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Key Points:
- New formulas to calculate the magnetic anomaly caused by two-dimensional irregular shape subsurface structure are derived
- The very first and most cited Talwani and Heirtzler (1964) algorithm is re-derived and compared to the new derivation
- Software for two-dimensional magnetic forward modeling is developed for educational and industry use

Supporting Information:
- Supporting Information S1

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Computation of Magnetic Anomalies Caused by Two-Dimensional Structures of Arbitrary Shape: Derivation and Matlab Implementation

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Abstract The very first computational and most referred in the literature algorithm of Talwani and Heirtzler (1964) for calculating of the magnetic anomaly caused by two-dimensional irregular shape subsurface structure has particular fundamental and educational significance in geophysics theory. We re-derive this algorithm from first principles and discuss previous derivation omissions. Our resulting solution differs from the original publication. Based on our new solution we present the two-dimensional forward magnetic modeling software and associated tutorials which are available for download from the website www.ualberta.ca/~vadim/software.htm. Additionally, we include the computation of the remnant magnetization which can be found using already published apparent polar wonder paths.

1. Introduction

Talwani and Heirtzler (1964) were first to examine a nonmagnetic space containing a uniformly magnetized two-dimensional structure approximated by a polygonal prism and to suggest a numerical and computational technique of the forward modeling. A magnetic anomaly above the magnetized body was calculated by analytical formulae using summation of the anomalies due to semiinfinite prisms limited on one side by a segment of the polygon. The derivation of the mathematical expression for the magnetic anomaly over a two-dimensional body of polygonal cross section was first done in Talwani and Heirtzler (1962). Certainly, it was not the first approach to the problem; a comprehensive review of algorithms and approaches previous to 1962 is given in Talwani and Heirtzler (1962). The algorithm was, however, derived specifically for the computation using digital computers and therefore was the first algorithm of such kind.

Since 1964, for the past more than five decades, forward calculations of magnetic anomalies caused by two-dimensional (2-D) and three-dimensional (3-D) bodies have progressed significantly. Talwani (1965) developed a new algorithm to compute a three-dimensional magnetic anomaly for geological bodies of arbitrary shape. Since, both 2- and 3-D forward problems have been developed in various alternative ways. A comprehensive overview of the progress and approaches of the 2-D modeling since 1964 is provided in introductions from Kostrov (2007) and Jeshvaghani and Darijani (2014).

Our initial motivation was to create a Matlab software for educational purposes and for rapid interpretation of magnetic data. The algorithm of Talwani and Heirtzler (1964) would provide a stable 2-D solution for variety of geological situations. This algorithm is a very effective for small-scale magnetic surveys, and the publication is the most cited among all existing magnetic forward modeling methods. The first version of our software, however, produced some unfitting anomalies in a number of theoretically modeled situations. Therefore, in this study, we reappraise the derivation that leads us to a different from Talwani and Heirtzler (1964) solution. Both solutions are compared and discussed below. Further we develop a Matlab p-coded and executable software that has user-friendly GUI. The software is a freeware for research and education purposes and can be redistributed among users. Any use of the software should refer to this publication. The software can be downloaded from www.ualberta.ca/vadim/software.htm.

2. Important Concepts

Here we introduce the important concepts and notation used for the derivation:

1. Magnetic susceptibility (X)—dimensionless. An object's magnetic susceptibility is the constant that indicates how much a material is magnetized in response to the local magnetic field.

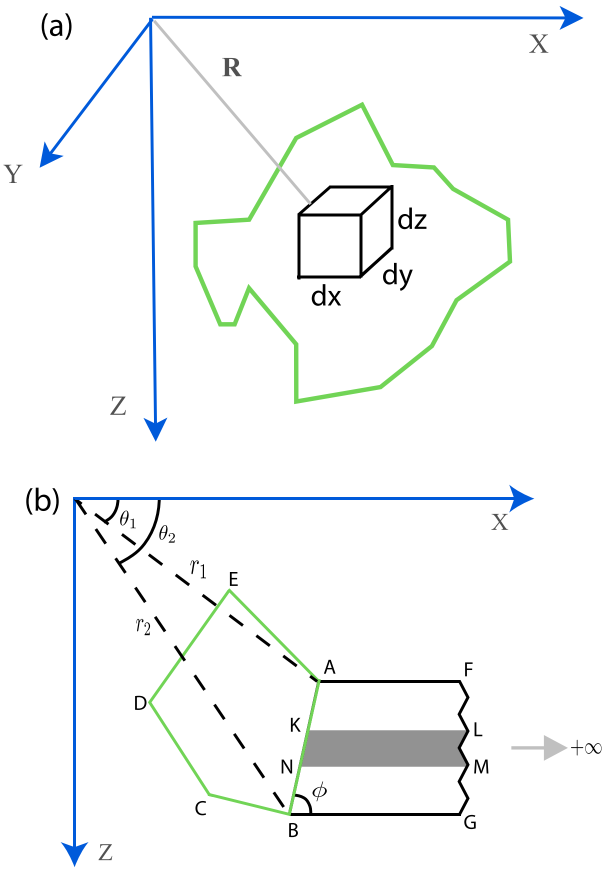

Figure 1. (a) A volume element with dimension *dx*, *dy*, *dz* within an irregular-shaped body. (b) AFGBA is a semiinfinite prism containing a rod (KLMNK) that extends to positive infinity. ABCDEA is an arbitrary polygon defined along the edge of AB. Modified from Talwani (1965).

2. Magnetization (M)—units = A/m. Magnetic fields can align the magnetic moments of individual atoms within a material based on that material's magnetic susceptibility. The net magnetic moment of the material per unit volume is magnetization.
3. Induced magnetization (M_I) is the magnetization associated with the proportion of the material that is aligned with the Earth's magnetic field according to its current inclination and declination.
4. Induced inclination/declination. Inclination is the angle the Earth's magnetic field makes with respect with the horizontal. Positive angles are defined as angles that are directed below the horizon. Declination is the difference in angle between true north and horizontal projection of Earth's present-day magnetic field. Values increase in the clockwise direction (0° for north, 90° for east, etc.).
5. Remnant magnetization (M_R) is any preserved magnetization not associated with induced magnetization. Often this is magnetization associated with the formation of the rock/sediment, or may be associated with recrystalization events (e.g., metamorphism); it is dependent on the direction of the Earth's magnetic field at the time of its acquisition.
6. Remnant inclination/declination. Remnant inclination is the angle the source of the remnant magnetization, makes with respect with the horizontal. Positive angles are defined as angles that are directed below the horizon. Remnant declination is the difference in angle between true north and horizontal projection of Earth's ancient magnetic field. Values increase in the clockwise direction (0° for north, 90° for east, etc.).

The values for remnant inclination and declination vary through time and location but can be estimated if a paleomagnetic pole (paleopole) is known for the object(s) in question. The paleopole latitude and paleopole longitude can be converted into inclination (I) and declination (D) using MagMod and is based on the following formulas:

$$P = \sin^{-1}\left[\sin[lat_s]\sin\left[lat_p\right] + \cos\left[lat_p\right]\cos\left[\left|long_s - long_p\right|\right]\right]$$
$$I = \tan^{-1}[2\tan[P]]$$
$$D = \sin^{-1}\left[\sin\left[\left|long_s - long_p\right|\right]\frac{\cos\left[lat_p\right]}{\cos[P]}\right]$$

where P is the paleolatitude, lat_s is the latitude of the site, lon_s is the longitude of the site, lat_p is the latitude of the paleopole, and lon_p is the longitude of the paleopole.

7. Total magnetization of the subsurface structure or small element is a superposition of the induced and remnant magnetizations:
8. A magnetic anomaly is the magnetic field associated with unknown bodies within the subsurface normalized against the local magnetic field (i.e., Earth's magnetic field).

3. Calculating Anomalies

Consider that there exists an elemental volume contained within an irregularly shaped body. This elemental volume extends from negative to positive infinity in the *y* direction. Bodies of irregular shapes can be approximated by a polygon, which can be and reduced to solving semiinfinite two-dimensional polygons (Talwani & Heirtzler, 1962). Now consider a small-volume element with dimensions *dx*, *dy*, and *dz* (Figure 1a) located in the geomagnetic field. The total magnetization of the volume is a superposition of both induced and remnant magnetizations which coexist.

The magnetic potential, Ω, at the origin is given by

$$\Omega = \frac{\vec{m}\cdot\vec{R}}{4\pi \ R^3} \tag{1}$$

where m is the magnetic moment of the volume element and R is the distance from the origin (Figure 1a).

Assuming that this volume element contains a uniform intensity of magnetization, J, the magnetic moment of a body can be represented as

$$\vec{m} = \vec{J}\, dxdydz \tag{2}$$

The magnetic moment in terms of Cartesian coordinates x, y, z can be written as

$$\Omega = \frac{J_x x + J_y y + J_z z}{4\pi(x^2 + y^2 + z^2)^{3/2}} dx \ \ dy \ \ dz \tag{3}$$

Using the assumption that the body extends from negative infinity to positive infinity in the y direction and then integrating equation (3) with respect to y, the magnetic potential has the form

$$\Omega = \int_{-\infty}^{\infty} \frac{J_x x + J_y y + J_z z}{4\pi(x^2 + y^2 + z^2)^{3/2}} dy = \frac{(J_x x + J_z z)}{2\pi(x^2 + z^2)} dxdz \tag{4}$$

The vertical (V) and horizontal (H) components of the magnetic strength can be derived by differentiating equation (4) with respect to z and x, respectively, and results in the following equations:

$$V = -\frac{\partial\Omega}{\partial z} = \frac{2J_x xz - J_z(x^2 - z^2)}{2\pi(x^2 + z^2)^2} dxdz \tag{5}$$

$$H = -\frac{\partial\Omega}{\partial x} = \frac{2J_z xz + J_x(x^2 - z^2)}{2\pi(x^2 + z^2)^2} dxdz \tag{6}$$

Assuming that the body extents to positive infinity in the x direction we can simplify equations (5) and (6) by integrating from x to positive infinity, which results in

$$V = \int_x^{\infty} \frac{2J_x xz - J_z(x^2 - z^2)}{2\pi(x^2 + z^2)^2} dxdz = \frac{J_x z - J_z x}{2\pi(x^2 + z^2)} dz \tag{7}$$

$$H = \int_x^{\infty} \frac{2J_z xz + J_x(x^2 - z^2)}{2\pi(x^2 + z^2)^2} dxdz = \frac{J_x x - J_z z}{2\pi(x^2 + z^2)} dz \tag{8}$$

Equations (7) and (8) are the components produced by the rod KLMNK in Figure 1b. The resulting integrating these equations from z_1 to z_2, the magnetic field strength for the prism AFGBA in Figure 1b produces equations that can be expressed in the simplified form as shown below (see detailed step by step derivation in the supporting information):

$$V = \frac{1}{2\pi}(J_x Q - J_z P) \tag{9}$$

$$H = \frac{1}{2\pi}(J_z Q + J_x P) \tag{10}$$

where

$$Q = \gamma_z^2 \ln\left(\frac{r_2}{r_1}\right) - \delta\gamma_z\gamma_x(\alpha_2 - \alpha_1)$$

$$P = \gamma_z\gamma_x \ln\left(\frac{r_2}{r_1}\right) + \delta\gamma_z^2(\alpha_2 - \alpha_1)$$

$$\gamma_z = \frac{z_{21}}{\sqrt{x_{21}^2 + z_{21}^2}}$$

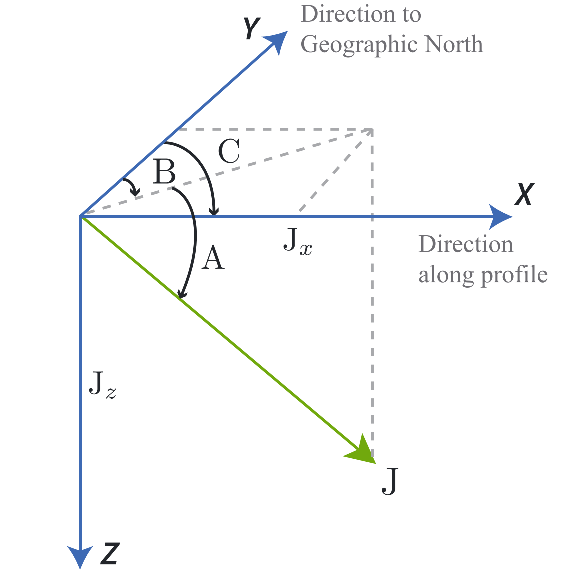

Figure 2. *A* is the angle that defines the horizontal projection of magnetic vector *J*. *B* is the angle measured from the geographic north clockwise toward the horizontal projection of *J*. *C* is the angle between geographic north and the positive *x* axis. Modified from Talwani (1965).

$$\gamma_x = \frac{x_{21}}{\sqrt{x_{21}^2 + z_{21}^2}}$$

$$r_1 = \sqrt{x_1^2 + z_1^2}, r_2 = \sqrt{x_2^2 + z_2^2}$$

$$\alpha_1 = \tan^{-1}\left(\frac{\delta(z_1 + gx_1)}{x_1 - gz_1}\right)$$

$$\alpha_2 = \tan^{-1}\left(\frac{\delta(z_2 + gx_2)}{x_2 - gz_2}\right)$$

$$g = \frac{x_2 - x_1}{z_2 - z_1} = \frac{x_{21}}{z_{21}}$$

$$\delta = 1 \text{ if } x_1 > gz_1$$

$$\delta = -1 \text{ if } x_1 < gz_1$$

Note that these equations differ from Talwani and Heirtzler (1962, 1964).

For an arbitrarily shaped polygon a point x_i, z_i represents a corner of the polygon and a point x_{i+1}, z_{i+1} to be the next corner of the polygon. Equations (9) and (10) represent the magnetic strength of the rectangular region AFGBA for only one side of the polygon. For a polygon with *n* sides there are a *n* number of prisms of the same form as AFGBA. Calculation for a positive anomaly requires calculation of the polygon clockwise with reference to the origin as depicted in Figure 2 and summing the contribution of each side.

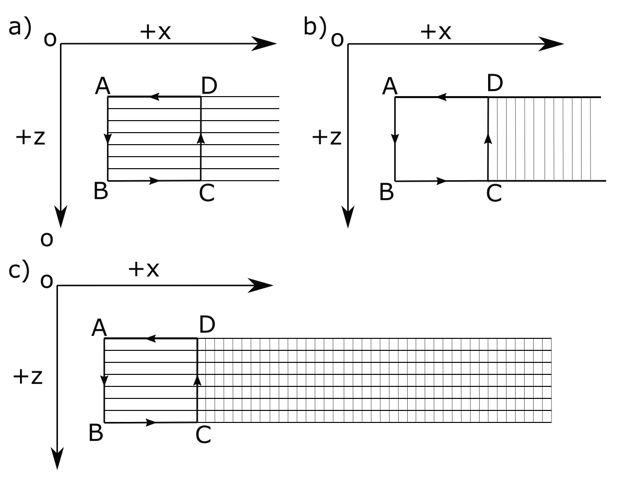

Figure 3. Visualization of the order of calculation of the magnetic field for a square ABCDA. Sides DA and CB provide no contribution as $z_2 - z_1 = 0$. (a) AB provides a positive contribution in the direction of the line integral with the horizontal lines shading the area of the semiinfinite prism. (b) CD provides a negative contribution in the direction of the line integral with the vertical lines shading the area of the semiinfinite prism. (c) Summation of the contributions from all sides results in only the magnetic field from the space enclosed in ABCDA, the area that has both horizontal and vertical lines cancel out.

To evaluate the total intensity anomaly, *T*, we need to sum the projection of *H* and *V* along the direction of the total field. This can be done by manipulating the magnetization vectors associated with total magnetization (*J*) while using the convention shown in Figure 3. In general, total magnetization is a superposition of induced (J_i) and remnant magnetization (J_r) which are given by

$$\vec{J}_i = J_i(\cos I \cos D\, \hat{n} + \cos I \sin D\, \hat{e} + \sin I\, \hat{v}) \tag{11}$$

$$\vec{J}_r = J_r(\cos I_r \cos D_r\, \hat{n} + \cos I_r \sin D_r\, \hat{e} + \sin I_r\, \hat{v}) \tag{12}$$

where $\hat{n}$ is north, $\hat{e}$ is east, $\hat{v}$ is vertical, *I* is the induced inclination, *D* is the induced declination, I_r is the remnant inclination, and D_r is the remnant declination. Using equations (11) and (12) the angle (Δ) between the two vectors can be determined as follows:

$$\Delta = \cos^{-1}\left(\frac{\vec{J}_i \cdot \vec{J}_r}{J_i J_r}\right)$$

$$\Delta = \cos^{-1}\left(\hat{J}_i \cdot \hat{J}_r\right)$$

$$\Delta = \cos^{-1}(\cos I \cos D \cos I_r \cos D_r + \cos I \sin D \cos I_r \sin D_r + \sin I \sin I_r)$$

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This angle Δ can be used to calculate the magnitude of the total magnetization (J) as well as its inclination (A) and declination (B). Using the cosine law the total magnetization J is defined as

$$J^2 = J_i^2 + J_r^2 - 2J_i\, J_r \cos\Delta$$

To determine the inclination (A) and declination (B) of J we split J_i and J_r into their horizontal (J_{iH} and J_{rH}, respectively) and vertical components (J_{iV} and J_{rV}, respectively). Inclination is then derived as follows:

$$\begin{aligned}
J_V &= J_{iV} + J_{rV} \\
J_V &= J_i \sin I + J_r \sin I_r \\
J \sin A &= J_i \sin I + J_r \sin I_r \\
\sin A &= \frac{J_i \sin I + J_r \sin I_r}{J} \\
A &= \sin^{-1}\left(\frac{J_i \sin I + J_r \sin I_r}{J}\right)
\end{aligned}$$

Similarly, declination is derived as follows:

$$\begin{aligned}
J_H &= J \cos A \\
J_H &= J_{iH} + J_{rH} \\
J_H &= J_i \cos I + J_r \cos I_r \\
J_H \widehat{n} &= J_H \cos B \\
J_H \cos B &= J_{iH} \widehat{n} + J_{rH} \widehat{n} \\
J_H \cos B &= J_{iH} \cos D + J_{rH} \cos D_r \\
J_H \cos B &= J_i \cos I \cos D + J_r \cos I_r \cos D_r \\
\cos B &= \frac{J_i \cos I \cos D + J_r \cos I_r \cos D_r}{J_H} \\
B &= \cos^{-1}\left(\frac{J_i \cos I \cos D + J_r \cos I_r \cos D_r}{J \cos A}\right)
\end{aligned}$$

The intensity of magnetization of magnetization in the x and z directions in the terms of total magnetization, J, in terms of A, B, and C, can be defined as

$$\begin{aligned}
J_x &= J \cos(A) \cos(C-B) \\
J_z &= J \sin(A)
\end{aligned}$$

The total intensity anomaly (T) can then be defined as

$$T = V \sin(A) + H \cos(A) \cos(C-B) \tag{13}$$

4. Discussion

Upon a rederivation of the original Talwani and Heirtzler (1964) algorithm we found three explicit differences and errors in Talwani and Heirtzler's (1964) derivation. The first error began in the definition of x. Figure 4 demonstrates the resultant difference between the two expressions for a polygon. Continuing derivation of the magnetic fields using Talwani and Heirtzler (1964) definition for x, it was evident that the definition for θ_1 and θ_2 are not equivalent to the angle the corners of the side make with the origin as depicted in Figure 1a. The final issue found in derivation was the definition of a δ term. In Talwani and Heirtzler (1964) this term was assumed to value 1, indicating that they did not account for the impact of the absolute value in the derivation.

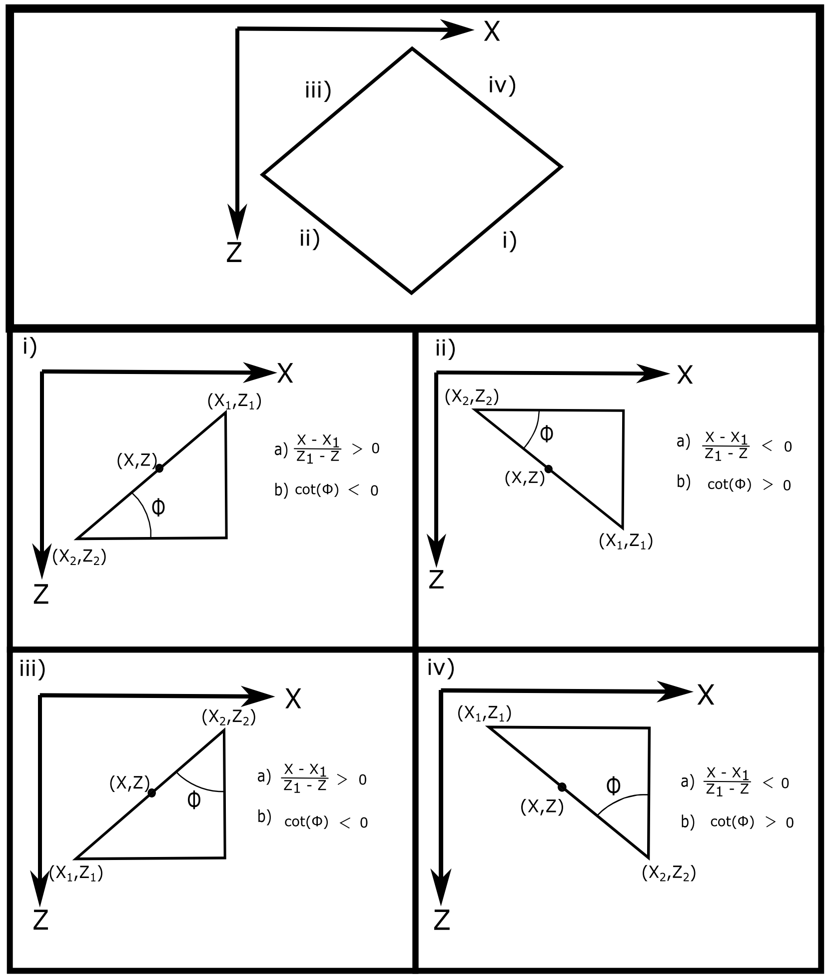

Figure 4. Depiction of a simple polygon shape (top) and (a) the resulting values of cot(ϕ) in our derivation and (b) in Talwani and Heirtzler (1964) definition of cot(ϕ) for each side, respectively. The figure demonstrates that the correct calculation for cot(ϕ) has an opposite sign of Talwani and Heirtzler's (1964) definition for this object.

Due to the many different shapes and sizes of polygons the resultant error is not broadly quantifiable but dependant on shape of the polygon, and inclination/declination of the induced magnetic field. To demonstrate the potential differences produced by different derivations, we have calculated the induced magnetic field produced from a diamond with three different inclinations. Figure 5 illustrates the comparison of the magnetic anomalies computed using the six different algorithms: (i) Talwani and Heirtzler (1964), (ii) our rederivation using Talwani and Heirtzler's (1964) definition of x and accounting for corrected δ, (iii) definition of x and corrected θ, (iv) definition of x and accounting for the corrected δ and θ term, (v) robust derivation from first principles, and (vi) Won and Bevis (1987). We find that the results for (i), (v), and (iv) are very similar. The errors inherent in the original derivation of Talwani and Heirtzler (1964), particularly the definitions of θ_1, θ_2, and δ, by removal compensate for each other to produce results that approximately agree with the properly derived solution provided by our derivation. However, when the corrections for θ_1, θ_2, and δ are applied independently they produce the same incorrect anomaly, which indicates that these errors had to be made dependently; otherwise, it would produce incorrect anomalies. We recommend the solution produced by Talwani and Heirtzler (1964) be avoided, as it cannot be guaranteed to work for all possible shapes and cases. It is, however, clear that the errors were fundamental and that when corrected the original algorithm of Talwani and Heirtzler (1964) produced significant differences in the modeled magnetic field (see supporting information).

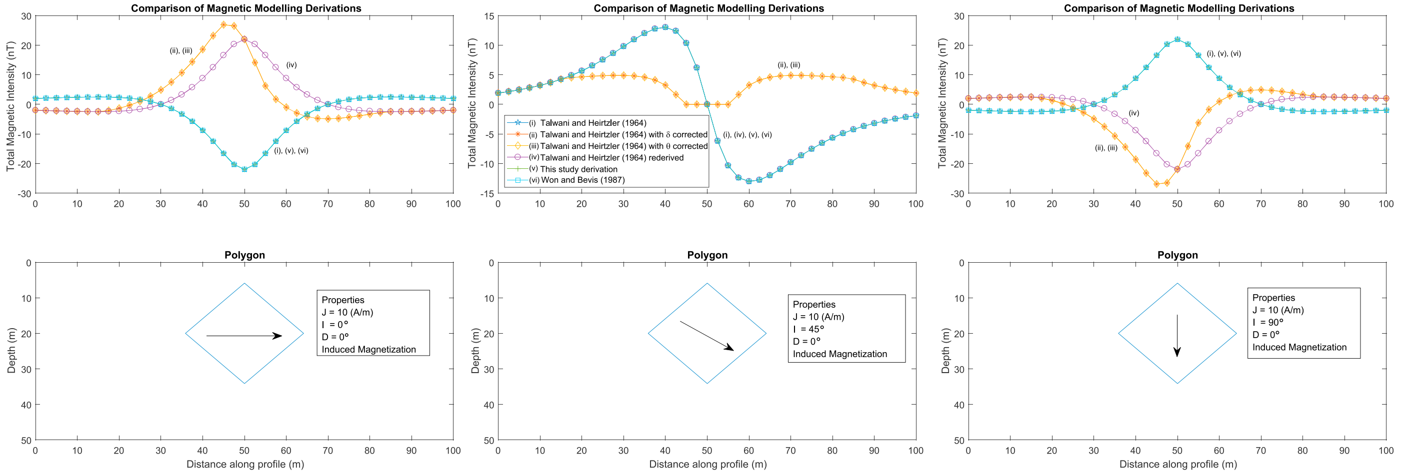

Figure 5. Comparison of the total magnetic intensity of a diamond at a magnetic inclination of 0°, 45°, and 90° of (i) Talwani and Heirtzler (1964), (ii) our rederivation using Talwani and Heirtzler's (1964) definition of *x* and accounting for corrected δ, (iii) definition of *x* and corrected θ, (iv) definition of *x* and accounting for the corrected δ and θ term, (v) our robust derivation in this study from first principles, and (vi) Won and Bevis (1987). The magnetization of the objects is 10 A/m. The original Talwani and Heirtzler (1964) algorithm produces results similar to this study algorithm and Won and Bevis (1987) algorithm in this example and a few other cases we have tried, although it cannot be guaranteed to work for all possible shapes and cases. The Talwani and Heirtzler (1964) rederived algorithm with the different corrections applied together and independently produces offset results when calculated to a diamond for different inclinations.

5. Conclusion

The resulting expressions for the components of the magnetic field (equations (9) and (10)) are not equal to the expressions derived by Talwani and Heirtzler (1964). The discrepancy between our derivation and Talwani and Heirtzler (1964) lies in the definition of the variable *x*, definition of the angles θ_1 and θ_2, and the dismissal of an absolute value. Talwani and Heirtzler (1964) have erroneous definitions. Detailed rederivation of Talwani and Heirtzler formulas to calculate magnetic anomalies caused by two-dimensional structures of arbitrary shape is given in the supporting information. The rederived final solution is different from the original published formulas of Talwani and Heirtzler (1964) and produces incorrect anomalies (Figure 5); therefore, we strongly recommend to use our derived in this study robust formulas from first principles to avoid any fundamental errors in calculating the anomalies.

Software and Data Availability

The free software and example data are available for download from www.ualberta.ca/~vadim/software.htm. This publication has to be referred with any use of the software.

Acknowledgments
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Geophysical Research Letters

Supporting Information for

Computation of magnetic anomalies caused by two dimensional structures of arbitrary shape: derivation and Matlab implementation

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Detailed derivation of the formulas to calculate magnetic anomalies caused by two dimensional structures of arbitrary shape

Consider that there exists an elemental volume contained within an irregularly shaped body. This elemental volume extends from negative to positive infinity in the y-direction. Bodies of irregular shapes can be approximated by a polygon, which can be and reduced to solving semi-infinite two dimensional polygons (*Talwani and Heirtzler, 1964*). Now consider a small volume element with dimensions dx, dy, dz (Fig. 1A) and its properties.

The magnetic potential, Ω, at the origin is given by:

$$\Omega = \frac{\vec{m} \cdot \vec{R}}{4\pi R^3} \qquad (1)$$

where m is the magnetic moment of the volume element and R is the distance from the origin (Fig. 1A).

Assuming that this volume element contains a uniform intensity of magnetization, J, the magnetic moment of a body can be represented as:

$$\vec{m} = \vec{J} dxdydz \qquad (2)$$

The magnetic moment in terms of Cartesian coordinates x, y, z, can then be written as:

$$\Omega = \frac{J_x x + J_y y + J_z z}{4\pi (x^2 + y^2 + z^2)^{3/2}} dxdydz \qquad (3)$$

Using the assumption that the body extends from negative infinity to positive infinity in the y-direction and then integrating equation 3 with respect to y, the magnetic potential has the form

$$\Omega = \int_{-\infty}^{\infty} \frac{J_x x + J_y y + J_z z}{4\pi(x^2 + y^2 + z^2)^{3/2}} dy = \frac{(J_x x + J_z z)}{2\pi(x^2 + z^2)} dxdz \qquad (4)$$

The vertical (V) and horizontal (H) components of the magnetic strength can be derived by differentiating equation (4) with respect to z and x respectively, and results in the following equations:

$$V = -\frac{\partial\Omega}{\partial z} = \frac{2J_x xz - J_z(x^2 - z^2)}{2\pi(x^2 + z^2)^2} dxdz \qquad (5)$$

$$H = -\frac{\partial\Omega}{\partial x} = \frac{2J_z xz + J_x(x^2 - z^2)}{2\pi(x^2 + z^2)^2} dxdz \qquad (6)$$

Assuming that the body extents to positive infinity in the x-direction we can simplify equations (5) and (6) by integrating from x to positive infinity, which results in:

$$V = \int_x^{\infty} \frac{2J_x xz - J_z(x^2 - z^2)}{2\pi(x^2 + z^2)^2} dxdz = \frac{J_x z - J_z x}{2\pi(x^2 + z^2)} dz \qquad (7)$$

$$H = \int_x^{\infty} \frac{2J_z xz + J_x(x^2 - z^2)}{2\pi(x^2 + z^2)^2} dxdz = \frac{J_x x - J_z z}{2\pi(x^2 + z^2)} dz \qquad (8)$$

Equations (7) and (8) are the components produced by the rod KLMNK in Fig. 1B. Integrating these equations from z_1 to z_2, the magnetic field strength for the prism AFGBA in Fig. 1B can be calculated.

$$V = \int_{z1}^{z2} \frac{J_x z - J_z x}{2\pi(x^2 + z^2)} dz \qquad (9)$$

In order to compute this integral, consider taking a point on the side of a polygon (ABCDEA) that makes up the region of interest (AFGBA). This enables us to find x as a function of the coordinates of the corners and z.

Let $g = \frac{z_2 - z_1}{x_2 - x_1} = \frac{x - x_1}{z - z_1}$ (10)

Equation (10) can then be rearranged into,

$x = g(z - z_1) + x_1$ (11)

and then inserted into equation (9), which results in the following sets of equations,

$$V = \frac{1}{2\pi}\int_{z1}^{z2} \frac{J_x z - J_z(g(z - z_1) + x_1)}{\left[g(z - z_1) + x_1\right]^2 + z^2} dz$$

$$V = \frac{1}{2\pi}\int_{z1}^{z2} \frac{(J_x z - gJ_z)z - J_z(x_1 - gz_1)}{(1 + g^2)z^2 + 2g(x_1 - gz_1)z + (x_1 - gz_1)^2} dz$$

We can rewrite this in simpler terms by letting

$$a = 1 + g^2$$
$$b = 2g(x_1 - gz_1)z$$
$$c = (x_1 - gz_1)^2$$

which results in,

$$V = \frac{1}{2\pi}\int_{z1}^{z2} \frac{(J_x - gJ_z)z - J_z(x_1 - gz_1)}{az^2 + bz + c} dz$$

These equations can be rewritten in terms of to 2 components,

$$V = \frac{1}{2\pi}\int_{z1}^{z2} \frac{(J_x - gJ_z)z}{az^2 + bz + c}dz - \frac{1}{2\pi}\int_{z1}^{z2} \frac{J_z(x_1 - gz_1)}{az^2 + bz + c}dz$$

$$V = \frac{(J_x - gJ_z)}{2\pi}\int_{z1}^{z2} \frac{z}{az^2 + bz + c}dz - \frac{J_z(x_1 - gz_1)}{2\pi}\int_{z1}^{z2} \frac{dz}{az^2 + bz + c}$$

$V = I_1 + I_2$ (12)

where,

$$I_1 = \frac{(J_x - gJ_z)}{2\pi}\int_{z1}^{z2} \frac{z}{az^2 + bz + c}dz$$
$$I_2 = -\frac{J_z(x_1 - gz_1)}{2\pi}\int_{z1}^{z2} \frac{dz}{az^2 + bz + c}$$

Equation (12) can be integrated using the following integral identities:

For $\begin{matrix} A \neq 0 \\ 4AC - B^2 > 0 \end{matrix}$

$$\int \frac{xdx}{Ax^2 + Bx + C} = \frac{1}{2A}\ln\left|Ax^2 + Bx + C\right| - \frac{B}{A\sqrt{4AC - B^2}}\tan^{-1}\left(\frac{2Ax + B}{\sqrt{4AC - B^2}}\right)$$
$$\int \frac{dx}{Ax^2 + Bx + C} = \frac{2}{\sqrt{4AC - B^2}}\tan^{-1}\left(\frac{2Ax + B}{\sqrt{4AC - B^2}}\right) \qquad (13)$$

To use these identities we first check that the criteria are met for equation (12). First by checking that $A \neq 0$. For the first criteria, equation (12) defines $A = 1 + g^2$, which requires that $A > 0$, which implies $A \neq 0$.

Checking $4AC - B^2 > 0$ is done as follows:

$$\begin{aligned} B &= 2g(x_1 - gz_1) \\ C &= x_1 - gz_1 \end{aligned}$$

$$\begin{aligned}
4AC - B^2 &= 4(1+g^2)(x_1 - gz_1)^2 - (2g(x_1 - gz_1))^2 \\
4AC - B^2 &= 4(1+g^2)(x_1 - gz_1)^2 - 4g^2(x_1 - gz_1))^2 \\
4AC - B^2 &= (x_1 - gz_1)^2(4(1+g^2) - 4g^2) \\
4AC - B^2 &= (x_1 - gz_1)^2(4(1+g^2) - 4g^2) \\
4AC - B^2 &= (x_1 - gz_1)^2(4(1+g^2 - g^2) \\
4AC - B^2 &= 4(x_1 - gz_1)^2 > 0
\end{aligned}$$

Since both criteria are met we use the above identities to solve for I_1 and I_2.

$$I_1 = \frac{J_x - gJ_z}{2\pi}\left[\begin{array}{l} \frac{1}{2a}\left(\ln\left|az_2^2 + bz_2 + c\right| - \ln\left|az_1^2 + bz_1 + c\right|\right) \\ -\frac{b}{a\sqrt{4ac - b^2}}\left(\tan^{-1}\left(\frac{2az_2 + b}{\sqrt{4ac - b^2}}\right) - \tan^{-1}\left(\frac{2az_1 + b}{\sqrt{4ac - b^2}}\right)\right) \end{array}\right] \tag{14}$$

where,

$$\begin{aligned}
az_2^2 + bz_2 + c &= (1+g^2)z_2^2 + 2g(x_1 - gz_1)z_2 + (x_1 - gz_1)^2 \\
az_2^2 + bz_2 + c &= z_2^2 + g^2z_2^2 + 2gx_1z_2 - 2g^2z_1z_2 + x_1^2 + g^2z_1^2 - 2gx_1z_1 \\
az_2^2 + bz_2 + c &= z_2^2 + g^2(z_2^2 - 2z_1z_2 + z_1^2) + 2gx_1z_2 - 2gx_1z_1 + x_1^2 \\
az_2^2 + bz_2 + c &= z_2^2 + g^2(z_2 - z_1)^2 + 2gx_1z_2 - 2gx_1z_1 + x_1^2 \\
az_2^2 + bz_2 + c &= z_2^2 + (g(z_2 - z_1))^2 + 2gx_1(z_2 - z_1) + x_1^2
\end{aligned}$$

but, $g(z_2 - z_1) = x_2 - x_1$, which then gives,

$$az_2^2 + bz_2 + c = z_2^2 + (x_2 - x_1)^2 + 2x_1(x_2 - x_1) + x_1^2 = z_2^2 + x_2^2$$

Similarly,

$$az_1^2 + bz_1 + c = z_1^2 + x_1^2$$

By inspection of Fig. 1B, the following relationship exists:

$$\begin{aligned} r_1^2 &= z_1^2 + x_1^2 = az_1^2 + bz_1 + c \\ r_2^2 &= z_2^2 + x_2^2 = az_2^2 + bz_2 + c \end{aligned} \qquad (15)$$

By inspection equation 15 is equivalent to its absolute value:

$$\begin{aligned} r_1^2 &= \left|az_1^2 + bz_1 + c\right| = \left|r_1^2\right| \\ r_2^2 &= \left|az_2^2 + bz_2 + c\right| = \left|r_2^2\right| \end{aligned}$$

therefore,

$$\begin{aligned} \ln\left|az_1^2 + bz_1 + c\right| &= \ln\left|r_1^2\right| = \ln r_1^2 \\ \ln\left|az_2^2 + bz_2 + c\right| &= \ln\left|r_2^2\right| = \ln r_2^2 \end{aligned}$$

The terms $2az_{1,2} + b$ in equation (14) can be rewritten as follows:

$$\begin{aligned} 2az_2 + b &= 2(g^2 + 1)z_2 + 2g(x_1 - gz_1) \\ 2az_2 + b &= 2g^2 z_2 + 2z_2 + 2gx_1 - 2g^2 z_1 \\ 2az_2 + b &= 2g^2(z_2 - z_1) + 2z_2 + 2gx_1 \end{aligned}$$

recall $g(z_2 - z_1) = x_2 - x_1$, so that,

$$2az_2 + b = 2g(x_2 - x_1) + 2z_2 + 2gx_1 = 2(z_2 + gx_2) \qquad (16)$$

similarly,

$$2az_1 + b = 2(z_1 + gx_1) \qquad (17)$$

The term $\sqrt{4ac-b^2}$ is rewritten as follows:

$$\begin{aligned}
\sqrt{4ac-b^2} &= \sqrt{4(1+g^2)(x_1-gz_1)^2-(2g(x_1-gz_1))^2} \\
\sqrt{4ac-b^2} &= \sqrt{4(x_1-gz_1)^2(1+g^2-g^2)} \\
\sqrt{4ac-b^2} &= 2\sqrt{(x_1-gz_1)^2} \\
\sqrt{4ac-b^2} &= 2|x_1-gz_1|
\end{aligned}$$

$$\sqrt{4ac-b^2} = 2\delta(x_1-gz_1) \qquad (18)$$

where,

$\delta = 1$ if $x_1 > gz_1$

$\delta = -1$ if $x_1 < gz_1$

Substituting equations (15), (16), (17), and (18) into equation (14), produces:

$$I_1 = \frac{J_x - gJ_z}{2\pi}\left[\left[\frac{1}{2(1+g^2)}\left(\ln r_2^2 - \ln r_1^2\right) - \frac{2g(x_1-gz_1)}{2(1+g^2)\delta(x_1-gz_1)}\left(\begin{array}{l}\tan^{-1}\left(\dfrac{2(z_2+gx_2)}{2\delta(x_1-gz_1)}\right)\\ -\tan^{-1}\left(\dfrac{2(z_1+gx_1)}{2\delta(x_1-gz_1)}\right)\end{array}\right)\right]\right]$$

recall that, $g = \dfrac{x_2-x_1}{z_2-z_1} \Rightarrow g(z_2-z_1) = x_2-x_1 \Rightarrow x_1-gz_1 = x_2-gz_2$,

which can be substituted into the above equation to produce:

$$I_1 = \frac{J_x - gJ_z}{2\pi}\left[\frac{1}{2(1+g^2)}\ln\left(\frac{r_2}{r_1}\right)^2 - \frac{\delta g}{(1+g^2)}\left(\tan^{-1}\left(\frac{\delta(z_2+gx_2)}{(x_2-gz_2)}\right) - \tan^{-1}\left(\frac{\delta(z_1+gx_1)}{(x_1-gz_1)}\right)\right)\right]$$

$$I_1 = \frac{J_x - gJ_z}{2\pi}\left[\frac{1}{2(1+g^2)}\ln\left(\frac{r_2}{r_1}\right)^2 - \frac{\delta g}{(1+g^2)}\left(\tan^{-1}(A_2) - \tan^{-1}(A_1)\right)\right]$$

where, $A_1 = \frac{\delta(z_1+gx_1)}{(x_1-gz_1)}$ and $A_2 = \frac{\delta(z_2+gx_2)}{(x_2-gz_2)}$

$$I_1 = \frac{J_x - gJ_z}{2\pi}\left[\frac{1}{2(1+g^2)}\ln\left(\frac{r_2}{r_1}\right)^2 - \frac{\delta g}{(1+g^2)}(\alpha_2 - \alpha_1)\right] \qquad (19)$$

where, $\alpha_1 = \tan^{-1}(A_1)$ and $\alpha_2 = \tan^{-1}(A_2)$ (20)

Now solving for I_2,

$$I_2 = \frac{-\left(J_z(x_1-gz_1)\right)}{2\pi}\int_{z_1}^{z_2}\frac{dz}{az^2+bz+c} = \frac{2}{\sqrt{4AC-B^2}}\tan^{-1}\left(\frac{2Ax+B}{\sqrt{4AC-B^2}}\right)$$

Using equations (14), (16), (17), and (18), as well as the appropriate integral identity we define:

$$I_2 = \frac{-\left(J_z(x_1-gz_1)\right)}{2\pi}\left[\frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2az+b}{\sqrt{4ac-b^2}}\right)\right]_{z_1}^{z_2}$$

$$I_2 = \frac{-\left(J_z(x_1-gz_1)\right)}{2\pi}\left[\frac{2}{\sqrt{4ac-b^2}}\left(\tan^{-1}\left(\frac{2az_2+b}{\sqrt{4ac-b^2}}\right) - \tan^{-1}\left(\frac{2az_1+b}{\sqrt{4ac-b^2}}\right)\right)\right]$$

$$I_2 = \frac{-\left(J_z(x_1-gz_1)\right)}{2\pi}\left[\frac{2}{2\delta(x_1-gz_1)}\left(\tan^{-1}(A_2) - \tan^{-1}(A_1)\right)\right]$$

$$I_2 = \frac{-\left(J_z(x_1 - gz_1)\right)}{2\pi}\left[\frac{1}{\delta(x_1 - gz_1)}(\alpha_2 - \alpha_1)\right]$$

$$I_2 = \frac{-J_z}{2\pi\delta}(\alpha_2 - \alpha_1)$$

$$I_2 = \frac{-J_z\delta}{2\pi}(\alpha_2 - \alpha_1) \qquad (21)$$

From equation (19) and (21), we can define V as:

$$V = I_1 + I_2$$

$$V = \frac{J_x - gJ_z}{2\pi}\left[\frac{1}{1+g^2}\ln\left(\frac{r_2}{r_1}\right) - \frac{\delta g}{(1+g^2)}(\alpha_2 - \alpha_1)\right] - \frac{J_z\delta}{2\pi}(\alpha_2 - \alpha_1)$$

Recall that $g = \frac{x_2 - x_1}{z_2 - z_1}$, then by letting $x_{21} = x_2 - x_1$ and $z_{21} = z_2 - z_1$ allows g to be defined as:

$$g = \frac{x_{21}}{z_{21}}$$

which produces:

$$V = \frac{z_{21}}{2\pi\sqrt{x_{21}^2 + z_{21}^2}}\left[\begin{array}{l} J_x\left(\frac{z_{21}}{\sqrt{x_{21}^2 + z_{21}^2}}\ln\left(\frac{r_2}{r_1}\right) - \frac{\delta x_{21}}{\sqrt{x_{21}^2 + z_{21}^2}}(\alpha_2 - \alpha_1)\right) \\ -J_z\left(\frac{x_{21}}{\sqrt{x_{21}^2 + z_{21}^2}}\ln\left(\frac{r_2}{r_1}\right) - \frac{\delta z_{21}}{\sqrt{x_{21}^2 + z_{21}^2}}(\alpha_2 - \alpha_1)\right)\end{array}\right] \qquad (22)$$

Solving for the horizontal component (H) can be done in a similar manner. Starting with equation (8) we integrate with respect to z to obtain:

$$H = \frac{1}{2\pi}\int_{z_1}^{z_2} \frac{J_x x + J_z z}{x^2 + z^2} dz$$

Recall that we defined $x = (z - z_1)g + x_1$, so subbing in this definition yields:

$$H = \frac{1}{2\pi}\int_{z_1}^{z_2} \frac{J_x\left((z - z_1)g + x_1\right) + J_z z}{\left((z - z_1)g + x_1\right)^2 + z^2} dz$$

which can then be split into two terms to create equation (23).

$$H = \frac{1}{2\pi}\int_{z_1}^{z_2} \frac{(J_z + gJ_x)z}{az^2 + bz + c} dz + \frac{1}{2\pi}\int_{z_1}^{z_2} \frac{J_x(x_1 - gz_1)}{az^2 + bz + c} dz \tag{23}$$

This can be written in short form using the following terms:

$$I_{1H} = \frac{1}{2\pi}\int_{z_1}^{z_2} \frac{(J_z + gJ_x)z}{az^2 + bz + c} dz$$

$$I_{1H} = \frac{J_z + gJ_x}{2\pi}\int_{z_1}^{z_2} \frac{z}{az^2 + bz + c} dz$$

$$I_{2H} = \frac{1}{2\pi}\int_{z_1}^{z_2} \frac{J_x(x_1 - gz_1)}{az^2 + bz + c} dz$$

$$I_{2H} = \frac{J_z + gJ_x}{2\pi}\int_{z_1}^{z_2} \frac{z}{az^2 + bz + c} dz$$

$$H = I_{1H} + I_{2H}$$

Using the appropriate identities from (13) we can integrate equation (23). For the first term integration yields:

$$I_{1H} = \frac{J_z + gJ_x}{2\pi}\left[\begin{array}{l} \frac{1}{2a}\left(\ln\left|az_2^2 + bz_2 + c\right| - \ln\left|az_1^2 + bz_1 + c\right|\right) \\ -\frac{b}{a\sqrt{4ac - b^2}}\left(\tan^{-1}\left(\frac{2az_2 + b}{\sqrt{4ac - b^2}}\right) - \tan^{-1}\left(\frac{2az_1 + b}{\sqrt{4ac - b^2}}\right)\right) \end{array}\right] \quad (24)$$

By applying the same transformations used in equations (14) and (19) we can transform equation (24) into the following expression:

$$I_{1H} = \frac{J_z + gJ_x}{2\pi}\left[\frac{1}{1+g^2}\ln\left(\frac{r_2}{r_1}\right) - \frac{\delta g}{1+g^2}(\alpha_2 - \alpha_1)\right] \qquad (25)$$

The 2nd term $I_{2H} = \frac{J_x(x_1 + gz_1)}{2\pi}\int_{z_1}^{z_2}\frac{dz}{az^2 + bz + c}$ is a similar to equation (21) except the term that lies outside the integral, thus:

$$I_{2H} = \frac{J_x}{2\pi}\delta(\alpha_2 - \alpha_1) \quad (26)$$

Combining I_{1H} and I_{2H} yields:

$$H = \frac{J_z + gJ_x}{2\pi}\left[\frac{1}{1+g^2}\ln\left(\frac{r_2}{r_1}\right) - \frac{\delta g}{1+g^2}(\alpha_2 - \alpha_1)\right] + \frac{J_x}{2\pi}\delta(\alpha_2 - \alpha_1)$$

Recalling that $x_{21} = x_2 - x_1$ and $z_{21} = z_2 - z_1$ allows g to be defined as in the following ways:

$$g = \frac{x_2 - x_1}{z_2 - z_1}$$

$$g = \frac{x_{21}}{z_{21}}$$

$$1 + g^2 = 1 + \left(\frac{x_{21}}{z_{21}}\right)^2$$

Using these definitions produces:

$$H = \frac{z_{21}}{2\pi\sqrt{x_{21}^2 + z_{21}^2}} \begin{bmatrix} J_z\left(\frac{z_{21}}{\sqrt{x_{21}^2 + z_{21}^2}}\ln\left(\frac{r_2}{r_1}\right) - \frac{\delta x_{21}}{\sqrt{x_{21}^2 + z_{21}^2}}(\alpha_2 - \alpha_1)\right) \\ + J_x\left(\frac{x_{21}}{\sqrt{x_{21}^2 + z_{21}^2}}\ln\left(\frac{r_2}{r_1}\right) + \frac{\delta z_{21}}{\sqrt{x_{21}^2 + z_{21}^2}}(\alpha_2 - \alpha_1)\right) \end{bmatrix} \qquad (27)$$

A simplified form of expressions (22) and (27) are as follows:

$$V = \frac{\gamma_z}{2\pi}\left[J_x\left(\gamma_z \ln\left(\frac{r_2}{r_1}\right) - \delta\gamma_x(\alpha_2 - \alpha_1)\right) - J_z\left(\gamma_x \ln\left(\frac{r_2}{r_1}\right) + \delta\gamma_z(\alpha_2 - \alpha_1)\right)\right]$$

$$H = \frac{\gamma_z}{2\pi}\left[J_z\left(\gamma_z \ln\left(\frac{r_2}{r_1}\right) - \delta\gamma_x(\alpha_2 - \alpha_1)\right) + J_x\left(\gamma_x \ln\left(\frac{r_2}{r_1}\right) + \delta\gamma_z(\alpha_2 - \alpha_1)\right)\right]$$

where,

$$\gamma_z = \frac{z_{21}}{\sqrt{x_{21}^2 + z_{21}^2}}$$

$$\gamma_x = \frac{x_{21}}{\sqrt{x_{21}^2 + z_{21}^2}}$$

$$r_1 = \sqrt{x_1^2 + z_1^2}$$

$$r_2 = \sqrt{x_2^2 + z_2^2}$$

$$\alpha_1 = tan^{-1}\left(\frac{\delta(z_1 + gx_1)}{x_1 - gz_1}\right)$$

$$\alpha_2 = tan^{-1}(\frac{\delta(z_2 + gx_2)}{x_2 - gz_2})$$

$\delta = 1$ if $x_1 > gz_1$

$\delta = -1$ if $x_1 < gz_1$

$$g = \frac{x_2 - x_1}{z_2 - z_1} = \frac{x_{21}}{z_{21}}$$

In a more simplified form our equations can be reduced to the following:

$$V = \frac{1}{2\pi}\left(J_x Q - J_z P\right) \qquad (28)$$

$$H = \frac{1}{2\pi}\left(J_z Q + J_x P\right) \qquad (29)$$

where,

$$Q = \gamma_z^2 \ln\left(\frac{r_2}{r_1}\right) - \delta\gamma_z\gamma_x(\alpha_2 - \alpha_1)$$
$$P = \gamma_z\gamma_x \ln\left(\frac{r_2}{r_1}\right) + \delta\gamma_z^2(\alpha_2 - \alpha_1)$$

Note that these equations differ from Talwani and Heirtzler (1962, 1964).

For an arbitrarily shaped polygon a point x_i, z_i represents a corner of the polygon and a point x_{i+1}, z_{i+1} to be the next nearest corner of the polygon. Equations (28) and (29) represent the magnetic strength of the rectangular region AFGBA for only one side of the polygon. For a polygon with n-sides there are a n number of prisms of the same form as AFGBA. By choosing the proper sign for each prism that comprise the polygon and summing their contribution of the magnetic field strength at the origin we can produce the magnetic anomaly for the entire polygon (AFGBA), at that point. Calculation for a positive anomaly requires calculation of the polygon clockwise with reference to the origin as depicted in Fig. 3 and summing the contribution of each side.

To evaluate the total intensity anomaly, T, we need to sum the projection of H and V along the direction of the total field. This can be done by manipulating the magnetization vectors associated with total magnetization (J) while using the convention shown in Fig. 2. In general, total magnetization is a superposition of induced (J_i) or remnant magnetization (J_r) which are given by:

$$\vec{J}_i = J_i\left(\cos I \cos D\hat{n} + \cos I \sin D\hat{e} + \sin I\hat{v}\right) \qquad (30)$$

$$\vec{J}_r = J_r\left(\cos I_r \cos D_r\hat{n} + \cos I_r \sin D_r\hat{e} + \sin I_r\hat{v}\right) \qquad (31)$$

where $\hat{n}$ = north, $\hat{e}$ = east, $\hat{v}$ = vertical, I = induced inclination, D = induced declination, I_r = remnant inclination, D_r = remnant declination. Using equations (30) and (31) the angle (Δ) between the two vectors can be determined as follows:

$$\Delta = \cos^{-1}\left(\frac{\vec{J}_i \cdot \vec{J}_r}{J_i J_r}\right)$$
$$\Delta = \cos^{-1}\left(\hat{J}_i \cdot \hat{J}_r\right)$$
$$\Delta = \cos^{-1}\left(\cos I \cos D Cos I_r \cos D_r + \cos I \sin D \cos I_r \sin D_r + \sin I \sin I_r\right)$$

This angle Δ can be used to calculate the magnitude of the total magnetization (J) as well as its inclination (A) and declination (B). Using the cosine law the total magnetization J is defined as:

$$J^2 = J_i^2 + J_r^2 - 2J_iJ_r \cos\Delta$$

To determine the inclination (A) and declination (B) of J we split J_i and J_r into their horizontal (J_{iH} and J_{rH}, respectively) and vertical components (J_{iV} and J_{rV}, respectively). Inclination is then derived as follows:

$$J_V = J_{iV} + J_{rV}$$
$$J_V = J_i \sin I + J_r \sin I_r$$
$$J \sin A = J_i \sin I + J_r \sin I_r$$
$$\sin A = \frac{J_i \sin I + J_r \sin I_r}{J}$$
$$A = \sin^{-1}\left(\frac{J_i \sin I + J_r \sin I_r}{J}\right)$$

Similarly, declination it derived as follows:

$$J_H = J \cos A$$
$$J_H = J_{iH} + J_{rH}$$
$$J_H = J_i \cos I + J_r \cos I_r$$
$$J_H \hat{n} = J_H \cos B$$
$$J_H \cos B = J_{iH} \hat{n} + J_{rH} \hat{n}$$
$$J_H \cos B = J_{iH} \cos D + J_{rH} \cos D_r$$
$$J_H \cos B = J_i \cos I \cos D + J_r \cos I_r \cos D_r$$
$$\cos B = \frac{J_i \cos I \cos D + J_r \cos I_r \cos D_r}{J_H}$$
$$B = \cos^{-1}\left(\frac{J_i \cos I \cos D + J_r \cos I_r \cos D_r}{J \cos A}\right)$$

The intensity of magnetization of magnetization in the x and z directions in the terms of total magnetization, J, in terms of A, B, C can be defined as:

$$J_x = J \cos(A) \cos(C - B)$$
$$J_z = J \sin(A)$$

The total intensity anomaly (T) can then be defined as:

$$T = V \sin(A) + H \cos(A) \cos(C - B) \qquad (32)$$

Detailed derivation of Talwani and Heirtzler formulas to calculate magnetic anomalies caused by two dimensional structures of arbitrary shape

Rederiving equations (3) and (4) from Talwani and Heirtzler (1964), and equations (9) and (23) from our derivation for the vertical and horizontal components of the magnetic intensity. Talwani and Heirtzler (1964) begin their derivation not by defining x and z as shown in Fig. 3, but defining x as:

$$x = x_1 + z_1 \cot(\phi) - zcot(\phi) \qquad (33)$$

which means

$$x = x_2 + z_2 \cot(\phi) - zcot(\phi) \qquad (34)$$

gives

$$\cot(\phi) = \frac{x_1 - x_2}{z_2 - z_1} \qquad (35)$$

This derivation fails as equation should be -cot(ϕ)

Using the equations for the Vertical and Horizontal Magnetic field

$$V = 2\int_{z_1}^{z_2} \frac{J_x z - J_z x}{x^2 + z^2} dz$$

$$H = 2\int_{z_1}^{z_2} \frac{J_x x + J_z z}{x^2 + z^2} dz$$

Subbing in for x using Equation (33) we obtain

$$V = 2\int_{z_1}^{z_2} \frac{J_x z - J_z\left(x_1 + z_1 \cot(\phi) - zcot(\phi)\right)}{\left(x_1 + z_1 \cot(\phi) - zcot(\phi)\right)^2 + z^2} dz \qquad (36)$$

$$H = 2\int_{z_1}^{z_2} \frac{J_x\left(x_1 + z_1 \cot(\phi) - zcot(\phi)\right) + J_z z}{\left(x_1 + z_1 \cot(\phi) - zcot(\phi)\right)^2 + z^2} dz \qquad (37)$$

Rearranging gives

$$V = 2(J_x + J_z \cot(\phi)) \int_{z_1}^{z_2} \frac{z}{\left(x_1 + z_1 \cot(\phi) - z cot(\phi)\right)^2 + z^2} dz$$
$$- 2J_z(x_1 + z_1 \cot(\phi)) \int_{z_1}^{z_2} \frac{dz}{\left(x_1 + z_1 \cot(\phi) - z cot(\phi)\right)^2 + z^2} dz$$

$$H = 2(J_z - J_x \cot(\phi)) \int_{z_1}^{z_2} \frac{z}{\left(x_1 + z_1 \cot \cot(\phi) - z cot(\phi)\right)^2 + z^2} dz$$
$$+ 2J_x(x_1 + z_1 \cot(\phi)) \int_{z_1}^{z_2} \frac{dz}{\left(x_1 + z_1 \cot(\phi) - z cot(\phi)\right)^2 + z^2} dz$$

Setting I_1 and I_2

$$I_1 = \int_{z_1}^{z_2} \frac{z}{\left(x_1 + z_1 \cot \cot(\phi) - z cot(\phi)\right)^2 + z^2} dz$$

$$I_2 = \int_{z_1}^{z_2} \frac{dz}{\left(x_1 + z_1 \cot \cot(\phi) - z cot(\phi)\right)^2 + z^2}$$

Solving the denominator

$$\left(x_1 + z_1 \cot(\phi) - z cot(\phi)\right)^2 + z^2$$
$$\left(1 + cot^2(\phi)\right)z^2 + \left(-2x_1 \cot(\phi) - 2z_1 cot^2(\phi)\right)z + (x_1 + z_1 \cot(\phi))^2 + cot^2(\phi))$$

$$A = \left(1 + cot^2(\phi)\right)$$
$$B = -2x_1 \cot(\phi) - 2z_1 cot^2(\phi) = 2\cot(\phi)(x_1 + z_1 cot(\phi))$$
$$C = (x_1 + z_1 \cot(\phi))^2 + cot^2(\phi))$$

Equation I_1 becomes

$$I_1 = \int_{z_1}^{z_2} \frac{z}{Az^2 + Bz + C} dz$$

Equation I_2 becomes

$$I_2 = \int_{z_1}^{z_2} \frac{dz}{Az^2 + Bz + C}$$

Solving Equation I_1

$$I_1 = \frac{1}{2A} \int_{z_1}^{z_2} \frac{2Az + B}{Az^2 + Bz + C} dz - \frac{B}{2A} \int_{z_2}^{z_2} \frac{dz}{Az^2 + Bz + C}$$

$$I_1 = \frac{1}{2A} I_3 - \frac{B}{2A} I_2$$

Solving I_3

$$I_3 = \int_{z_1}^{z_2} \frac{2Az + B}{Az^2 + Bz + C} dz$$

Substituting in

$$u = Az^2 + Bz + C$$

$$du = (2Az + B)dz$$

We get

$$I_3 = \int \frac{1}{u} du$$

Solved as

$$I_3 = \ln|u|$$

Subbing back in for u

$$I_3 = \ln |\mathrm{A}z^2 + Bz + C| \;\Big|_{z_1}^{z_2}$$

$$I_3 = \ln|Az_2^2 + Bz_2 + C| - \ln|Az_1^2 + Bz_1 + C|$$

Solving $Az_1^2 + Bz_1 + C$

$$\left(1 + cot^2(\phi)\right)z_1^2 + \left(-2x_1 \cot(\phi) - 2z_1 cot^2(\phi)\right)z_1 + (x_1 + z_1 \cot(\phi))^2$$

$$+ z_1^2 cot^2(\phi) - 2x_1 z_1 \cot(\phi) - 2\, z_1^2 cot^2(\phi)$$

$$+ x_1^2 + 2x_1 z_1 - 2z_1^2 cot^2(\phi)$$

$z_1^2 + x_1^2$

Where $r_1^2 = z_1^2 + x_1^2$

Solving $Az_2^2 + Bz_2 + C$

$$x_1 = x_2 + (z_2 - z_1)\cot(\phi)$$

$$\left(1 + cot^2(\phi)\right)z_2^2 + \left(-2x_1\cot(\phi) - 2z_1 cot^2(\phi)\right)z_2 + (x_1 + z_1\cot(\phi))^2$$

Rearranging

$$(z_2^2 + z_2^2 cot^2(\phi))$$

Becomes

$$r_2^2 = z_2^2 + x_2^2$$

Plugging back into

$$I_3 = 2\ln\left|\frac{r_2}{r_1}\right|$$

Solving I_2

Checking A is not equal to 0

$$A = 1 + cot^2(\phi) > 0$$

Checking that 4AC – B2 > 0 for I_2

$$4\left(1 + cot^2(\phi)\right)(x_1 + z_1\cot(\phi))^2 - 4cot^2(\phi)(x_1 + z_1\cot(\phi))^2$$

$$\left(1 + cot^2(\phi) - cot^2(\phi)\right)(x_1 + z_1 cot(\phi))^2 = (x_1 + z_1 cot(\phi))^2 > 0$$

Completing the square for the denominator

$$Az^2 + Bz + C = \sqrt{A}z + B + C - \frac{B^2}{A}$$

$$I_2 = \int_{z_1}^{z_2} \frac{dz}{\left(\sqrt{A}z + \frac{B}{2\sqrt{A}}\right)^2 + C - \frac{B^2}{4A}}$$

$$u = \sqrt{A}z + \frac{B}{2\sqrt{A}}$$

$$v^2 = C - \frac{B^2}{4A}$$

$$du = \sqrt{A}dz$$

Subbing in u and v gives

$$I_2 = \frac{1}{\sqrt{A}} \int \frac{du}{u^2 + v^2}$$

$$u = vtan(\beta)$$

$$du = vsec^2(\beta) + v^2 = v^2sec^2(\beta)$$

$$I_2 = \frac{1}{v\sqrt{A}} \int d\beta$$

$$I_2 = \frac{1}{\left(v\sqrt{A}\right)}(\beta_2 - \beta_1)$$

Checking β

$$\theta = tan^{-1}\left(\frac{u}{v}\right)$$

subbing back in u and v

$$\theta = tan^{-1}\left(\frac{\sqrt{A}z + \frac{B}{2\sqrt{A}}}{\sqrt{C - \frac{B^2}{A}}}\right)$$

Solving $z = z_1$

$$2Az_1 + B = 2\left(1 + cot^2(\phi)\right)z_1 - 2x_1\cot(\phi) - 2z_1cot^2(\phi)1 - x_1\,cot(\phi))$$

Solving $z = z_2$

$$x_1 = x_2 + (z_2 - z_1)\cot(\phi)$$

$$2Az_2 + B = 2\left(1 + cot^2(\phi)\right)z_1 - 2x_1\cot(\phi) - 2z_1cot^2(\phi)$$

$$2(2(z_2 - x_2\cot(\phi))$$

Solving the denominator

$$\sqrt{4AB - C^2} = \sqrt{4(x_1 + z_1 \cot(\phi))^2}$$

$$2|x_1 + z_1 \cot(\phi)|$$

Using $x_1 = x_2 + (z_2 - z_1)\cot(\phi)$

$$2|x_2 + z_2 \cot(\phi)|$$

which gives

$$\theta_1 = tan^{-1}\left(\frac{z_1 - x_1 \cot(\phi)}{|x_1 + z_1 \cot(\phi)|}\right) \quad (38)$$

$$\theta_2 = tan^{-1}\left(\frac{z_2 - x_2 \cot(\phi)}{|x_2 + z_2 \cot(\phi)|}\right) \quad (39)$$

Which are not the same as defined in Fig. 1b.

$$V = 2(J_x + J_z \cot(\phi))I_1 - 2J_z(x_1 + z_1 \cot(\phi))I_2 \quad (40)$$

$$H = 2(J_z - J_x \cot(\phi))I_1 + 2J_x(x_1 + z_1 \cot(\phi))I_2 \quad (41)$$

Subbing I_1 and I_2 back into V and H gives

$$V = 2(J_x + J_z \cot(\phi))I_1 - 2J_z(x_1 + z_1 \cot(\phi))I_2$$

$$H = 2(J_z - J_x \cot(\phi))I_1 + 2J_x(x_1 + z_1 \cot(\phi))I_2$$

$$V = 2(J_x + J_z \cot(\phi))\left(\frac{1}{2A}\ln\left|\frac{r_2}{r_1}\right| - \frac{B}{2A^{\frac{3}{2}}}(\theta_2 - \theta_1)\right) - \frac{2J_z(x_1 + z_1 \cot(\phi))B}{2A^{\frac{3}{2}}}(\theta_2 - \theta_1)$$

$$H = 2(J_z - J_x \cot(\phi))\left(\frac{1}{2A}\ln\left|\frac{r_2}{r_1}\right| - \frac{B}{2A^{\frac{3}{2}}}(\theta_2 - \theta_1)\right) + \frac{2J_x(x_1 + z_1 \cot(\phi))B}{2A^{\frac{3}{2}}}(\theta_2 - \theta_1)$$

Rearranging V and H and subbing in values for A, B, and C

$$V = 2(J_x\,[sin^2\,(\phi)\ln\left|\frac{r_2}{r_1}\right| + \sin(\phi)\cos(\phi)\,\frac{x_1 + z_1 \cot(\phi)}{|x_1 + z_1 \cot(\phi)|}(\theta_2 - \theta_1)] + J_z[-\sin(\phi)\cos(\phi)\ln\left|\frac{r_2}{r_1}\right| + sin^2\,(\phi)\frac{x_1 + z_1 \cot(\phi)}{|x_1 + z_1 \cot(\phi)|}(\theta_2 - \theta_1)\,]$$

$$H = 2(J_x\,[-\sin(\phi)\cos(\phi)\ln\left|\frac{r_2}{r_1}\right| + sin^2\,(\phi)\frac{x_1 + z_1 \cot(\phi)}{|x_1 + z_1 \cot(\phi)|}(\theta_2 - \theta_1)] + J_z[sin^2(\phi)\ln\left|\frac{r_2}{r_1}\right| + \sin(\phi)\cos(\phi)\frac{x_1 + z_1 \cot(\phi)}{|x_1 + z_1 \cot(\phi)|}(\theta_2 - \theta_1)]$$

Comparing to equations (3) and (4) in Talwani and Heirtzler (1964) we get the same answer with the exception of:

$$\delta = \frac{x_1+z_1\cot(\phi)}{|x_1+z_1\cot(\phi)|} \qquad (42)$$

And θ from Talwani and Heirtzler (1964) ≠ θ derived here.

Rewriting to get Q and P

$$P = -\sin(\phi)\cos(\phi)\ln\left|\frac{r_2}{r_1}\right| + sin^2(\phi)\frac{x_1 + z_1\cot(\phi)}{|x_1 + z_1\cot(\phi)|}(\theta_2 - \theta_1)$$

$$Q = sin^2(\phi)\ln\left|\frac{r_2}{r_1}\right| + \sin(\phi)\cos(\phi)\frac{x_1 + z_1\cot(\phi)}{|x_1 + z_1\cot(\phi)|}(\theta_2 - \theta_1)$$

using Talwani and Heirtzler definition of ϕ gives

$$P = \frac{z_{21}x_{12}}{z_{21}^2 + x_{12}^2}\ln\left|\frac{r_2}{r_1}\right| - \frac{z_{21}^2}{z_{21}^2 + x_{12}^2}\frac{x_1 + z_1\cot(\phi)}{|x_1 + z_1\cot(\phi)|}(\theta_2 - \theta_1)$$

$$Q = \frac{z_{21}^2}{z_{21}^2 + x_{12}^2}\ln\left|\frac{r_2}{r_1}\right| + \frac{z_{21}x_{12}}{z_{21}^2 + x_{12}^2}\frac{x_1 + z_1\cot(\phi)}{|x_1 + z_1\cot(\phi)|}(\theta_2 - \theta_1)$$

This shows dissimilarity with our derivation of the P and Q terms due to the different definition of the angle θ and the δ term in Talwani and Heirtzler (1964).

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Appendix

Click here to download the open source code on GitHub.
Click here to download the software manual.
Click here to download the software zipped p-code. This code is verified for Matlab 2015-2021.
Click here to go to the software web page.