Maximum and Minimum Values

Let f be a function of two variables x and y. We say that f has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in a neighborhood of (a, b). We say that f has a **local minimum** at (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) in a neighborhood of (a, b).

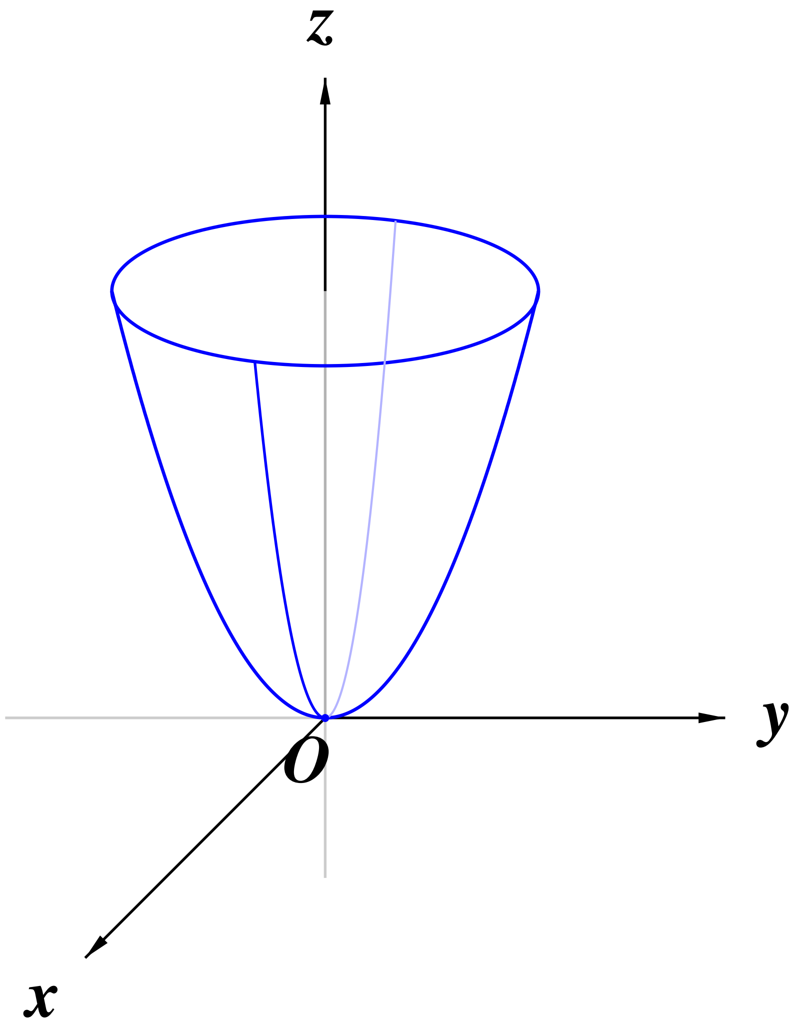
If f has a local extremum (that is, a local maximum or minimum) at (a, b), and if the first order partial derivatives of f exist there, then

$$\frac{\partial f}{\partial x}(a, b) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(a, b) = 0.$$

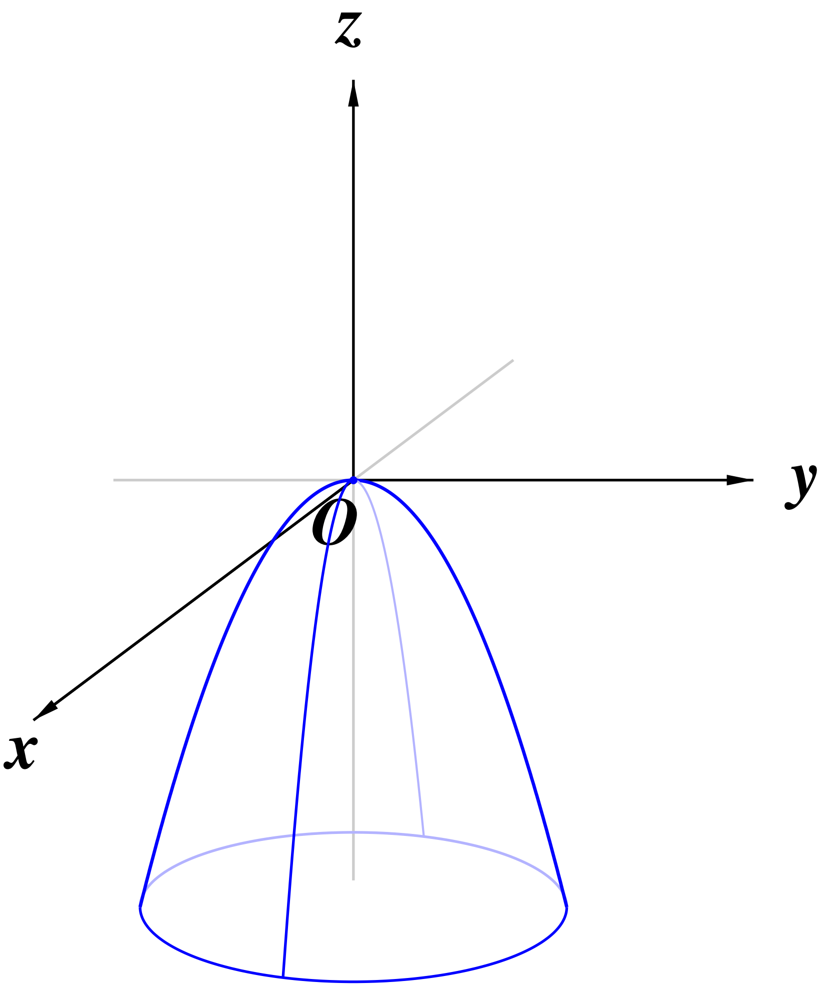
Critical Points

A point (a, b) such that $\nabla f(a, b) = \mathbf{0}$, or one of the partial derivatives does not exist, is called a **critical point** of f. At a critical point, a function could have a local maximum, or a local minimum, or neither.

Example. Let $z = f(x, y) = x^2 + y^2$. The only critical point is $(0, 0)$. At $(0, 0)$, f has a minimum.



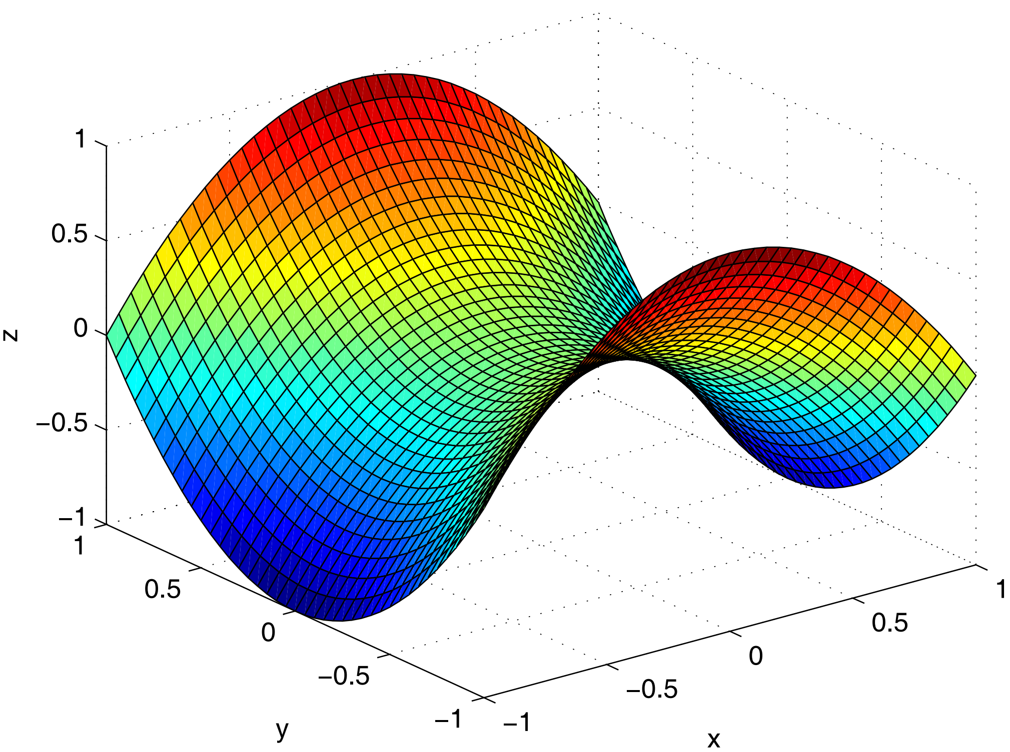
Example. Let $z = g(x, y) = -x^2 - y^2$. The only critical point is $(0, 0)$. At $(0, 0)$, g has a maximum.



Example. Let $z = h(x, y) = y^2 - x^2$. The only critical point is $(0, 0)$. At $(0, 0)$, h has neither maximum, nor minimum. Indeed, for $a \neq 0$ we have

$$h(0, a) = a^2 > 0 \quad \text{and} \quad h(a, 0) = -a^2 < 0.$$

The point $(0, 0)$ is called a **saddle point** of the graph of h.



Quadratic Forms

Let $Q(h,k) = ah^2 + 2bhk + ck^2$ be a quadratic form. Suppose $a \neq 0$. We have

$$\begin{aligned} Q(h,k) &= a\Big[h^2 + \frac{2b}{a}hk + \frac{c}{a}k^2\Big] \\ &= a\Big[h^2 + 2h\frac{bk}{a} + \Big(\frac{bk}{a}\Big)^2 - \Big(\frac{bk}{a}\Big)^2 + \frac{c}{a}k^2\Big] \\ &= a\Big(h + \frac{b}{a}k\Big)^2 + \frac{ac - b^2}{a}k^2. \end{aligned}$$

Let $D = ac - b^2$. We have the following properties:

(1) If $D > 0$ and $a > 0$, then $Q(h,k) > 0$ for $(h,k) \neq (0,0)$.

(2) If $D > 0$ and $a < 0$, then $Q(h,k) < 0$ for $(h,k) \neq (0,0)$.

(3) If $D < 0$, then $Q(h,k)$ takes on both positive and negative values in a neighborhood of $(0,0)$.

The Second Derivative Test

Let f be a function of two variables x and y. Suppose that all the second-order partial derivatives of f are continuous in a neighborhood of (a, b) and

$$\frac{\partial f}{\partial x}(a, b) = 0, \quad \frac{\partial f}{\partial y}(a, b) = 0.$$

Let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2.$$

We have the following results:

(1) If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(2) If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(3) If $D(a, b) < 0$, then (a, b) is a saddle point.

(4) If $D(a, b) = 0$, no conclusion may be drawn.

Example. Find and classify all critical points for the function $f(x, y) = x^4 + y^4 - 4xy$.

Solution. The partial derivatives of f are

$$\frac{\partial f}{\partial x} = 4x^3 - 4y \quad \text{and} \quad \frac{\partial f}{\partial y} = 4y^3 - 4x.$$

Setting $f_x = 0$ and $f_y = 0$, we obtain $y = x^3$ and $x = y^3$. It follows that $x = (x^3)^3 = x^9$. We observe that

$$\begin{aligned} x^9 - x &= x(x^8 - 1) = x(x^4 - 1)(x^4 + 1) \\ &= x(x^2 - 1)(x^2 + 1)(x^4 + 1). \end{aligned}$$

Hence, the equation $x^9 - x = 0$ has three real roots: $x = 0, 1, -1$. Thus, the critical points of f are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

Next, we find the second-order partial derivatives of f:

$$f_{xx} = 12x^2, \quad f_{xy} = -4, \quad f_{yy} = 12y^2.$$

At the critical point $(1, 1)$, we have

$$D(1,1) = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 144 - 16 = 128 > 0.$$

Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, $f(1,1) = -2$ is a local minimum of f. Similarly, $f(-1,-1) = -2$ is also a local minimum of f.

At the point $(0,0)$ we have

$$D(0,0) = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0.$$

Therefore, $(0,0)$ is a saddle point of the graph of f.

Absolute Maximum and Minimum Values

If f is a continuous function of two variables x and y on a closed and bounded region E in $\mathbb{R}^2$, then f attains an absolute maximum value and an absolute minimum value at some points in E.

To find the absolute maximum and minimum values of f on E:

(1) Find the values of f at the critical points of f in the interior of E.

(2) Find the extreme values of f on the boundary of E.

(3) The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example. Find the absolute maximum and minimum values of the function

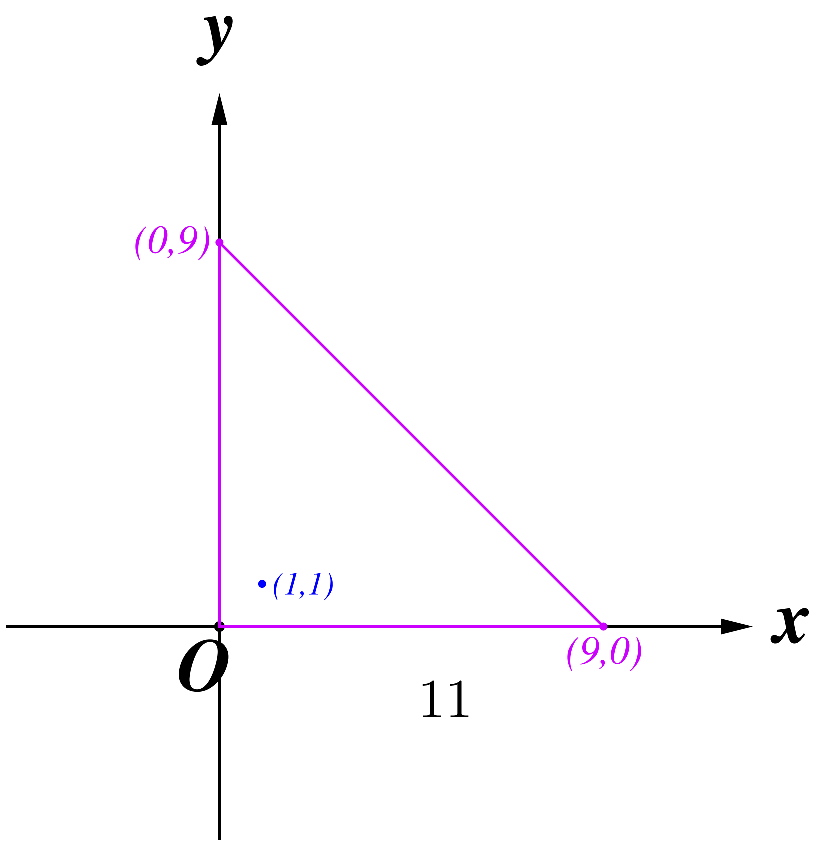
$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.

Solution. Set $f_x = 2 - 2x = 0$ and $f_y = 2 - 2y = 0$. The only critical point is $(1, 1)$. We have $f(1, 1) = 4$.

At three vertices we have

$$f(0, 0) = 2, \quad f(9, 0) = -61, \quad f(0, 9) = -61.$$



Let us consider values of f on three edges. On the line segment $y = 0$, $0 \leq x \leq 9$, we have $g_1(x) = f(x, y) = 2 + 2x - x^2$. Hence, $g_1'(x) = 2 - 2x$ and $g_1'(x) = 0$ yields $x = 1$. We have $f(1, 0) = 3$.

On the line segment $x = 0$, $0 \leq y \leq 9$, we have $g_2(x) = f(x, y) = 2 + 2y - y^2$. Hence, $g_2'(y) = 2 - 2y$ and $g_2'(y) = 0$ yields $y = 1$. We have $f(0, 1) = 3$.

On the line segment $y = 9 - x$, $0 \leq x \leq 9$, we have $g_3(x) = f(x, y) = -2x^2 + 18x - 61$. Hence, $g_3'(x) = -4x + 18$ and $g_3'(x) = 0$ yields $x = 9/2$. We have $f(9/2, 9/2) = -41/2$.

We conclude that f achieves the maximum value 4 at $(1, 1)$ and the minimum value -61 at the points $(0, 9)$ and $(9, 0)$.