaré-Bendixson Theorem from which it can be readily deduced if we consider an autonomous 2-dimensional system $\dot{r} = f(\theta, r)$, $\dot{\theta} = 1$ in polar coordinates with $\omega = 2\pi$. When $n = 2$, [20, Theorem 2] shows that same conclusion holds if the additional assumption is made that each solution of (5.1) exists on a ray of the form (t_0, ∞). However this result does not hold without additional hypotheses for $n > 2$. Many interesting results provide examples of additional hypotheses which yield higher dimensional versions of Massera's Theorem. Massera shows in [20] Theorem 4, with a proof which he attributes to Bohnenblust, that when $f(t, x) = A(t)\,x + b(t)$, where $A(\cdot)$ and $b(\cdot)$ are $n \times n$ and $n \times 1$ continuous, ω-periodic matrix-valued functions respectively, the existence of bounded solution of (5.1) implies the existence of a periodic solution. Smith provides an analogue of his higher-dimensional Poincaré-Bendixson Theory in terms of guiding functions to obtain a similar conclusion in [33]. Practical criteria are developed for a nonautonomous version of the feedback control equation. In [29] Theorem 4, Sell proves that, if $U = \mathbb{R}^n$ and (5.1) has a solution $\psi(\cdot)$ which is bounded and uniformly asymptotically stable, then it has a *harmonic solution*: a periodic solution of period $k\omega$ where k is an integer $\geqslant 1$. This solution is also uniformly asymptotically stable. Here we show that a weaker restriction than uniform asymptotic stability of $\psi(\cdot)$ is sufficient to imply the existence of a harmonic solution. We also provide a Poincaré-type sufficient condition for the existence and uniform asymptotic stability of a harmonic solution in terms of the variational equation of (5.1) with respect to $\psi(\cdot)$ when $f(t, \cdot)$ is C^1 for each t:

$$\dot{y} = \frac{\partial f}{\partial x}(t, \psi(t))\, y. \tag{5.2}$$

A comprehensive discussion of Massera's Theorem is given in Yoshizawa 31. Chap. VII]. Yoshizawa [37] further develops Sell's result and proves a similar theorem for functional differential equations; he also has many references to earlier work and brief descriptions of the results. Pliss [26, Chap. I] also gives results of this type together with many interesting applications.

We first develop a Poincaré criterion for discrete semiflows in $\mathbb{R}^n$ and deduce the result for (5.1) from this. Let $x \mapsto \varphi(x)$ be a continuous function with open domain U in $\mathbb{R}^n$ and range in $\mathbb{R}^n$. Then $\varphi(0, x) = x$, $\varphi(k, x) = \varphi \circ \varphi(k-1, x)$, $k = 1, 2, ...,$ defines a semiflow as described in §2 with $\mathbb{T} = \mathbb{Z}_+$, $X = U_0$ where U_0 is any subset of U such that $\varphi(U_0) \subset U_0$. The semiflow will be called *asymptotic* at $S \subset U$ if there is a $\rho > 0$ such that $x_0 \in S$, $x \in U$ and $|x - x_0| < \rho$ implies $\lim_{k \to \infty} |\varphi(k, x) - \varphi(k, x_0)| = 0$. This means that the semiflow is phase asymptotic at S as defined in Section 2 with phase function $x \mapsto h(x) = 0$ in (2.1) for each $x_0 \in S$. When φ is C^1 and $(\partial\varphi/\partial x)$ is nonsingular on U, then the nonautonomous recursion

$$y_{k+1} = \frac{\partial \varphi}{\partial x}(x_k)\, y_k \tag{5.3}$$

is the variational equation of the semiflow with respect to $\Gamma_+(x_0)$, if $x_0 \in U$, and $x_k = \varphi(k, x_0)$ exists for each $k \in \mathbb{Z}_+$.

THEOREM 5.1. *Suppose the discrete semiflow φ has a Lagrange stable orbit Γ_+ with omega limit set Ω.*

(a) *If φ is asymptotic (and Lyapunov stable) at Γ_+, then Ω is a periodic orbit and φ is asymptotic (and Lyapunov stable) at $cl\,\Gamma_+$.*

(b) *If φ is C^1 and $(\partial\varphi/\partial x)$ is nonsingular, suppose the recursion (5.3) is uniformly asymptotically stable for some $x_0 \in cl\,\Gamma_+$. Then Ω is a periodic orbit and there exist positive constants H, α, ρ such that $\alpha < 1$ and*

$$|\varphi(k, x) - \varphi(k, x_0)| \leqslant H\,|x - x_0|\,\alpha^k, \qquad k = 0, 1, 2, ...,$$

if $x_0 \in \Omega$ and $|x - x_0| < \rho$.

Proof. Part (a) may be proved as in Theorem 2.1 with $h(x) = 0$. To prove part (b), recall that solutions $\{y_k\}$ of (5.3) satisfy

$$y_k = \frac{\partial\varphi}{\partial x}(x_{k-1})\,\frac{\partial\varphi}{\partial x}(x_{k-2})\cdots\frac{\partial\varphi}{\partial x}(x_j)\,y_j = \frac{\partial}{\partial x}(k-j, x_j)\,y_j, \qquad 0 \leqslant j \leqslant k.$$

The recursion is uniformly asymptotically stable if and only if

$$\left|\frac{\partial}{\partial x}\varphi(k, x)\right| \leqslant K\alpha^k \tag{5.4}$$

for each $x \in \Gamma_+(x_0)$, $k \in \mathbb{Z}_+$, where K, α are constants, $0 < \alpha < 1$. If $x \in U$ and $\varphi(k, x)$ exists define z_k by $\varphi(k, x) = z_k + x_k$. If z_{k+1} exists, then $z_{k+1} = \varphi(z_k + x_k) - \varphi(x_k)$ and

$$z_{k+1} = \frac{\partial\varphi}{\partial x}(x_k)\,z_k + F(k, z_k) \tag{5.5}$$

where $F(k, z) = \varphi(z + x_k) - \varphi(x_k) - (\partial\varphi/\partial x)(x_k)\,z$. It follows that

$$z_k = \frac{\partial}{\partial x}\varphi(k, x_0)\,z_0 + \sum_{j=1}^{k}\frac{\partial}{\partial x}\varphi(k-j, x_j)\,F(j-1, z_{j-1}) \tag{5.6}$$

if z_k exists. Conversely, if a sequence $\{z_k\}$ satisfies (5.6), it is a solution of the recursion (5.5) and $\varphi(k, x) = z_k + x_k$, with $x = z_0 + x_0$, for each $k \in \mathbb{Z}_+$. If $cl\,\Gamma_+(x_0)$ is a compact subset of U, then for each $\varepsilon > 0$ there exists $\delta > 0$

such that $|z|<\delta$ implies $z+x_k \in U$ and $|F(k, z)| < \varepsilon\,|z|$. Therefore, if $|z_j| < \delta$, $j=0, ..., k-1$ and (5.4) is satisfied, then (5.6) implies

$$|z_k| \leqslant K\left[\,|z_0|\,\alpha^k + \varepsilon \sum_{j=1}^{k} |z_{j-1}|\,\alpha^{k-j}\right]. \tag{5.7}$$

Choose $\varepsilon < \alpha/K$ and, with corresponding δ, let $|z_0| < (\alpha/K - \varepsilon)\delta = \rho$. Then, by induction from (5.7),

$$\begin{aligned} |z_k| &\leqslant [1 + K\varepsilon/\alpha + (K\varepsilon/\alpha)^2 + \cdots + (K\varepsilon/\alpha)^k]\; K\,|z_0|\alpha^k \\ &< \frac{K\,|z_0|\,\alpha^{k+1}}{\alpha - \varepsilon K} < \delta \end{aligned} \tag{5.8}$$

and z_k is defined by (5.6) for all $k \in \mathbb{Z}_+$. Furthermore (5.8) implies

$$|\varphi(k, x) - \varphi(k, x_0)| \leqslant H\,|x - x_0|\,\alpha^k \tag{5.9}$$

if $|x - x_0| < \rho$ and $K\alpha(\alpha - \varepsilon K)^{-1} \leqslant H$ so that the semiflow is asymptotic at x_0. Since φ is C^1, (5.4) is satisfied for each $x \in cl\,\Gamma_+(x_0)$ and it may be assumed without loss of generality that $x_0 \in \Omega$. As in Theorem 2.1(b), Ω is a periodic orbit at which the semiflow is asymptotic. Since Ω is finite, the constants H, ρ may be chosen so that (5.9) is satisfied if $|x - x_0| < \rho$ for any $x_0 \in \Omega$. ▮

A solution $\psi(\cdot)$ of (5.1) will be called *asymptotic* if there exists $\rho > 0$ such that any solution $x(\cdot)$ with $|x(s) - \psi(s)| < \rho$ for some $s \geqslant 0$ satisfies $\lim_{t\to\infty} |x(t) - \psi(t)| = 0$.

Let $t \mapsto x(t, x_0)$ denote the solution $x(\cdot)$ of (5.1) such that $x(0) = x_0$. Then $\varphi(x_0) = x(\omega, x_0)$ defines a discrete semiflow as in the preceding discussion with $\varphi(k, x_0) = x(k\omega, x_0)$. Applying Theorem 5.1 to this semiflow we obtain the corollary.

Corollary 5.2. *Suppose $cl\{\psi(t) \colon t \in \mathbb{R}_+\}$ is a compact subset of U for the solution $\psi(\cdot)$ of* (5.1).

(a) *If $\psi(\cdot)$ is asymptotic* (*and uniformly stable*), *then there is a harmonic solution which is asymptotic* (*and uniformly stable*).

(b) *If $f(t, \cdot)$ is C^1 for each $t \geqslant 0$ and the variational equation* (5.2) *is uniformly asymptotically stable, then there is a harmonic solution $\theta(\cdot)$ and positive constants H, γ, ρ such that any solution $x(\cdot)$ with $|x(s) - \theta(s)| < \rho$ for some $s \geqslant 0$ satisfies*

$$|x(t) - \theta(t)| \leqslant H\,|x(s) - \theta(s)|\,e^{-\gamma(t-s)}, \qquad t \geqslant s.$$

We note that, when the compactness condition is satisfied, concrete examples of conditions which imply that (5.2) is uniformly asymptotically stable as required in Corollary 5.2(b) are given by

$$\int_s^t \mu(u, \psi(u))\, du \leqslant -\gamma(t-s), \qquad \text{if} \quad t-s \geqslant T, \tag{5.10}$$

for some constants $T, \gamma > 0$, where $(t, x) \mapsto \mu(t, x)$ is any one of the expressions

$$\text{(i)} \quad \mu = \sup \left\{ \frac{\partial f_r}{\partial x_r} + \sum_{q \neq r} \left| \frac{\partial f_q}{\partial x_r} \right| : 1 \leqslant r \leqslant n \right\}$$

$$\text{(ii)} \quad \mu = \sup \left\{ \frac{\partial f_r}{\partial x_r} + \sum_{q \neq r} \left| \frac{\partial f_r}{\partial x_q} \right| : 1 \leqslant r \leqslant n \right\}$$

$$\text{(iii)} \quad \mu = \lambda_1$$

where λ_1 is the largest eigenvalue of $1/2((\partial f^*/\partial x) + (\partial f/\partial x))$. The formula (5.10) is the analogue for periodic systems of the concrete Poincaré conditions (4.4) for autonomous systems. The preceding observation may be verified by considering the expressions $|y| = \sup_i |y_i|$, $\sum_i |y_i|$, $(y^*y)^{1/2}$ respectively as Lyapunov functions for (5.2).

Acknowledgments

The authors thank the referee for several helpful suggestions and comments. M.Y.L. is grateful for the hospitality of Centre de Recherches Mathématiques, Université de Montréal, 1993–1994, where his research was supported by a NSERC Postdoctoral Fellowship. The research of J.S.M. was supported by NSERC Grant 7197.

References

1. N. P. Bhatia and G. P. Szego, "Dynamical Systems: Stability Theory and Application," Springer-Verlag, Berlin, 1967.
2. J. Cronin, "Differential Equations, Introduction and Qualitative Theory," Dekker, New York, 1980.
3. J. Cronin, A criterion for asymptotic stability, *J. Math. Anal. Appl.* **74** (1980), 247–269.
4. E. A. Coddington and N. Levinson, "Theory of Ordinary Differential Equations," McGraw–Hill, New York, 1955.
5. W. A. Coppel, "Stability and Asymptotic Behavior of Differential Equations," Heath, Boston, 1965.
6. W. A. Coppel, "Dichotomies in Stability Theory," Springer-Verlag, New York, 1978.

7. Ju. L. Daleckiĭ and M. G. Krein, "Stability of Solutions of Differential Equations in Banach Spaces," Amer. Math. Soc. Translations, Providence, RI, 1974.
8. B. Fiedler and J. Mallet-Paret, A Poincaré-Bendixson theorem for scalar reaction diffusion equations, *Arch. Rational Mech. Anal.* **107** (1989), 325–345.
9. M. Fiedler, Additive compound matrices and inequality for eigenvalues of stochastic matrices, *Czechoslovak Math. J.* **24** (99) (1974), 392–402.
10. J. K. Hale, "Ordinary Differential Equations," Wiley-Interscience, New York, 1969.
11. P. Hartman, "Ordinary Differential Equations," Wiley, New York, 1964.
12. M. W. Hirsch, Systems of differential equations which are competitive or cooperative I: limit sets, *SIAM J. Math. Anal.* **13** (1982), 432–439.
13. U. Kirchgraber and K. J. Palmer, "Geometry in the neighborhood of invariant manifolds of maps and semiflows and linearization," Longman, Harlow, 1990.
14. Y. Li and J. S. Muldowney, On Bendixon's criterion, *J. Differential Equations* **106** (1993), 27–39.
15. M. Y. Li and J. S. Muldowney, On R. A. Smith's autonomous convergence theorem, *Rocky Mountain J. Math.* **25** (1995), 365–379.
16. N. G. Lloyd, "Degree Theory," Cambridge Univ. Press, Cambridge, U.K., 1978.
17. N. G. Lloyd, A note on the number of limit cycles in certain two-dimensional systems, *J. London Math. Soc.* (2) **20** (1979), 277–287.
18. J. Mallet-Paret and H. L. Smith, The Poincaré-Bendixson Theorem for monotone cyclic feedback systems, *J. Dynam. Diff. Eqns.* **2** (1990), 367–421.
19. A. W. Marshall and I. Olkin, "Inequalities: Theory of Majorization and its Applications," Academic Press, New York, 1979.
20. J. L. Massera, The existence of periodic solutions of systems of differential equations, *Duke Math. J.* **17** (1950), 457–475.
21. J. S. Muldowney, On the dimension of the zero or infinity tending sets for linear differential equations, *Proc. Amer. Math. Soc.* **83** (1981), 705–709.
22. J. S. Muldowney, Dichotomies and asymptotic behaviour for linear differential systems, *Trans. Amer. Math. Soc.* **283** (1984), 465–484.
23. J. S. Muldowney, Compound matrices and ordinary differential equations, *Rocky Mountain J. Math.* **20** (1990), 857–871.
24. V. V. Nemytskii and V. V. Stepanov, "Qualitative Theory of Differential Equations," Princeton Univ. Press, Princeton, NJ, 1960.
25. R. Ortega, Topological degree and stability of periodic solutions for certain differential equations, *J. London Math. Soc. (2)* **42** (1990), 505–516.
26. V. A. Pliss, "Nonlocal Problems of the Theory of Oscillations," Academic Press, New York, 1966.
27. S. H. Saperstone, "Semidynamical Systems in Infinite Dimensional Spaces," Applied Mathematical Sciences, Vol. 37, Springer-Verlag, New York, 1981.
28. B. Schwarz, Totally positive differential systems, *Pacific J. Math.* **32** (1970), 203–229.
29. G. R. Sell, Periodic solutions and asymptotic stability, *J. Differential Equations* **2** (1966), 143–157.
30. H. L. Smith, Systems of ordinary differential equations which generate an order preserving semiflow, *SIAM Rev.* **30** (1988), 87–113.
31. R. A. Smith, The Poincaré-Bendixson Theorem for certain differential equations of higher order, *Proc. Roy. Soc. Edinburgh Sect. A* **83** (1979), 63–79.
32. R. A. Smith, Existence of periodic orbits of autonomous ordinary differential equations, *Proc. Roy. Soc. Edinburgh Sect. A* **85** (1980), 153–172.
33. R. A. Smith, Massera's convergence theorem for periodic nonlinear differential equations, *J. Math. Anal. Appl.* **120** (1986), 679–708.

34. R. A. SMITH, Poincaré-Bendixson theory for certain retarded functional differential equations, *Differential Integral Equations* **5** (1992), 213–240.
35. R. TEMAM, "Infinite-Dimensional Dynamical Systems in Mechanics and Physics," Springer-Verlag, New York, 1988.
36. T. YOSHIZAWA, "Stability Theory by Liapunov's Second Method," Math. Soc. Jap., Tokyo, 1966.
37. T. YOSHIZAWA, "Stability and existence of periodic and almost periodic solutions," Proceeding United States–Japan Seminar on Differential and Functional Equations (W. A. Harris and Y. Sibuya, Eds.), pp. 411–427, Benjamin, New York, 1967.