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POINCARÉ'S STABILITY CONDITION FOR QUASI-PERIODIC ORBITS

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ABSTRACT. A criterion for the orbital asymptotic stability of a quasi-periodic solution to autonomous systems in any finite dimension is derived. A principal tool used is the theory of compound matrices and compound linear differential systems. Results generalize a stability criterion of Poincaré for periodic solutions of planar autonomous systems.

1. Introduction. Let function $x \mapsto f(x) \in \mathbf{R}^n$ be defined and C^1 in an open subset $D \subset \mathbf{R}^n$. A solution $\varphi(t)$ to the autonomous system

$$x' = f(x) \tag{1.1}$$

is said to be *quasi-periodic* with m basic frequencies if there exists a function $u \mapsto \Phi(u) \in \mathbf{R}^n$ which is Lipschitz continuous for $u \in \mathbf{R}^m$ and periodic of period 1 in each of its arguments, and m real numbers $\omega_1, \dots, \omega_m$ linearly independent over the rationals, such that

$$\varphi(t) = \Phi(\omega_1 t, \dots, \omega_m t). \tag{1.2}$$

Any such choice of $\omega_1, \dots, \omega_m$ will be called a set of *basic frequencies* for $\varphi(t)$.

Quasi-periodic functions belong to a larger class of uniformly almost periodic functions which can be characterized as functions that have a Fourier series. An equivalent definition of quasi-periodicity is to require that the frequency module for its Fourier series have a finite integral base, cf. [**1, 7**]. A quasi-periodic function with a single basic frequency ω is periodic with period $1/\omega$. A simple example of quasi-periodic function that is not periodic is $\sin t + \sin\sqrt{2}t$, which has two basic

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frequencies $1/2\pi$ and $1/\sqrt{2}\pi$. It is well known that, see [**1, 7**], if $\varphi(t)$ is as given in (1.2), then its orbit closure is the same as

$$\mathbf{T}^m = \mathrm{cl}\,\{\Phi(u_1, \dots, u_m) \in \mathbf{R}^n : (u_1, \dots, u_m) \in \mathbf{R}^m\}$$

which is a nondegenerate topological m-torus in $\mathbf{R}^n$. For this reason, quasi-periodic orbits are also called invariant tori.

As is the case for periodic orbits, study of stability is central to the investigation of many important properties of quasi-periodic orbits such as persistence under small perturbations of the vector field and bifurcations with respect to changes of certain parameters. For periodic orbits, stability is described in terms of the Floquet multipliers, see [**7, 8**]. In the case of quasi-periodic orbits, however, a general Floquet theory no longer exists. The stability is described using exponential rates in various solution subspaces of the linearized equation in terms of Lyapunov numbers, see [**6, 9, 17**], or using exponential dichotomies, see [**2, 15, 18**]. In both cases, verification of stability conditions for the concrete system is of great importance and still a great challenge, see [**17**].

When the space dimension $n = 2$, the following well-known stability criterion for periodic orbits is due to Poincaré, cf. [**8**].

Poincaré's stability criterion. *A periodic solution* $x = \varphi(t)$ *to* (1.1) *with period* ω *is orbitally stable with asymptotic phase if*

$$\int_0^{\omega} \operatorname{div} f(\varphi(t))\, dt < 0. \tag{1.3}$$

A generalization of Poincaré's criterion to periodic orbits in a space of any finite dimension is given in [**14**], which is subsequently applied to resolve a global stability problem in an epidemic model [**12**]. In the present paper we give in Theorem 4.1 a stability condition in the spirit of Poincaré's criterion for quasi-periodic orbits in $\mathbf{R}^n$ for $n \geq 3$. The proof uses certain dichotomy properties for the hull of the linear variational equation with respect to a quasi-periodic orbit. The key tool in our study, as in [**14**], is the theory of compound matrices and compound linear differential systems, which we will briefly describe in the next section. Our main results are given in Section 4, after we discuss properties of the linear variational equation for a quasi-periodic orbit in Section 3.

2. Compound matrices and compound equations. Let A be an $n\times n$ matrix, and also denote the linear operator on $\mathbf{R}^n$ it represents with respect to the standard basis of $\mathbf{R}^n$. Let "$\wedge$" denote the exterior product in $\mathbf{R}^n$. Two linear operators $A^{(k)}$ and $A^{[k]}$ on $\wedge^k\mathbf{R}^n$, the k-th exterior power of $\mathbf{R}^n$, are induced from A as follows; set

$$A^{(k)}(u_1\wedge\cdots\wedge u_k) = (Au_1)\wedge\cdots\wedge(Au_k)$$

and

$$A^{[k]}(u_1\wedge\cdots\wedge u_k) = \sum_{i=1}^{k} u_1\wedge\cdots\wedge(Au_i)\wedge\cdots\wedge u_k$$

for $u_1,\ldots,u_k\in\mathbf{R}^n$, and extend the definition linearly to all of $\wedge^k\mathbf{R}^n$. Their matrix representations with respect to the canonical basis of $\wedge^k\mathbf{R}^n$, also denoted by $A^{(k)}$ and $A^{[k]}$, are called the *k-th multiplicative and k-th additive compound matrix* of A, respectively. Both $A^{(k)}$ and $A^{[k]}$ are $N\times N$ matrices with $N=\binom{n}{k}$. The compounds have properties

$$(AB)^{(k)} = A^{(k)}B^{(k)}, \quad (A+B)^{[k]} = A^{[k]} + B^{[k]},$$

and

$$A^{(n)} = \det A, \qquad A^{[n]} = \operatorname{tr} A.$$

If $Y(t)$ is a fundamental matrix for a linear system

$$y'(t) = A(t)y(t), \tag{2.1}$$

with $A(\cdot)$ a continuous $n\times n$ matrix-valued function, then $Z(t) = Y^{(k)}(t)$ is a fundamental matrix for the *k-th compound system*

$$z'(t) = A^{[k]}(t)z(t). \tag{2.2}$$

From this, it follows that $z(t) = y_1(t)\wedge\cdots\wedge y_k(t)$ is a solution to (2.2) whenever $y_1(t),\ldots,y_k(t)$ are solutions of (2.1). If $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of A, then $\lambda_{i_1}\cdots\lambda_{i_k}$ and $\lambda_{i_1}+\cdots+\lambda_{i_k}$, $1\le i_1\le\cdots\le$

$i_k \leq n$, are the eigenvalues of $A^{(k)}$ and $A^{[k]}$, respectively. For a more detailed discussion of these compound matrices and compound differential systems, the reader is referred to [**14**].

The following Proposition 2.1 is a slightly more general version of a result (Proposition 3.1) in [**11**]. Its proof is essentially the same as in [**11**] so we omit it and refer the reader to [**11**]. Condition (a) in Proposition 2.1 requires that the k-th compound equation (2.2) be uniformly asymptotically stable, condition (b) requires that the solution space of equation (2.1) have a $(k-1)$-dimensional strongly stable subspace and assumption (c) specifies that solutions to (2.1) have bounded growth. Under these assumptions (2.1) has a dichotomy which splits its solution space into the strongly stable subspace and a complementary $(n-k+1)$-dimensional subspace which is uniformly asymptotically stable. This is also closely related to a result in [**14**] on the dimension of the subspace of zero-tending solutions to (2.1).

Proposition 2.1. *Assume that*

(a) *There exist constants* $K, \alpha > 0$ *such that*

$$|z(t)| < K|z(s)|e^{-\alpha(t-s)}, \quad 0 \leq s \leq t$$

for each solution $z(t)$ *of* (2.2).

(b) *There is a constant* $L > 1$ *and* $k-1$ *linearly independent solutions* $y_i(t)$, $i = 1, \dots, k-1$, *of* (2.1) *such that*

$$|y_i(t)| \leq L|y_i(s)|, \quad i = 1, \dots, k-1, \quad 0 \leq s, t.$$

(c) *There exist constants* $M > 0$ *and* β *such that*

$$|y(t)| \leq M|y(s)|e^{\beta(t-s)}, \quad 0 \leq s \leq t$$

for each solution $y(t)$ *of* (2.1).

Then, if $Y(t)$ *is a fundamental matrix of* (2.1), *there exists a projection* P *on* $\mathbf{R}^n$, $\operatorname{rank} P = k-1$, *and a constant* C *such that*

$$|Y(t)PY^{-1}(s)| \leq C, \quad 0 \leq s, t$$

and

$$|Y(t)(I-P)Y^{-1}(s)| \leq Ce^{-\alpha(t-s)}, \quad 0 \leq s \leq t.$$

In particular, (2.1) *is uniformly stable.*

Remark. Vectors $y_1(0), \dots, y_{k-1}(0)$ form a basis for the range of P.

3. The linear variational equation. Let $t \mapsto F(t)$ be an $r \times s$ matrix-valued function, bounded and uniformly continuous on $\mathbf{R}$. Arzéla-Ascoli's theorem implies that any sequence of real numbers $\{p_j\}$ contains a subsequence $\{q_j\}$ such that the translates $F(t+q_j)$ converge uniformly on every compact interval, and the limit function $\tilde{F}(t)$ is also bounded and uniformly continuous over $\mathbf{R}$. The set of all such limit functions $\tilde{F}(t)$ is called the *hull* of $F(t)$.

Let $[a] = a \pmod 1$ for a real number a, and let $\mathbf{Q}_m = \{(u_1, \dots, u_m) \in \mathbf{R}^m : |u_i| \leq 1, i = 1, \dots, m\}$. The following result is standard, see [**1**].

Lemma 3.1. *If* $\omega_1, \dots, \omega_m$ *are linearly independent over the rationals, then the set* $\{([\omega_1 t], \dots, [\omega_m t]) : t \in \mathbf{R}\}$ *is dense in* $\mathbf{Q}_m$.

For a continuous function $u \mapsto \Psi(u)$ from $\mathbf{R}^m$ to $\mathbf{R}^d$ that is periodic of period 1 in each of its arguments, the following result follows immediately from Lemma 3.1.

Lemma 3.2. *Assume that* $\omega_1, \dots, \omega_m$ *are linearly independent over the rationals. Then the hull of* $\Psi(\omega_1 t, \dots \omega_m t)$ *consists of functions of form* $\Psi(\omega_1 t + h_1, \dots, \omega_m t + h_m)$ *for* $(h_1, \dots, h_m) \in \mathbf{R}^m$.

Let $\varphi(t)$ be a quasi-periodic solution to (1.1) as given in (1.2). The linear variational equation of (1.1) with respect to $\varphi(t)$

$$y'(t) = \frac{\partial f}{\partial x}(\varphi(t))y(t), \tag{3.1}$$

where $\partial f/\partial x$ is the Jacobian matrix of f, is a quasi-periodic linear system. It is well known that the Floquet theory no longer holds in general for such systems, see [**2**]. What becomes essential in the study of stability of $\varphi(t)$ is the dichotomy of (3.1) and its hull.

By Lemma 3.2, the hull of $\varphi(t)$ consists of functions of the form

$$\varphi_h(t) \stackrel{\Delta}{=} \Phi(\omega_1 t + h_1, \dots, \omega_m t + h_m) \tag{3.2}$$

for $h = (h_1, \dots, h_m) \in \mathbf{R}^m$, all of which are solutions to (1.1). By a continuity argument, we see that the hull of $(\partial f/\partial x)(\varphi(t))$ consists of functions of form $(\partial f/\partial x)(\varphi_h(t))$. In the rest of the section, we are interested in the following family of linear systems, which is said to be the *hull* of linear system (3.1),

$$y'(t) = \frac{\partial f}{\partial x}(\varphi_h(t))y(t), \quad h \in \mathbf{R}^m. \tag{3.3}$$

From the Lipschitz continuity and periodicity of Φ, its partial derivatives $(\partial\Phi/\partial u_i)(u)$, $i = 1, \dots, m$, exist for almost all $u \in \mathbf{Q}_m$. Together with Lemma 3.1, this implies that the functions

$$\frac{\partial\Phi}{\partial u_i}(\omega_1 t, \dots, \omega_m t), \quad i = 1, \dots, m \tag{3.4}$$

are defined for almost all $t \in \mathbf{R}$.

Proposition 3.3. *The m functions in* (3.4) *are linearly independent quasi-periodic solutions to* (3.1).

Proof. The fact that $\partial\Phi/\partial u_i$ is a solution to (3.1) may be seen from formally taking the first variation in the equation

$$\frac{d}{dt}\Phi(\omega_1 t, \dots, \omega_i t + s, \dots, \omega_m t) = f(\Phi(\omega_1 t, \dots, \omega_i t + s, \dots, \omega_m t))$$

with respect to s at $s = 0$. In fact, a standard argument shows that $(\partial\Phi/\partial u_i)(\omega_1 t, \dots, \omega_m t)$ is continuous for all t and satisfies the equation (3.1).

To see the linear independence of $\partial\Phi/\partial u_i$, note that Φ is Lipschitz continuous from $\mathbf{Q}_m$ to $\mathbf{R}^n$ so that $\Phi(\mathbf{Q}_m)$ is a rectifiable m-set in $\mathbf{R}^n$, and thus has m-dimensional tangent space at all points except a set of zero m-dimensional Hausdorff measure, see [**4, 5, 13**]. Vectors $(\partial\Phi/\partial ui)(\omega_1 t, \dots, \omega_m t)$, $i = 1, \dots, m$, form a basis for the tangent space of $\Phi(\mathbf{Q}_m)$ at $\Phi(\omega_1 t, \dots, \omega_m t)$, and thus are linearly independent for some t. Consequently, they are linearly independent for all t from the uniqueness of solutions to linear systems. □

Corollary 3.4. *Linear system* (3.3), *for* $h = (h_1, \dots, h_m)$, *has* m *linearly independent solutions*

$$\frac{\partial\Phi}{\partial u_i}(\omega_1 t + h_1, \dots, \omega_m t + h_m), \quad i = 1, \dots, m. \tag{3.5}$$

Let $\varphi(t)$ be a quasi-periodic solution to (1.1) with m basic frequencies as given in (1.2). Since the matrix $(\partial f/\partial x)(\varphi_h(t))$ is bounded in t and the bound is uniform for h, each equation in the hull (3.3) satisfies condition (c) in Proposition 2.1 with $k = m + 1$ and the constants M, β are independent of h. The m quasi-periodic solutions to (3.3) as given in (3.5) are bounded in norm both from above and below. It thus follows from Corollary 3.4 that (3.3) also satisfies the condition (b) and the constant L is independent of h. The following result about the dichotomy of the hull (3.3) follows from Proposition 2.1.

Proposition 3.5. *Assume that the* $(m+1)-th$ *compound equation of the linear variational equation* (3.1)

$$z'(t) = \frac{\partial f}{\partial x}^{[m+1]}(\varphi(t))y(t) \tag{3.6}$$

is uniformly asymptotically stable. Then there exist constants $C, \alpha > 0$ *and a uniquely determined projection* P_h *such that*

(1) P_h *depends on* h *continuously,* $\operatorname{rank} P_h = m$, *and*

(2) *the following relations hold*

$$|Y_h(t) P_h Y_h^{-1}(s)| \le C, \quad 0 \le s, t \tag{3.7}$$

and

$$|Y_h(t)(I - P_h)Y_h^{-1}(s)| \le Ce^{-\alpha(t-s)}, \quad 0 \le s \le t. \tag{3.8}$$

Furthermore, the vectors $(\partial\Phi/\partial u_i)(0)$, $i = 1, \dots, m$, *form a basis for the range of* P_0.

Proof. The uniform asymptotic stability assumption on system (3.6) implies the existence of constants K and α such that

$$|z(t)| < K|z(s)|e^{-\alpha(t-s)}, \quad 0 \le s \le t$$

for each solution $z(t)$ of (3.6). This, together with the discussion in the preceding paragraph, implies that a fundamental matrix $Y_0(t)$ of (3.1) satisfies the conditions (a), (b) and (c) of Proposition 2.1 with $k = m + 1$. Therefore, there exists a projection P_0 with $\operatorname{rank} P_0 = m$ and a constant C, such that

$$|Y_0(t)P_0Y_0^{-1}(s)| \le C, \quad 0 \le s, t$$

and

$$|Y_0(t)(I - P_0)Y_0^{-1}(s)| \le Ce^{-\alpha(t-s)}, \quad 0 \le s \le t.$$

Thus (3.7) and (3.8) hold for system (3.1). In particular, $|Y_0(t)Y_0^{-1}(t)|$ is bounded. For each equation in the hull (3.3), its coefficient matrix $A_h(t) = (\partial f/\partial x)(\varphi_h(t))$ satisfies $A_h(t) = \lim A_0(t + t_k)$ locally uniformly in t for some sequence t_k, where $A_0(t) = (\partial f/\partial x)(\varphi(t))$. The linear system

$$y'(t) = A_0(t + t_k)y(t)$$

has a fundamental matrix $Y(t,k) = Y_0(t + t_k)Y_0^{-1}(t_k)$, which satisfies

$$|Y(t,k)P_kY^{-1}(s,k)| \le C, \quad -t_k \le s, t$$

and

$$|Y(t,k)(I - P_k)Y^{-1}(s,k)| \le Ce^{-\alpha(t-s)}, \quad -t_k \le s \le t,$$

where $P_k = Y_0(t_k)P_0Y_0^{-1}(t_k)$. Since $|P_k|$ has a uniform bound for k, and $\operatorname{rank} P_k = m$, by restricting to a subsequence we can assume $P_k \to P_h$ where P_h is a projection of rank m. Since $Y(t,k) \to Y_h(t)$ for every t, where $Y_h(t)$ is the fundamental matrix for (3.3), it follows that

$$|Y_h(t)P_hY_h^{-1}(s)| \le C, \quad -\infty \le s, t < +\infty$$

and

$$|Y_h(t)(I-P_h)Y_h^{-1}(s)| \le Ce^{-\alpha(t-s)}, \quad -\infty \le s \le t < +\infty$$

where all constants are independent of h. Thus systems in the hull (3.3) have a dichotomy over $\mathbf{R}$ with projection P_h. Since the projection associated with such a dichotomy is uniquely determined, it follows that P_h is well defined and depends on h continuously. □

4. Orbital stability. Let $\varphi(t)$ be a quasi-periodic solution to (1.1) with m basic frequencies $\omega_1, \dots, \omega_m$ as given in (1.2) and the m-torus $\mathbf{T}^m$ in (1.3) its orbit closure. We say $\varphi(t)$ is *orbitally stable* if the m-torus $\mathbf{T}^m$ is a stable compact invariant set. It is called *orbitally asymptotically stable with asymptotic phase* if it is orbitally stable and there exists $b > 0$ such that $d(x_0, \mathbf{T}^m) < b$ implies that $d(x(t,x_0), \varphi_h(t)) \to 0$ exponentially for some $h \in \mathbf{R}^m$.

Theorem 4.1. *A quasi-periodic solution with m basic frequencies as given in* (1.2) *is orbitally asymptotically stable with asymptotic phase if the* $(m+1)-$*th compound equation* (3.6) *is uniformly asymptotically stable.*

Remarks. (1) A periodic solution is a quasi-periodic solution with one basic frequency. Setting $m = 1$ in Theorem 4.1, we obtain an orbital stability condition of Muldowney [**14**, Theorem 3.4], which says that a nonconstant periodic solution $x = p(t)$ of (1.1) is orbitally asymptotically stable with asymptotic phase if the second compound equation

$$z'(t) = \frac{\partial f}{\partial x}^{[2]}(p(t))z(t) \tag{4.1}$$

is asymptotically stable. For this, we note that uniform asymptotic stability is equivalent to asymptotic stability for a periodic linear system. For an application of Muldowney's result in a global stability problem, see [**12**].

(2) When $n = 2$, $\partial f^{[2]}/\partial x = \operatorname{tr}(\partial f/\partial x) = \operatorname{div} f$, and thus (4.1) becomes a scalar equation. Poincaré's stability condition (1.4) is

a sufficient and necessary condition for the asymptotic stability of the scalar equation (4.1). Thus, Theorem 4.1 is a generalization of Poincaré's stability criterion to quasi-periodic solutions in $\mathbf{R}^n$, $n \geq 3$.

Let $|\cdot|$ denote a vector norm in $\mathbf{R}^n$ and the matrix norm in the space $C(\mathbf{R}^n \to \mathbf{R}^n)$ which it induces. The *Lozinskiĭ measure* $\mu(B)$ of an $n \times n$ matrix B is defined, see [**3**, p. 41],

$$\mu(B) = \lim_{h \to 0^+} \frac{|I + hB| - 1}{h}.$$

The Lozinskiĭ measure $\mu(B)$ is also referred to as the *logarithmic norm* of B. For a quasi-periodic solution $\varphi(t)$ of m basic frequencies, we define

$$q_{m+1} = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mu\left(\frac{\partial f}{\partial x}^{[m+1]}(\varphi(s))\right) ds \tag{4.2}$$

with respect to certain Lozinskiĭ measure μ. The limit in (4.2) exists since the integrand is quasi-periodic in t. Standard linear theory, cf. [**3**], implies that system (3.6) is uniformly stable if there exists T, $\alpha > 0$, such that

$$\int_s^t \mu\left(\frac{\partial f}{\partial x}^{[m+1]}(\varphi(\tau))\right) d\tau \leq -\alpha(t - s), \quad \text{if } t - s \geq T. \tag{4.3}$$

We thus have the following result.

Corollary 4.3. *A quasi-periodic solution with m basic frequencies is orbitally asymptotically stable with asymptotic phase if $q_{m+1} < 0$ for some Lozinskiĭ measure μ.*

Remark. The formula for the Lozinskiĭ measure with respect to some usual norms can be found in [**3**, p. 41], which can be used to derive concrete expressions of $\mu(\partial f^{[m+1]}/\partial x)$ in terms of partial derivatives of f. The reader is referred to [**10**] for these expressions. Also see [**11**] for these expressions in the case $m = 1$. In particular, if the vector norm is chosen as the Euclidean norm $\|z\| = (z^*z)$, then q_{m+1} takes the form

$$q_{m+1} = \lim_{t \to \infty} \frac{1}{t} \int_0^t (\lambda_1(s) + \cdots + \lambda_{m+1}(s))\, ds \tag{4.4}$$

where $\lambda_1(s) \geq \cdots \geq \lambda_n(s)$ are the eigenvalues of

$$\frac{1}{2}\left(\frac{\partial f^*}{\partial x}(\varphi(s)) + \frac{\partial f}{\partial x}(\varphi(s))\right),$$

and thus are quasi-periodic in s. This and other forms of q_{m+1} provide upper bounds for the sum of the first $m+1$ Lyapunov exponents for the solution $\varphi(t)$, see [**16**].

For instance, if $n = 3$, then $(\partial f^{[3]}/\partial x) = \operatorname{div} f$. The quantity q_3 for a 2-torus in $\mathbf{R}^3$ is

$$q_3 = \lim_{t\to\infty} \frac{1}{t}\int_0^t \operatorname{div} f(\Phi(\omega_1 s, \omega_2 s))\, ds.$$

Corollary 4.2. *Suppose that* $n = 3$. *Assume that* $\operatorname{div} f(x) < 0$ *in* D. *Then every quasi-periodic solution with two basic frequencies whose orbit stays in* D *is orbitally asymptotically stable with asymptotic phase.*

Proof of Theorem 4.1. By Proposition 3.5, there exists a projection P_h so that (3.7) and (3.8) hold for the hull (3.3). Substituting $x = z + \varphi_h(t)$ in (1.1), we obtain the equation

$$z' = \frac{\partial f}{\partial x}(\varphi_h(t))z + F_h(t, z) \tag{4.5}$$

where

$$F_h(t,z) = f(z + \varphi_h(t)) - f(\varphi_h(t)) - \frac{\partial f}{\partial x}(\varphi_h(t))z,$$

which is quasi-periodic in t and $F_h(t, 0) = 0$. Since f is of class C^1 and $\mathbf{T}^m$ is compact, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|z_1| \leq \delta$, $|z_2| \leq \delta$ implies

$$|F_h(t, z_2) - F_h(t, z_1)| \leq \varepsilon|z_2 - z_1|, \quad \text{for all } t \text{ and } h. \tag{4.6}$$

Let K, α be the constants in (3.7) and (3.8). Choose γ so that $0 < \gamma < \alpha$ and fix ε so small that

$$\theta = \varepsilon K[\gamma^{-1} + (\alpha - \gamma)^{-1}] < 1. \tag{4.7}$$

Consider the Banach space

$$\mathcal{B}_\gamma = \{z \in C([0,\infty) \to \mathbf{R}^n) : |z|_\gamma < \infty\},$$

where $|z|_\gamma = \sup_{t\geq 0} e^{\gamma t}|z(t)|$. For $z \in \mathcal{B}_\gamma$, $|z|_\gamma \leq \delta$ and ξ in the kernel of P_0 such that $|\xi| < (1-\theta)K^{-1}\delta$, define an operator $\mathcal{P}$ by

$$\begin{aligned} \mathcal{P}z(t) = {} & Y_h(t)(I-P_h)\xi \\ & + \int_0^t Y_h(t)(I-P_h)Y_h^{-1}(s)F_h(s,z(s))\,ds \\ & - \int_t^\infty Y_h(t)P_hY_h^{-1}(s)F_h(s,z(s))\,ds. \end{aligned} \tag{4.8}$$

Using (3.7), (3.8) and (4.6)–(4.8), it is easily verified that

$$|\mathcal{P}z|_\gamma \leq K|\xi| + \theta|z|_\gamma < \delta$$

and

$$|\mathcal{P}z_2 - \mathcal{P}z_1|_\gamma \leq \theta|z_2 - z_1|_\gamma.$$

It follows, by the contraction mapping principle, that, for any given vector ξ, the operator $\mathcal{P}$ has a unique fixed point $z_h(\cdot,\xi) \in \mathcal{B}_\gamma$ which is continuous in ξ, and

$$|z_h(t,\xi)|_\gamma \leq (1-\theta)^{-1}K|\xi|.$$

Moreover, for $t = 0$,

$$Z_h(0,\xi) = (I-P_h)\xi - \int_0^\infty P_hY_h^{-1}(s)F_h(s,z_h(s,\xi))\,ds$$

and thus

$$z_h(0,\xi) = (I-P_h)\xi + o(|\xi|) = \xi \quad \text{uniformly in } h$$

as $|\xi| \to 0$.

Since the fixed point $z_h(t,\xi)$ is a solution to (4.5), the function $x_h(t,\xi) = z_h(t,\xi) + \varphi_h(t)$ is a solution to (1.1). Hence, for each h, we have an $(n-m)$-dimensional family of solutions of (1.1) which converge to $\varphi_h(t)$ exponentially as $t \to \infty$. To establish the asymptotic orbital stability of $\varphi_h(t)$, we show that the mapping $(h, \xi) \mapsto \varphi_h(0, \xi)$ maps a neighborhood of the origin onto a neighborhood of $\varphi(0)$. By the continuous dependence of P_h on h and the Lipschitz continuity of Φ, when $|h| + |\xi| \to 0$, we have

$$z_h(0, \xi) = \xi + o(|\xi|),$$
$$\varphi_h(0) = \Phi(h) = \Phi(0) + \frac{\partial \Phi}{\partial u}(0)h + o(|h|),$$

and thus

$$x_h(0, \xi) = \varphi(0) + \frac{\partial \Phi}{\partial u}(0)h + \xi + o(|h| + |\xi|).$$

The linear map $(h, \xi) \mapsto (\partial\Phi/\partial u)(0)h + \xi$ is invertible, since $(\partial\Phi/\partial u)(0)h$ and ξ belong to the range and kernel of the projection P_0, respectively. Therefore, our assertion follows from the inverse mapping theorem. Consequently, for each solution $x(t)$ to (1.1) such that $d(x(0), \mathbf{T}^m)$ is sufficiently small, there exists a vector $h = (h_1, \ldots, h_m)$ such that $d(x(t), \Phi(\omega_1 t + h_1, \ldots, \omega_m t + h_m))$ tends to zero exponentially. This establishes Theorem 4.1. $\square$

Theorem 4.1 can be interpreted in terms of *generalized Lyapunov numbers*, see [**6, 17**]. The Lipschitz condition on Φ ensures that the linearized flow of (1.1), when restricted to the tangent bundle $T\mathbf{T}^m$ of $\mathbf{T}^m$, is neither contracting nor stretching. The uniform asymptotic stability condition on the $(m+1)$-th compound equation (3.6) implies that the linearized flow contracts in the normal direction of $\mathbf{T}^m$. This is a special case of "normal hyperbolicity" in that $\mathbf{T}^m$ only has a stable manifold. More precisely using terminologies and notations of [**17**], let $\phi(t)$ denote the flow generated by (1.1), and $T\mathbf{R}^n|_{\mathbf{T}^m} = T\mathbf{T}^m \oplus N$ be a bundle splitting with respect to the usual metric in $\mathbf{R}^n$ with N the normal bundle and

$$\Pi : T\mathbf{R}^n|_{\mathbf{T}^m} \longrightarrow N$$

be the orthogonal projection. Two linear operators $A_t(p)$ and $B_t(p)$ can be defined, cf. [**17**, p. 57],

$$A_t(p) \equiv D\phi_{-t}|_{\mathbf{T}^m}(p) : T_p\mathbf{T}^m \longrightarrow T_{\phi_t(p)}\mathbf{T}^m,$$
$$B_t(p) \equiv \Pi D\phi_t(\phi_{-t}(p))|_N : N_{\phi_{-t}(p)} \longrightarrow N_p,$$

at each point $p \in \mathbf{T}^m$, respectively. Here $T_p\mathbf{T}^m$ denotes the tangent space of $\mathbf{T}^m$ at p. Let $P = P_0$ and $Y(t) = Y_0(t)$ be the projection and fundamental matrix as in the proof of Theorem 4.1, the following relation holds: $A_t(p) = Y(-t)P$ and $B_t(p) = \Pi Y(t)|_N$. Therefore, we have $|A_t(p)| \le K$ and $|B_t(p)| \le Ke^{-\alpha t}$. To see the latter, we note that, for a vector $u \in N_{\phi_{-t}(p)}$, $u = Pu + (I - P)u$, and thus $B_t(p)u = \Pi Y(t)(I - P)u$, since $Y(t)Pu$ belongs to $T_p\mathbf{T}^m$, the range of P. Using Lemma 3.1.1 of [**17**], the generalized Lyapunov-type numbers can be calculated as

$$\nu(p) = \limsup_{t\to\infty} |B_t(p)|^{1/t} \le e^{-\alpha} < 1,$$

$$\sigma(p) = \limsup_{t\to\infty} \frac{\log|A_t(p)|}{-\log|B_t(p)|} \le \frac{\log|K|}{\alpha t} \to 0.$$

We thus may invoke the theory on the persistence of invariant manifolds, see [**6, 9, 17**], to arrive at the following result. The statement of the theorem follows that given in [**17**, Theorem 3.3.1].

Theorem 4.3. *Assume that*

(a) *f is of class C^r, $r \ge 1$.*

(b) *$\varphi(t)$ is a quasi-periodic solution to* (1.1) *with m basic frequencies as given in* (1.2).

(c) *The $(m+1)-$th compound equation* (3.6) *is uniformly asymptotically stable.*

Then

(1) *the orbit closure of $\varphi(t)$ is a C^r smooth m-torus $\mathbf{T}^m$,*

(2) *$\varphi(t)$ is orbitally asymptotically stable with asymptotic phase,*

(3) *For any C^r vector field $g(x)$ that is C^1 θ-close to $f(x)$, with θ sufficiently small, there exists a C^r smooth m-torus invariant for $g(x)$.*

REFERENCES

1. A.S. Besicovitch, *Almost periodic functions*, Dover, New York, 1954.

2. W.A. Coppel, *Dichotomies in stability theory*, Springer-Verlag, Berlin, 1978.

3. ———, *Stability and asymptotic behavior of differential equations*, Heath, Boston, 1965.

4. K.J. Falconer, *The geometry of fractal sets*, Cambridge Univ. Press, Cambridge, 1985.

5. H. Federer, *Geometric measure theory*, Springer-Verlag, New York, 1969.

6. N. Fenichel, *Persistence and smoothness of invariant manifolds for flows*, Indiana Univ. J. **21** (1971), 193–225.

7. J.K. Hale, *Ordinary differential equations*, John Wiley & Sons, New York, 1969.

8. P. Hartman, *Ordinary differential equations*, John Wiley & Sons, Baltimore, 1973.

9. M.W. Hirsch, C.C. Pugh and M. Shub, *Invariant manifolds*, Springer-Verlag, New York, 1977.

10. M.Y. Li and J.S. Muldowney, *Evolution of surface functionals and differential equations*, in *Ordinary and delay differential equations*, Pitman Research Notes in Math. **272** (1992), 144–148.

11. ———, *Phase asymptotic semiflows, Poincaré's condition, and the existence of stable limit cycles*, J. Differential Equations **124** (1996), 425–448.

12. ———, *Global stability for the SEIR Model in Epidemiology*, Math. Biosci. **125** (1995), 155–164.

13. F. Morgan, *Geometric measure theory, A beginner's guide*, Academic Press, Boston, 1998.

14. J.S. Muldowney, *Compound matrices and ordinary differential equations*, Rocky Mountain J. Math. **20** (1990), 857–872.

15. R.J. Sacker and G.R. Sell, *Existence of dichotomies and invariant splittings for linear differential systems*, J. Differential Equations **15** (1974), 429–458.

16. R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York, 1988.

17. S. Wiggins, *Normally hyperbolic invariant manifolds in dynamical systems*, Springer-Verlag, New York, 1994).

18. Y. Yi, *Stability of integral manifold and orbital attractivity of quasi-periodic motion*, J. Differential Equations **103** (1993), 278–322.

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