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A Criterion for Stability of Matrices

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A necessary and sufficient condition for the stability of $n \times n$ matrices with real entries is proved. Applications to asymptotic stability of equilibria for vector fields are considered. The results offer an alternative to the well-known Routh–Hurwitz conditions. © 1998 Academic Press

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1. INTRODUCTION

Let A be an $n \times n$ matrix and let $\sigma(A)$ be its spectrum. The stability modulus of A is $s(A) = \max\{\text{Re}\,\lambda: \lambda \in \sigma(A)\}$, and A is said to be *stable* if $s(A) < 0$. The stability of a matrix is related to the Routh–Hurwitz problem on the number of zeros of a polynomial that have negative real parts. Much research has been devoted to the latter. The first solution dates back to Sturm [21, p. 304]. Using Sturm's method, Routh developed a simple algorithm to solve the problem. Hurwitz independently discovered necessary and sufficient conditions for all of the zeros to have negative real parts, which are known today as the Routh–Hurwitz conditions. A good and concise account of the Routh–Hurwitz problem can be found in [5]. According to the Routh–Hurwitz conditions, a 2×2 real matrix A is stable if and only if $\text{tr}(A) < 0$ and $\det(A) > 0$; a 3×3 real matrix A is stable if and only if $\text{tr}(A) < 0$, $\det(A) < 0$, and $\text{tr}(A) \cdot a_2 < \det(A)$, where a_2 is the sum of all 2×2 principal minors of A.

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Stability of matrices is intimately related to the stability of stationary solutions of various kinds in the theory and applications of dynamical systems. Let f be a vector field defined in an open set of $\mathbf{R}^n$. An equilibrium point $\bar{x}$ of f is such that $f(\bar{x}) = 0$. It is *asymptotically stable* if, for each neighborhood U of $\bar{x}$, there exists a neighborhood V such that $\bar{x} \in V \subset U$, and $x(0) \in V$ implies that the solution $x(t)$ satisfies $x(t) \in U$ for all $t > 0$, and that $x(t) \to \bar{x}$ as $t \to \infty$ (see [11]). If f is C^1, then the asymptotic stability of $\bar{x}$ is closely related to the stability of $Df(\bar{x})$, the Jacobian matrix of f at $\bar{x}$; it is necessary that $s(Df(\bar{x})) \leq 0$ and sufficient that $s(Df(\bar{x})) < 0$ for $\bar{x}$ to be asymptotically stable. In many applications, the entries of $Df(\bar{x})$ contain system parameters, and the stability of $\bar{x}$ may have to be verified without knowing its explicit coordinates. The verification of the Routh–Hurwitz conditions for $Df(\bar{x})$ can be technically nontrivial, especially when $n \geq 3$.

In the present paper, a necessary and sufficient condition for the stability of an $n \times n$ matrix with real entries is derived (Theorem 3.1), using a simple spectral property of compound matrices. As an application and demonstration of the effectiveness of our criteria, the asymptotic stability of a unique endemic equilibrium of an epidemic model of SEIR type with varying total population is proved. The verification of the Routh–Hurwitz conditions for this problem, on the other hand, has presented substantial technical difficulties.

We outline in the next section the preliminaries for our main results, which are given in Section 3. In Section 4, we show how the conditions in Section 3 can be relaxed in the presence of certain constraints on the matrix. Stability of equilibria of differential equations that possess first integrals are considered as an example. An application to a system arising from an epidemic model is presented in Section 5.

2. PRELIMINARIES

Let $\mathbf{M}_m(\mathbf{T})$ be the linear space of $m \times m$ matrices with entries in $\mathbf{T}$, where $\mathbf{T}$ is either the field of real numbers $\mathbf{R}$ or complex numbers $\mathbf{C}$. An $M \in \mathbf{M}_m(\mathbf{T})$ will be identified with the linear operator on $\mathbf{T}^m$ that it represents. Let $\wedge$ denote the exterior product in $\mathbf{T}^m$, and let $1 \leq k \leq m$ be an integer. With respect to the canonical basis in the kth exterior product space $\wedge^k \mathbf{T}^m$, the *kth additive compound matrix* $M^{[k]}$ of M is a linear operator on $\wedge^k \mathbf{T}^m$ whose definition on a decomposable element $u_1 \wedge \cdots \wedge u_k$ is

$$M^{[k]}(u_1 \wedge \cdots \wedge u_k) = \textstyle\sum_{i=1}^{k} u_1 \wedge \cdots \wedge M u_i \wedge \cdots \wedge u_k. \qquad (2.1)$$

Definition over the whole $\wedge^k \mathbf{T}^m$ is done by linear extension. The entries of $M^{[k]}$ are linear relations of those of M. Let $M = (a_{ij})$. For any integer $i = 1, \dots, \binom{m}{k}$, let $(i) = (i_1, \dots, i_k)$ be the ith member in the lexicographic ordering of integer k-tuples such that $1 \le i_1 < \cdots < i_k \le m$. Then the entry in the ith row and the jth column of $Z = M^{[k]}$ is

$$z_{ij} = \begin{cases} a_{i_1 i_1} + \cdots + a_{i_k i_k}, & \text{if } (i) = (j), \\ (-1)^{r+s} a_{j_r i_s}, & \text{if exactly one entry } i_s \text{ of } (i) \text{ does not} \\ & \text{occur in } (j) \text{ and } j_r \text{ does not occur in } (i), \\ 0, & \text{if } (i) \text{ differs from } (j) \text{ in two or more entries}. \end{cases} \tag{2.2}$$

As special cases, we have $M^{[1]} = M$ and $M^{[m]} = \operatorname{tr}(M)$. The second compound matrix of an $m \times m$ matrix is given in the Appendix for $m = 2$, 3, and 4. For detailed discussions on compound matrices, the reader is referred to [8, 19]. Pertinent to the purpose of the present paper is a spectral property of $M^{[k]}$. Let $\sigma(M) = \{\lambda_i : i = 1, \dots, m\}$. Then the spectrum of $M^{[k]}$, $\sigma(M^{[k]}) = \{\lambda_{i_1} + \cdots \lambda_{i_k} : 1 \le i_1 < \cdots < i_k \le m\}$.

Let $|\cdot|$ denote a vector norm in $\mathbf{T}^m$ and the operator norm it induces in $\mathbf{M}_m(\mathbf{T})$. The *Lozinskiĭ measure* (or *logarithmic norm*) μ on $\mathbf{M}_m(\mathbf{T})$ with respect to $|\cdot|$ is defined by (see [5, p. 41]), for $M \in \mathbf{M}_n(\mathbf{T})$,

$$\mu(M) = \lim_{h \to 0^+} \frac{|I + hM| - 1}{h}. \tag{2.3}$$

A Lozinskiĭ measure $\mu(M)$ dominates the stability modulus $s(M)$, as the following lemma states. A simple proof can be found in [5].

LEMMA 2.1. *Let μ be a Lozinskiĭ measure. Then $s(M) \le \mu(M) \le |M|$.*

The Lozinskiĭ measures of $M = (a_{ij})$ with respect to the three common norms $|x|_\infty = \sup_i |x_i|$, $|x|_1 = \sum_i |x_i|$, and $|x|_2 = (\sum_i |x_i|^2)^{1/2}$ are

$$\begin{aligned} \mu_\infty(M) &= \sup_i \Big(\operatorname{Re} a_{ii} + \sum_{k, k \ne i} |a_{ik}| \Big), \\ \mu_1(M) &= \sup_k \Big(\operatorname{Re} a_{kk} + \sum_{i, i \ne k} |a_{ik}| \Big), \end{aligned} \tag{2.4}$$

and

$$\mu_2(M) = s\left(\frac{M + M^*}{2} \right),$$

respectively (see [5]), where M^* denotes the Hermitian adjoint of M. If M is real symmetric, then $\mu_2(M) = s(M)$. For a real matrix M, conditions $\mu_\infty(M) < 0$ or $\mu_1(M) < 0$ can be interpreted as $a_{ii} < 0$ for $i = 1, \dots, m$, and M is diagonally dominant in rows or in columns, respectively.

Let $P \in \mathbf{M}_m(\mathbf{T})$ be invertible. Define a new norm $|\cdot|_P$ by $|x|_P = |Px|$ and denote the corresponding Lozinskiĭ measure by μ_P. The next lemma follows directly from the definition of μ.

LEMMA 2.2. *Let P be an invertible matrix. Then*

$$\mu_P(M) = \mu(PMP^{-1}). \tag{2.5}$$

PROPOSITION 2.3. *For any matrix $M \in \mathbf{M}_m(\mathbf{T})$,*

$$s(M) = \inf\{\mu(M): \mu \text{ is a Lozinski\u{\i} measure on } \mathbf{M}_m(\mathbf{T})\}. \tag{2.6}$$

Proof. We first prove the case when $\mathbf{T} = \mathbf{C}$. The relation (2.6) obviously holds for diagonalizable matrices by Lemma 2.2 and the first two relations in (2.4). Furthermore, the infimum in (2.6) can be achieved if M is diagonalizable. The general case can be shown based on this observation, the fact that M can be approximated by diagonalizable matrices in $\mathbf{M}_m(\mathbf{C})$, and the continuity of $\mu(\cdot)$, which is implied by the property $|\mu(A) - \mu(B)| \leq |A - B|$ (see [5, p. 41]). Next, let $M \in \mathbf{M}_m(\mathbf{R})$. Then $s(M) = \inf\{\mu_C(M)\}$, where $\mu_{\mathbf{C}}$ are Lozinskiĭ measures with respect to vector norms $|\cdot|_{\mathbf{C}}$ in $\mathbf{C}^m$. When restricted to $\mathbf{R}^m$, each $|\cdot|_{\mathbf{C}}$ induces a vector norm $|\cdot|_{\mathbf{R}}$ on $\mathbf{R}^m$. Let $\mu_{\mathbf{R}}$ be the corresponding Lozinskiĭ measure. Then $\mu_{\mathbf{C}}(M) \geq \mu_{\mathbf{R}}(M)$ holds for $M \in \mathbf{M}_m(\mathbf{R})$, since $|M|_{\mathbf{C}} \geq |M|_{\mathbf{R}}$ holds for the induced matrix norms. This shows that (2.6) is valid for the case $\mathbf{T} = \mathbf{R}$ also. ∎

Remark. From the above proof, one can see that, if $\mathbf{T} = \mathbf{C}$, then

$$s(M) = \inf\left\{\mu_\infty(PMP^{-1}): P \in \mathbf{M}_m(\mathbf{C}) \text{ is invertible}\right\}.$$

The same relation holds if μ_∞ is replaced by μ_1. However, we would like to point out that similar relations no longer hold for the case $\mathbf{T} = \mathbf{R}$.

COROLLARY 2.4. *Let $M \in \mathbf{M}_m(\mathbf{T})$. Then $s(M) < 0 \Leftrightarrow \mu(M) < 0$ for some Lozinskiĭ measure μ on $\mathbf{M}_m(\mathbf{T})$.*

3. STABILITY CRITERIA

In this section, we assume that $A \in \mathbf{M}_n(\mathbf{R})$.

THEOREM 3.1. *For* $s(A) < 0$, *it is sufficient and necessary that* $s(A^{[2]}) < 0$ *and* $(-1)^n \det(A) > 0$.

Proof. By the spectral property of $A^{[2]}$, the condition $s(A^{[2]}) < 0$ implies that at most one eigenvalue of A can have a nonnegative real part. We thus may assume that all of the eigenvalues are real. It is then simple to see that the existence of one and only one nonnegative eigenvalue is precluded by the condition $(-1)^n \det(A) > 0$. ∎

Theorem 3.1 and Corollary 2.4 lead to the following result.

THEOREM 3.2. *Assume that* $(-1)^n \det(A) > 0$. *Then* A *is stable if and only if* $\mu(A^{[2]}) < 0$ *for some Lozinskiĭ measure* μ *on* $\mathbf{M}_N(\mathbf{R})$, $N = \binom{n}{2}$.

EXAMPLE. Let

$$A = \begin{bmatrix} -1 & -t^2 & -1 \\ t & -t-1 & t \\ t^2 & 1 & -t^2-1 \end{bmatrix}.$$

We show that $A(t)$ is stable for all $t > 0$. From the Appendix,

$$A^{[2]}(t) = \begin{bmatrix} -2-t & t & 1 \\ 1 & -2-t^2 & -t^2 \\ -t^2 & t & -2-t-t^2 \end{bmatrix}.$$

Considering that $A^{[2]}(t)$ is diagonally dominant in rows and $N = \binom{3}{2} = 3$, we let μ be the Lozinskiĭ measure on $\mathbf{M}_3(\mathbf{R})$ with respect to the norm $|x| = \sup\{|x_1|, |x_2|, |x_3|\}$. Then $\mu(A^{[2]}) = -1 < 0$ by (2.4). Moreover, $\det(A(t)) = -2t^5 - 3t^3 - 2t^2 - t - 1 < 0$. The stability of $A(t)$ follows from Theorem 3.2.

Let $\alpha \mapsto A(\alpha) \in \mathbf{M}_n(\mathbf{R})$ be a function that is continuous for $\alpha \in (a, b) \in \mathbf{R}$. An $\alpha_0 \in (a, b)$ is said to be a *Hopf bifurcation point* for $A(\alpha)$ if $A(\alpha)$ is stable for $\alpha < \alpha_0$, and there exists a pair of complex eigenvalues $\operatorname{Re} \lambda(\alpha) \pm \operatorname{Im} \lambda(\alpha)$ of $A(\alpha)$ such that $\operatorname{Re} \lambda(\alpha) > 0$, while the rest of the eigenvalues of $A(\alpha)$ have nonzero real parts for $\alpha > \alpha_0$. From the proof of Theorem 3.1 we see that $s(A^{[2]}) \leq 0$ precludes the existence of a pair of eigenvalues of A having positive real parts. This leads to the following result.

THEOREM 3.3. *If* $s(A^{[2]}(\alpha)) \leq 0$ *for* $\alpha \in (a,b)$, *then* (a,b) *contains no Hopf bifurcation points of* $A(\alpha)$.

COROLLARY 3.4. *No Hopf bifurcation points of* $A(\alpha)$ *exist in* (a,b) *if* $\mu(A^{[2]}(\alpha)) \leq 0$ *for some Lozinskiĭ measure* μ *and all* $\alpha \in (a,b)$.

Let $(x,\alpha) \mapsto f(x,\alpha) \in \mathbf{R}^n$ be a function defined for $(x,\alpha) \in \mathbf{R}^n \times \mathbf{R}$. Assume that f is C^1 in x, and both f and $\partial f/\partial x$ are continuous in α. We also assume that $f(x,\alpha) = 0$ has a solution $(\bar{x}(\alpha), \alpha)$ for all $\alpha \in (a,b)$. Then $\bar{x}(\alpha)$ is an equilibrium for the ordinary differential equation

$$x' = f(x,\alpha), \qquad \alpha \in (a,b). \tag{3.1}$$

Let $J(\alpha) = \partial f/\partial x\,(\bar{x}(\alpha),\alpha)$. We say that the differential equation (3.1) has a *Hopf bifurcation* from the equilibrium $\bar{x}(\alpha)$ at $\alpha = \alpha_0$ if α_0 is a Hopf bifurcation point for $J(\alpha)$.

THEOREM 3.5. *Assume that* $s(J^{[2]}(\alpha)) < 0$ *for* $\alpha \in (a,b)$. *Then* (3.1) *has no Hopf bifurcation from* $\bar{x}(\alpha)$ *in* (a,b). *Moreover, if* $(-1)^n \det(J(\alpha)) > 0$ *for* $\alpha \in (a,b)$, *then* $\bar{x}(\alpha)$ *is an asymptotically stable equilibrium of* (3.1) *for* $\alpha \in (a,b)$.

Remark. The condition $s(J^{[2]}(\alpha)) < 0$ for $\alpha \in (a,b)$, or equivalently, $\mu(J^{[2]}(\alpha) < 0$ for some Lozinskiĭ measure, does not preclude the possibility of $\bar{x}(\alpha)$ losing stability in (a,b). However, by Theorem 3.3, if $\bar{x}(\alpha)$ loses its stability at $\alpha = \alpha_0 \in (a,b)$, then the bifurcation that may occur when α passes through α_0 involves only equilibria.

We end this section by a brief discussion on the connection of the above results with the Markus–Yamabe Conjecture for an autonomous system in $\mathbf{R}^n$,

$$x' = f(x) \qquad x \in \mathbf{R}^n. \tag{3.2}$$

An equilibrium $\bar{x}$ of (3.2) is said to be *globally asymptotically stable* in $\mathbf{R}^n$ if it is asymptotically stable and all solutions of (3.2) satisfy $x(t) \to \bar{x}$ as $t \to \infty$. In such a case, $\bar{x}$ is necessarily the only equilibrium in $\mathbf{R}^n$.

Markus–Yamabe Conjecture. If $f(0) = 0$ and the eigenvalues of $Df(x)$ all have negative real parts for each $x \in \mathbf{R}^n$, then $x = 0$ is globally asymptotically stable in $\mathbf{R}^n$.

Markus and Yamabe formulated the conjecture in [18] for the case $n = 2$, which has recently been given an affirmative answer independently by several authors (see [7, 9, 10]). For $n \geq 3$, the Markus–Yamabe Conjecture has been proved to be false (see [1, 2, 4]). However, it is still of interest to see that how the conditions can be strengthened so that the conclusion may still hold in the case $n \geq 3$. This question has been considered by many authors (see [5, 6, 12, 13]). In the spirit of our Theorems 3.1 and 3.2, we formulate the following conjecture.

Conjecture. Assume that $f \in C^1(\mathbf{R}^n \to \mathbf{R}^n)$ satisfies $f(0) = 0$, $(-1)^n \det(Df(x)) > 0$, and $\mu(Df^{[2]}(x)) < 0$ for some Lozinskiĭ measure μ and for all $x \in \mathbf{R}^n$. Then $x = 0$ is globally asymptotically stable in $\mathbf{R}^n$.

Remarks. (1) Note that $Df^{[2]} = \operatorname{tr}(Df)$ when $n = 2$. In this case, the last two assumptions in our conjecture become $\det(Df(x)) > 0$ and $\operatorname{tr}(Df(x)) < 0$, respectively. It follows that, for planar systems, our conjecture is equivalent to the Markus–Yamabe Conjecture and thus holds true.

(2) If the condition $\mu(Df^{[2]}(x)) < 0$ is replaced by $s(Df^{[2]}(x)) < 0$, then the above conjecture is equivalent to the Markus–Yamabe Conjecture by Theorem 3.1. An autonomous convergence theorem of R. A. Smith [20] and Li and Muldowney [17] states that the condition $\mu(Df^{[2]}(x)) < 0$ for $x \in \mathbf{R}^n$ implies that each bounded semi-orbit of (3.2) converges to a single equilibrium. To prove our conjecture, it remains to see whether its conditions can ensure (a) all solutions are forwardly bounded, and (b) $x = 0$ is the only equilibrium.

(3) Any solution that establishes the validity of our conjecture may also offer a new proof of the Markus–Yamabe Conjecture for the case $n = 2$.

4. STABILITY IN THE PRESENCE OF CONSTRAINTS

Let $A \in \mathbf{M}_n(\mathbf{R})$. Assume that $n \geq 3$. A subspace $V \subset \mathbf{R}^n$ is *invariant* under A if $A(V) \subset V$. A is said to be stable with respect to an invariant subspace V if the restriction of A to V, $A|_V \colon V \to V$, is stable. Let matrix B be such that rank $B = r$, $0 < r < n - 1$, and

$$BA = 0. \tag{4.1}$$

Then $\ker B = \{x \in \mathbf{R}^n \colon Bx = 0\}$ satisfies $A(\mathbf{R}^n) \subset \ker B$. In particular, $\ker B$ is an $n - r$ dimensional invariant subspace of A. It is of interest to study the stability of A with respect to $\ker B$ when (4.1) holds.

LEMMA 4.1. *Let $V \subset \mathbf{R}^n$ be a subspace such that $A(\mathbf{R}^n) \subset V$ and* $\dim V = k < n$. *Then* 0 *is an eigenvalue of* A, *and there exist* $n - k$ *eigenvectors of* 0 *that do not belong to* V.

Proof. Let W be the quotient space $\mathbf{R}^n / V$. Then $\mathbf{R}^n \cong V \oplus W$ and $A(W) = \{0\}$, since $A(\mathbf{R}^n) \subset V$. This establishes the lemma. ∎

THEOREM 4.2. *Assume that A and B satisfy* (4.1) *and rank* $B = r$, $0 < r < n - 1$. *Then, for A to be stable with respect to* $\ker B$, *it is necessary and sufficient that*

(a) $s(A^{[r+2]}) < 0$, *and*

(b) $\limsup_{\epsilon \to 0^+} \text{sign}[\det(\epsilon I + A)] = (-1)^{n-r}$.

Proof. Let $\lambda_1, \dots, \lambda_{n-r}$ be the eigenvalues of $A|_{\ker B}$. By Lemma 4.1, the eigenvalues of A can be written as

$$\lambda_1, \dots, \lambda_{n-r}, \overbrace{0, \dots, 0}^{r},$$

and thus $\{\lambda_i + \lambda_j : 1 \le i < j \le n - r\} \subset \sigma(A^{[r+2]})$ by the spectral property of additive compound matrices discussed in Section 2. It follows that $s(A^{[r+2]}) < 0$ precludes the possibility of more than one λ_i, $1 \le i \le n - r$, have nonnegative real parts. For $\epsilon > 0$ sufficiently small,

$$\text{sign}(\det(\epsilon I + A)) = \text{sign}(\epsilon^r \lambda_1 \cdots \lambda_{n-r}).$$

The theorem can be proved using the same argument as in the proof of Theorem 3.1. ∎

Remark. If $r = n$ in (4.1), then B is of full rank and hence $A = 0$. If $r = n - 1$, then $\ker B$ is of dimension 1, and thus the eigenvalues of A are λ_1 and 0 of multiplicity $n - 1$. From the above proof, we know that Theorem 4.2 still holds in this case, if condition (a) is replaced by $\text{tr}(A) < 0$.

Let matrix B_1 be such that rank $B_1 = r$, $0 < r < n - 1$, and

$$B_1 A = \nu B_1 \tag{4.2}$$

for some scalar $\nu \neq 0$. Then $\ker B_1$ is an invariant subspace of A. A class of differential equations that give rise to the consideration of (4.2) can be found in [15, 16]. Noting that (4.2) is equivalent to $B_1(A - \nu I) = 0$, one can apply Theorem 4.2 to $A - \nu I$ and obtain the following result.

COROLLARY 4.3. *Assume that A and B_1 satisfy* (4.2) *and* rank $B_1 = r$, $0 < r < n - 1$. *Then A is stable with respect to* $\ker B_1$ *if and only if the following conditions hold*:

(a) $s(A^{[r+2]}) < (r + 2)\nu$.

(b) $(\text{sign}\, \nu)^r (-1)^{n-r} \det(A) > 0$.

A function $x \mapsto H(x) \in \mathbf{R}$ defined for $x \in \mathbf{R}^n$ is said to be a *first integral* for system (3.2) if $H(x(t, x_0)) = H(x_0)$ for all $t \ge 0$, where $x(t, x_0)$ denotes the solution to (3.2) such that $x(0, x_0) = x_0$. Suppose that (3.2) has

r first integrals $H_i(x)$, $i = 1, \dots, r$. Then each trajectory of (3.2) stays on a level set

$$\Gamma = \{x : H_i(x) = c_i,\ i = 1, \dots, r\},$$

where $c_i = H_i(x_0)$. Assume that H_i is smooth and that the gradient vectors ∇H_i satisfy

$$\nabla H_1(x) \wedge \cdots \wedge \nabla H_r(x) \neq 0, \qquad \text{for all } x \in \Gamma,$$

which is equivalent to the linear independence of $\nabla H_i(x)$. Then Γ is a $n - r$-dimensional submanifold of $\mathbf{R}^n$. If $r = n$, then $f \equiv 0$. If $r = n - 1$, then (3.2) is integrable and the case is trivial. We assume that $0 < r < n - 1$.

Let $\varphi_t : \mathbf{R}^n \to \mathbf{R}^n$ be the flow of (3.2) defined by $\varphi_t(x_0) = x(t, x_0)$. Then φ_t is a diffeomorphism on $\mathbf{R}^n$ for each t and $\varphi_t(\Gamma) \subset \Gamma$. This implies that the tangent space $T_{x_0}\Gamma$ of Γ at x_0 satisfies

$$D\varphi_t(x_0)(T_{x_0}\Gamma) \subset T_{\varphi_t(x_0)}\Gamma.$$

If $\bar{x} \in \Gamma$ is an equilibrium, then $T_{\bar{x}}\Gamma$ is an invariant subspace of $D\varphi_t(\bar{x})$. An equilibrium $\bar{x} \in \Gamma$ is said to be *asymptotically stable with respect to* Γ if it is so with respect to the restriction $\varphi_t|_\Gamma$ of the flow φ_t on Γ. Using a local chart on Γ, one can show that a sufficient condition for $\bar{x}$ to be asymptotically stable with respect to Γ is that it is so under the restriction $D\varphi_t(\bar{x})|_{T_{\bar{x}}\Gamma}$ of the linearized flow $D\varphi_t(\bar{x})$ on the invariant subspace $T_{\bar{x}}\Gamma$.

Differentiating the identity $H_i(x(t, x_0)) = H_i(x_0)$ with respect to x_0 leads to

$$\nabla H_i(x(t, x_0))^* D\varphi_t(x_0) = \nabla H_i(x_0)^*.$$

In particular, when $x_0 = \bar{x}$, this becomes

$$\nabla H_i(\bar{x})^* D\varphi_t(\bar{x}) = \nabla H_i(\bar{x})^*.$$

This relation and the fact that $D\varphi_t(\bar{x})$ is a fundamental matrix to the linear variational equation of (3.2),

$$y'(t) = Df(\bar{x})y(t), \tag{4.3}$$

lead to the relations

$$\nabla H_i(\bar{x})^* Df(\bar{x}) = 0, \qquad i = 1, \dots, r, \tag{4.4}$$

and hence $Df(\bar{x})$ satisfies (4.1) with $B = (\nabla H_1(\bar{x}), \dots, \nabla H_r(\bar{x}))^*$. Theorem 4.2 can be applied to obtain the following result.

THEOREM 4.4. *Let $H_i(x)$, $i = 1, \ldots, r$, be first integrals of* (3.2) *such that* $\nabla H_1(x) \wedge \cdots \wedge \nabla H_r(x) \neq 0$ *near an equilibrium* $\bar{x} \in \Gamma$. *Then $\bar{x}$ is asymptotically stable with respect to Γ if*

(a) $s(Df(\bar{x})^{[r+2]}) < 0$, *and*

(b) $\limsup_{\epsilon \to 0^+} \operatorname{sign}(\det(\epsilon I + Df(\bar{x}))) = (-1)^{n-r}$.

Remarks. (1) If all of the eigenvalues of $D\varphi_t(\bar{x})|_{T_{\bar{x}}\Gamma}$ have nonzero real parts, in which case $\bar{x}$ is said to be hyperbolic with respect to Γ, the conditions (a) and (b) in Theorem 4.4 are also necessary.

(2) When $r + 2 = n$, we have $Df^{[r+2]} = \operatorname{tr}(Df) = \operatorname{div} f$, and condition (a) in Theorem 4.4 becomes

$$\operatorname{div} f(\bar{x}) < 0. \tag{4.5}$$

This is so if $n = 3$ and (3.2) possesses a first integral $H(x)$ such that $\nabla H(x) \neq 0$ in Γ.

5. AN APPLICATION

Consider the following system:

$$\begin{aligned} s' &= b - bs - \lambda is + \alpha is \\ e' &= \lambda is - (\epsilon + b)e + \alpha ie \\ i' &= \epsilon e - (\gamma + \alpha + b)i + \alpha i^2, \end{aligned} \tag{5.1}$$

which arises from an epidemic model of SEIR type with varying total population, where s, e, and i denote the fractions in the population that are susceptible, exposed (in the latent period), and infectious, respectively. Then $1 - s - e - i$ is the fraction of individuals who are recovered from infection with immunity, which is assumed to be permanent. All of the parameters in (5.1) are assumed to be nonnegative. In particular, b is the exponential birth rate of the host population, λ is the per capita contact rate, and α is the exponential rate constant for the disease-caused death. A detailed description of the model can be found in [14], and a special case is studied in [3]. Based on biological considerations, system (5.1) will be studied in the following region:

$$G = \left\{(s, e, i) \in \mathbf{R}^3_+ : s + e + i \leq 1\right\}, \tag{5.2}$$

which can be shown to be positively invariant with respect to (5.1). For all nonnegative values of the parameters, (5.1) has an equilibrium $P_0 = (1, 0, 0)$ on the boundary of G that corresponds to the case of no disease. It is of interest to investigate the existence, number, and stability of equilibria in the interior of G, which correspond to disease being endemic.

As a remark on the range of the parameters, we note that if $\lambda \le \alpha$, then $s'(t) \ge b - bs(t)$ from (5.1), which implies that $s(t) \to 1$, and hence $e(t)$, $i(t) \to 0$ as $t \to \infty$; no interior equilibrium can exist. Since only interior equilibria are considered in the rest of the section, we assume that $\lambda > \alpha$.

5.1. *Existence of a Unique Interior Equilibrium*

For a possible interior equilibrium $P^* = (s^*, e^*, i^*) \in G$, its coordinates satisfy

$$\begin{aligned} b - bs - \lambda is + \alpha is &= 0 \\ \lambda is - (\epsilon + b)e + \alpha ie &= 0 \\ \epsilon e - (\gamma + \alpha + b)i + \alpha i^2 &= 0, \end{aligned} \tag{5.3}$$

and $s^* > 0$, $e^* > 0$, and $i^* > 0$. Adding the above equations leads to

$$(b - \alpha i)(1 - s - e - i) = \gamma i,$$

which implies that i^* has the range

$$0 < i^* < \frac{b}{\alpha}. \tag{5.4}$$

Eliminating s and e from (5.3), one obtains the following equation satisfied by i^*:

$$f(i) = \sigma, \tag{5.5}$$

where

$$f(i) = \left(1 - \frac{\alpha}{\epsilon + b} i\right)\left(1 - \frac{\alpha}{\gamma + \alpha + b} i\right)\left(1 + \frac{\lambda - \alpha}{b} i\right)$$

and

$$\sigma = \frac{\lambda \epsilon}{(\epsilon + b)(\gamma + \alpha + b)}. \tag{5.6}$$

Furthermore, s^* and e^* can be uniquely determined from i^* by

$$s^* = \frac{b}{b + \lambda i^* - \alpha i^*} \quad \text{and} \quad e^* = \frac{(\gamma + \alpha + b - \alpha i^*) i^*}{\epsilon}, \tag{5.7}$$

respectively. None of the three roots of $f(i)$, considering $\lambda > \alpha$,

$$i_1 = \frac{\epsilon + b}{\alpha}, \qquad i_2 = \frac{\gamma + \alpha + b}{\alpha}, \qquad i_3 = -\frac{\lambda - \alpha}{b},$$

lies in $[0, b/\alpha]$. Furthermore, $f(0) = 1$ and $f(b/\alpha) = \sigma(\alpha + \gamma)/\alpha > \sigma$. These observations lead to the conclusion that, when $\sigma > 1$, the line $y = \sigma$ has exactly one intersection $(i^*, f(i^*))$ with the graph of $y = f(i)$ that satisfies $i^* \in [0, b/\alpha]$ (see Fig. 1). We thus established the existence and uniqueness of the endemic equilibrium of (5.1) when $\sigma > 1$, which, we believe, have not been established in the literature of epidemic models.

PROPOSITION 5.1. *When* $\sigma > 1$, (5.1) *has a unique interior equilibrium* $P^* = (s^*, e^*, i^*)$, *and* s^*, e^*, *and* i^* *satisfy* (5.3)–(5.5).

5.2. *Asymptotic Stability of the Interior Equilibrium*

The Jacobian matrix of (5.1) at $P^* = (s^*, e^*, i^*)$ is

$$J(P^*) = \begin{bmatrix} -b - \lambda i^* + \alpha i^* & 0 & -\lambda s^* + \alpha s^* \\ \lambda i^* & -(\epsilon + b) + \alpha i^* & \lambda s^* + \alpha e^* \\ 0 & \epsilon & -(\gamma + \alpha + b) + 2\alpha i^* \end{bmatrix},$$

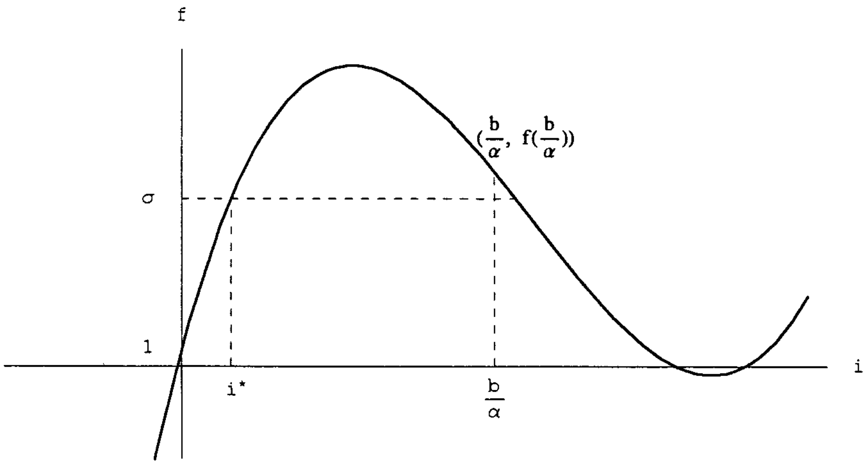

FIG. 1. Existence and uniqueness of i^* in the interval $[0, \frac{b}{\alpha}]$.

whose second additive compound matrix $J^{[2]}(P^*)$ is, by the Appendix,

$$\begin{bmatrix} -2b - \lambda i^* - \epsilon + 2\alpha i^* & \lambda s^* + \alpha e^* & \lambda s^* - \alpha s^* \\ \epsilon & -2b - \lambda i^* - \gamma - \alpha + 3\alpha i^* & 0 \\ 0 & \lambda i^* & -2b - \epsilon - \gamma - \alpha + 3\alpha i^* \end{bmatrix}.$$

We rewrite (5.3) as

$$\begin{aligned} \frac{b}{s^*} &= b + \lambda i^* - \alpha i^* \\ \frac{\lambda i^* s^*}{e^*} &= (\epsilon + b) - \alpha i^* \\ \frac{\epsilon e^*}{i^*} &= \gamma + \alpha + b - \alpha i^*. \end{aligned} \tag{5.8}$$

Thus

$$\begin{aligned} \det(J(P^*)) &= \begin{vmatrix} -\dfrac{b}{s^*} & 0 & \dfrac{bs^* - b}{i^*} \\ \lambda i^* & -\dfrac{\lambda i^* s^*}{e^*} & \lambda s^* + \alpha e^* \\ 0 & \epsilon & -\dfrac{\epsilon e^*}{i^*} + \alpha i^* \end{vmatrix} \\ &= -\lambda b\epsilon(1 - s^*) + \lambda b i^* \frac{\alpha i^*}{e^*} + \frac{b\alpha\epsilon e^*}{s^*} \\ &= -\lambda b\epsilon(1 - s^*) + \lambda b\epsilon i^* \frac{\alpha i^*}{\epsilon e^*} + \lambda b\epsilon e^* \frac{\alpha}{\lambda s^*} \\ &\le -\lambda b\epsilon(1 - s^* - i^* - e^*) < 0, \end{aligned}$$

since from (5.8),

$$\frac{\epsilon e^*}{i^*} = \gamma + \alpha + b - \alpha i^* > \alpha$$

$$\lambda s^* = \frac{b - bs^*}{i^*} + \alpha > \alpha.$$

This leads to the following lemma.

LEMMA 5.2. $\det(J(P^*)) < 0$.

Next, we set $Q = \mathrm{diag}(i^*, e^*, s^*)$. Then $QJ^{[2]}(P^*)Q^{-1}$ is

$$\begin{bmatrix} -2b - \lambda i^* - \epsilon + 2\alpha i^* & \dfrac{\lambda i^* s^*}{e^*} + \alpha i^* & \lambda i^* - \alpha i^* \\ \dfrac{\epsilon e^*}{i^*} & -2b - \lambda i^* - \gamma - \alpha + 3\alpha i^* & 0 \\ 0 & \dfrac{\lambda i^* s^*}{e^*} & -2b - \epsilon - \gamma - \alpha + 3\alpha i^* \end{bmatrix}.$$

Let μ be the Lozinskiĭ measure on $\mathbf{M}_3(\mathbf{R})$ with respect to the norm

$$|(X, Y, Z)| = \sup\{|X|, |Y|, |Z|\}.$$

Then from (2.4) and (2.5), $\mu_Q(J^{[2]}(P^*)) = \mu(QJ^{[2]}(P^*)Q^{-1}) = \max\{g_1, g_2, g_3\}$, where

$$\begin{aligned} g_1 &= -2b - \epsilon + 2\alpha i^* + \frac{\lambda i^* s^*}{e^*}, \\ g_2 &= -2b - \lambda i^* - \gamma - \alpha + 3\alpha i^* + \frac{\epsilon e^*}{i^*}, \\ g_3 &= -2b - \epsilon - \gamma - \alpha + 3\alpha i^* + \frac{\lambda i^* s^*}{e^*}. \end{aligned} \tag{5.9}$$

Therefore,

$$\mu_Q\big(J^{[2]}(P^*)\big) = \max\{-b + \alpha i^*,\ -b - \lambda i^* + 2\alpha i^*,\ -b - \gamma - \alpha + 2\alpha i^*\} < 0,$$

by (5.8) and (5.4). This establishes the following result.

LEMMA 5.3. $\mu_Q(J^{[2]}(P^*)) < 0$.

From Lemma 5.2, Lemma 5.3, and Theorem 3.1 follows the main result of this section.

THEOREM 5.4. *If $\sigma > 1$, then (5.1) has a unique equilibrium P^* in the interior of G and P^* is asymptotically stable.*

APPENDIX

For $m = 2, 3$, and 4, the second additive compound matrix of an $m \times m$ matrix $M = (a_{ij})$ is, respectively,

$$m = 2: \quad a_{11} + a_{22} \quad (= \operatorname{tr}(M))$$

$$m = 3: \quad \begin{bmatrix} a_{11}+a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11}+a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22}+a_{33} \end{bmatrix}$$

$$m = 4: \quad \begin{bmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{bmatrix}.$$

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